

## CHAPTER V

### LOGARITHMS AND MUSICAL INTERVALS

The logarithm allows us to convert ratios into cents or semitones, which are the most natural representations of intervals. We will review some basic facts. In this discussion,  $b$  will be a positive number  $\neq 1$  which will be called the *base* of the logarithm.

**Exponents.** If  $n$  is a positive integer, then  $b^n$  is the  $n$ -fold product  $b \cdot b \cdots b$ ,  $b^{-n} = 1/b^n$ , and  $b^{1/n} = \sqrt[n]{b}$ . These facts, together with the the rule of exponents

$$b^{st} = (b^s)^t,$$

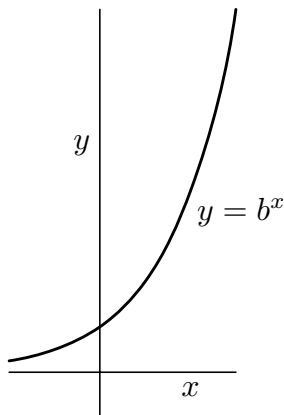
give meaning to  $b^x$  for all rational numbers  $x$ . For example,  $b^{-2/3}$  can be calculated as

$$b^{(-2) \cdot (\frac{1}{3})} = (b^{-2})^{\frac{1}{3}} = \left(\frac{1}{b^2}\right)^{\frac{1}{3}} = \sqrt[3]{\frac{1}{b^2}}$$

**Exponential Functions.** The calculus concept of limit provides a definition  $b^x$  for all real numbers  $x$  in such a way that  $f(x) = b^x$  is a continuous function. Its domain is the set of real numbers  $\mathbb{R}$  and, (since  $b \neq 1$ ) its range is the set of positive real numbers  $\mathbb{R}^+$ .

$$f : \mathbb{R} \rightarrow \mathbb{R}^+$$

For  $b > 1$  the function is increasing, hence it gives a one-to-one correspondents between the sets  $\mathbb{R}$  and  $\mathbb{R}^+$ . The graph of  $f(x) = b^x$  is:



The number  $b$  is called the *base* of the exponential function. It will always be a positive real number,  $\neq 1$ , and we generally take it to be  $> 1$ .

**Logarithmic Functions.** Since the function  $f$  is one-to-one and onto, it has an inverse function. The function  $g(x) = \log_b(x)$  is defined as the inverse function of  $f(x) = b^x$ , that is to say

$$f(g(x)) = x, \text{ which says } b^{\log_b x} = x$$

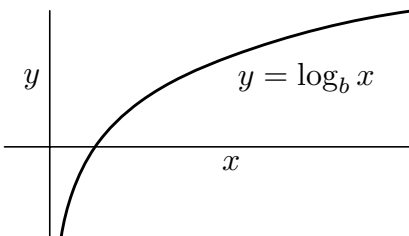
and

$$g(f(x)) = x, \text{ which says } \log_b(b^x) = x.$$

Thus the statement  $\log_b x = y$  means exactly the same as  $b^y = x$ . The domain of  $g(x)$  (= the range of  $f(x)$ ) is  $\mathbb{R}^+$ ; the range of  $g(x)$  (= the domain of  $f(x)$ ) is  $\mathbb{R}$ .

$$g : \mathbb{R}^+ \rightarrow \mathbb{R}$$

The graph of  $g(x) = \log_b x$  is obtained by flipping the graph of  $f(x) = b^x$  around the line  $y = x$ . Again assuming  $b \geq 1$ , we see that  $g(x) = \log_b x$  is an increasing, hence 1 to 1, function whose graph is:



The number  $b$  is called the *base* of the logarithm. Remember that it is always positive,  $\neq 1$ , and we usually take it to be  $> 1$ .

If we recognize a number  $x$  as a power of  $b$  then we can say immediately what  $\log_b x$  is. For example,  $\log_3 9 = 2$  (since  $3^2 = 9$ ) and  $\log_b \sqrt{b} = \frac{1}{2}$  (since  $b^{\frac{1}{2}} = \sqrt{b}$ ).

