CHAPTER VI

CHROMATIC SCALES

As we have seen, the interval of an octave is represented as a ratio by the number 2. Two pitches an octave apart, played simultaneously, present the ear with a two against one pattern that the brain easily recognizes as a pleasing consonance. Therefore it is easy to understand what led Western music, as well other musical traditions, to embrace the octave and to incorporate octave identification into its notation.

What is not so apparent is what led to the subdivision of the octave into twelve equal intervals, a custom which is less universal, and which only came into acceptance within the last 200 years. It is quite natural to wonder if the subdivision of the octave into 12 equal intervals is purely arbitrary, or if some natural phenomenon brought Western music toward this practice. This will be discussed later.

Meanwhile, we can explore the sound of a chromatic scales which equally divide the octave differently. For example we might want to design and listen to a chromatic scale which divides the octave into 19, 10, or 5 equal intervals.

Non-standard chromatic scales. If we obtain a chromatic unit by dividing the octave into n equal intervals, where n is a positive integer, this unit measured as a ratio is $2^{\frac{1}{n}}$, and measured in cents is $1200/n$. We will refer to the resulting chromatic scale as the *n-chromatic scale,* and the smallest chromatic interval ($= \frac{1}{n}$ octaves) as the *n-chromatic unit.* Thus the 12-chromatic unit is the usual semitone. The n-chromatic unit is measured in cents by $1200/n$ and has interval ratio $\sqrt[n]{2}$.

Detuning. Many synthesizers allow notes of the chromatic scale to be individually detuned in cents, and this feature will allow one to experience the sound of such non-standard chromatic scales where $n < 12$.

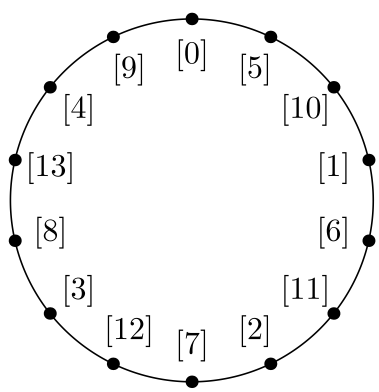
If we chose $n = 4$ the smallest interval would be $1200/4 = 300$ cents, which is the keyboard's minor third. So the scale can be played on a keyboard without detuning. For example, in G we would play G, B$^\flat$, D$^\flat$, and E. For $n = 3$ the smallest chromatic interval would be the major third, and for $n = 6$ it would be the (whole) step.

If we chose $n = 5$, some detuning is required. We could detune the notes A,B,C,D,E so that the five keys G,A,B,C,D play the five-note chromatic scale. The interval in cents would be $1200/5 = 240$. We need the interval between G and A to be 240. The default interval is one step, or 200 cents, so A should be detuned upward by 40 cents. B needs to be 240 cents above the detuned A, so B should be detuned upward by 80 cents. C, which be default is only 100 cents above the default B, will need to be detuned upward by 220

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX cents. Detuning D upward by 260 cents completes the task. With this accomplished, and using only these five keys, we can listen to the sound of the five-note chromatic scale and experiment with melody and harmony in this tuning environment.

Generating intervals. For a fixed positive integer n, the generating intervals are those modular n-chromatic intervals I for which all modular n-chromatic intervals can be expressed as iterations of I. We will see later that the generating intervals correspond to those $[m] \in \mathbb{Z}_n$ which are generators for the group $\mathbb{Z}_n$, which, it will be shown, is equivalent to saying $(m,n) = 1$. In number theory, the *Euler phi function* ϕ is defined as the function from $\phi : \mathbb{Z}^+ \to \mathbb{Z}^+$ which takes n to the number of integers $m < n$ in $\mathbb{Z}^+$ such that $(m,n) = 1$. Thus $\phi(n)$ also counts the number of generating intervals in the equally tempered n-scale. For any such I the circle based on I contains all intervals in the chromatic scale.

Example. Consider the case $n = 14$. The numbers 1, 3, 5, 9, and 11 are the positive integers < 14 which are relatively prime to 14, so $\phi(14) = 5$. These five numbers, modulo 14, give the five generating intervals in the 14-chromatic scale. For the interval I corresponding to [5], its circle of intervals is:



Approximating Standard Keyboard Intervals. Let us determine how closely some of the standard intervals of the standard 12-scale's intervals can be approximated using the chromatic 14-scale. Clearly the tritone is precisely 7 chromatic steps in the 14-scale, being one-half of an octave. More generally, ℓ semitones can be calculated by:

$$\ell \text{ semitones} = \ell \cdot \frac{\text{14-chromatic units}}{\text{12 semitones}} = \tfrac{7}{6}\ell \text{ 14-chromatic units.}$$

For example, the keyboard's interval of a fourth, being 5 semitones, is $(\frac{7}{6}) \cdot 5 = \frac{35}{6} \approx 5.833$ 14-chromatic units. Hence it is best approximated in the 14-scale by 6 units. Now, since the chronmatic unit is $(1/14)^{th}$ of an octave, it is measured in cents by $1200/14 \approx 85.714$. Therefore 6 units is $6 \cdot (1200/14) \approx 514.29$ cents, which is 14.29 cents greater than the fourth.

To calculate a ratio r in 14-chromatic units, we reason as follows: A 14-chromatic unit has ratio $2^{\frac{1}{14}} = \sqrt[14]{2}$. If x is the measurement of r in 14-chromatic units, then $r = (\sqrt[14]{2})^x = 2^{\frac{x}{14}}$.

Solving for x using the logarithm, we have

$$x = 14 \log_2 r = 14 \frac{\ln r}{\ln 2}.$$

For example, the ratio 0.75 is $14 \ln(0.75)/\ln 2 \approx -5.81$ chromatic units (i.e., 5.81 units downward).

Twelve-Tone Music. In the 1920s Arnold Schoenberg (1874-1951) began developing the twelve-tone technique of composition, a techique that is heavily based on the subdivision of the octave into 12 equal units. It was continued by Anton Webern (1883-1945), Alban Berg (1885-1935), and Milton Babbitt (b. 1916). In this music, the consonance is largely abandoned in favor of combinatorics. The reader is referred to Chapter 2 of [1], which gives an excellent exposition on methods of twelve-tone composing, and from which the twelve-tone examples which follow are taken.

A twelve-tone composition is based on a *row chart,* which is a 12 by 12 array having the following