**Properties of Logarithms.** In a certain sense, logarithms transform multiplication to addition; this is why they are useful in understanding and measuring intervals. The basic properties which underlie this are:

(L1)

$$\log_b xy = \log_b x + \log_b y$$

(L2)

$$\log_b \frac{x}{y} = \log_b x - \log_b y$$

(L3)

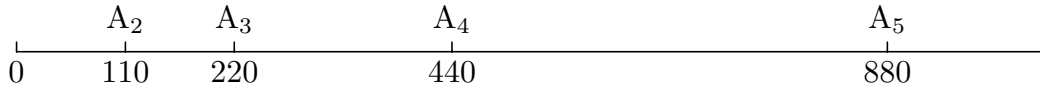
$$\log_b(x^p) = p \log_b x$$

for any real numbers  $x, y > 0$  and any real number  $p$ . Property (L1) derives from the law of exponents  $b^{s+t} = b^s b^t$  as follows: Let  $s = \log_b x$  and  $t = \log_b y$ . Then

$$b^{s+t} = b^s b^t = b^{\log_b x} b^{\log_b y} = xy,$$

according to the above principle. But  $b^{s+t} = xy$  means  $s + t = \log_b xy$ , completing the proof.

**Logarithmic Scale for Pitch.** Property (L2) assures us a pleasing outcome if we plot pitches on an axis corresponding to the *logarithm* of their frequency: Pairs of pitches which have the same interval will lie the same distance apart on the axis. For suppose pitches (i.e., frequencies)  $x$  and  $y$  create the same interval as the two  $x'$  and  $y'$ . This means the ratio of the frequencies is the same, i.e.,  $x/y = x'/y'$ . According to (L2), then, we have  $\log_b x - \log_b y = \log_b x' - \log_b y'$ , which says the distance between  $\log_b x$  and  $\log_b y$  is the same as the distance between  $\log_b x'$  and  $\log_b y'$ . Recall that when we plot  $A_2, A_3, A_4$ , and  $A_5$  according to their frequencies we get:



If we instead plot these notes according to the logarithm of their frequencies we find that they are equally spaced. For example, choosing  $b = 10$ , we get:



**Different Bases.** We will need to compare logarithms of different bases. If  $a$  is another positive number  $\neq 1$ , we have the following relationship between  $\log_b x$  and  $\log_a x$ :

(L4)  $\log_b x = \frac{\log_a x}{\log_a b}$

This is established as follows. Let  $u = \log_a x$ ,  $v = \log_b x$ , and  $w = \log_a b$ . Then  $a^u = x$ ,  $b^v = x$ , and  $a^w = b$ . The last two equations give us  $x = (a^w)^v = a^{wv}$ . This establishes that  $wv = \log_a x = u$  from which (L4) is immediate.

The result is that the functions  $\log_b x$  and  $\log_a x$  are proportional as functions, with constant of proportionality  $1/\log_a b$ . For example, if we compare the graphs of  $g(x) = \log_6 x$  and  $\log_3 x$  we see that the latter is obtained by “stretching” the former vertically by a factor of  $\log_3 6 \approx 1.631$ .

**Calculating Using the Natural Logarithm.** Scientists often prefer use the *natural logarithm*, which has as its base the transcendental number  $e$ , approximated by 2.71828. This number and its logarithm are highly significant in mathematics for reasons that will not be explained here. It is common to denote  $\log_e x$  by  $\ln x$ . Any calculator that has  $\ln$  as

a supplied function can be used to evaluate any logarithm, using (L4). Setting  $a = e$  the formula reads:

$$(L5) \quad \boxed{\log_b x = \frac{\ln x}{\ln b}}$$

Similarly, one can calculate any logarithm using  $\log_{10}$ , which is supplied with many calculators.

**Converting Intervals from Multiplicative to Additive Measurement.** Suppose we want the octave interval to appear as the distance 1 on the logarithmic axis. If two frequencies  $x$  and  $y$  are an octave apart,  $x$  being the greater frequency, then we know  $x/y = 2$ . We need, then,  $1 = \log_b x - \log_b y = \log_b(x/y) = \log_b 2$ . But  $\log_b 2 = 1$  means  $b^1 = 2$ , i.e.,  $b = 2$ . Therefore 2 is our desired base.

Returning to a problem posed in the last section, suppose we are given a musical interval represented as a ratio  $r$  and we wish to convert it to one of the standard measurements for intervals such as octaves, steps, semitones, or cents. We have noted that if  $x$  is the measurement of the interval in cents, then  $r = 2^{x/1200}$ . Applying the function  $\log_2$  to both sides of this equation yields  $\log_2 r = \log_2(2^{x/1200}) = x/1200$ , i.e.,  $x = 1200 \log_2 r$ . Thus we have:

$$(4.1) \quad \boxed{\text{The interval ratio } r \text{ is measured in cents by } 1200 \log_2 r.}$$

Similar reasoning shows:

$$(4.2) \quad \boxed{\text{The interval ratio } r \text{ is measured in semitones by } 12 \log_2 r.}$$

and:

$$(4.3) \quad \boxed{\text{The interval ratio } r \text{ is measured in octaves by } \log_2 r.}$$

Using (L4) we can make these conversions using any base. For example, if our calculator only provides the natural logarithm, we appeal to (L5) to make the conversion by evaluating  $x = 1200 \log_2 r$  as

$$x = 1200 \left( \frac{\ln r}{\ln 2} \right).$$

Note that if  $r$  is less than 1, then  $\ln r < 0$ , hence measurement  $x$  in cents is negative. This is logical, for if  $r$  is the interval from frequency  $f_1$  to frequency  $f_2$  we have  $r = f_2/f_1 < 1$ . This says  $f_2$  is less than  $f_1$ , so that the interval in cents is given by a negative number.

We note that the conversion in (4.1) and (4.2) can be expressed as  $\log_b r$  for an appropriate base  $b$ . For example, if we wish to the ratio  $r$  is  $x$  semitones, we have

$$r = 2^{\frac{x}{12}} = \left( 2^{\frac{1}{12}} \right)^x = \left( \sqrt[12]{2} \right)^x.$$

Applying  $\log_b$  with  $b = \sqrt[12]{2}$  we get

$$x = \log_{\sqrt[12]{2}} r.$$

**Example.** Let us measure in cents the interval given by the ratio  $3/2$  and find the chromatic interval which best approximates this interval. If  $x$  is the measurement in cents, we have

$$\begin{aligned} x &= 1200 \left( \frac{\ln(3/2)}{\ln 2} \right) \\ &= 1200 \left( \frac{\ln 3 - \ln 2}{\ln 2} \right) && \text{by (L2)} \\ &= 1200 \left( \frac{\ln 3}{\ln 2} - 1 \right) \\ &\approx 701.955 && \text{using a calculator.} \end{aligned}$$

Thus the ratio  $3/2$  is very close to 702 cents. A fifth is 700 cents ( $= 7$  semitones), so our interval is 2 cents greater than a fifth. The fifth is the chromatic interval that gives the best approximation.

### Exercises

- (1) Evaluate without a calculator by writing the argument of  $\log$  as a power of the base. Write down each step of the simplification, e.g.,  $\log_3 3\sqrt{3} = \log_3 3^{3/2} = \frac{3}{2} \log_3 3 = \frac{3}{2}$ :
  - (a)  $\log_{10}(0.01)$
  - (b)  $\log_2 16$
  - (c)  $\log_5 \sqrt[3]{25}$
  - (d)  $\log_c \sqrt[r]{c^\ell}$
- (2) Express as a single logarithm without coefficient, i.e., in the form  $\log_b x$  (do not evaluate with a calculator):
  - (a)  $\log_2 5 + \log_2 3$
  - (b)  $\log_4 7 - 2 \log_4 11$
  - (c)  $\log_3 10 + \log_9 16$
  - (d)  $2 \log_a x^2 - \frac{1}{2} \log_{\sqrt{a}} x$
- (3) Sketch the graphs of:
  - (a)  $f(x) = 10^x$
  - (b)  $g(x) = \log_{10} x$
  - (c)  $r(x) = 2^x$
  - (d)  $s(x) = \log_2 x$
- (4) For a base  $b$  with  $0 < b < 1$ , sketch the graph of  $f(x) = b^x$  and  $g(x) = \log_b x$  and explain what happens if we plot pitches according to the logarithm of their frequency using the base  $b$ .
- (5) Prove these properties of logarithms

$$\begin{aligned} \log_b \frac{x}{y} &= \log_b x - \log_b y \\ \log_b(x^p) &= p \log_b x \end{aligned}$$

using laws of exponents.

(6) Suppose  $n \in \mathbb{Z}^+$  and we want the interval of an octave to correspond to a distance of  $n$  on the logarithmic axis. What base should we choose? Justify your answer.


(7) Convert to semitones the intervals given by the following ratios: (Round off to 2 digits to the right of the decimal.)


(a) 3            (b) 0.8            (c)  $\frac{4}{3}$             (d)  $\sqrt[3]{2}$             (e)  $e$


(8) Convert to cents the intervals given by the following ratios, rounding off to the nearest whole cent:

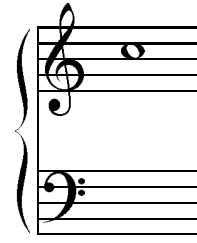
(a) 1.25            (b) 1.1            (c)  $\frac{7}{4}$             (d)  $\frac{2}{3}$             (e)  $\pi$

(9) Write on the staff the note which best approximates the frequency having the given interval ratio  $r$  from the given note:

(a)   $r = 3$

(b)   $r = \frac{2}{5}$

(c)   $r = 2.3$

(d)   $r = \pi^{-1}$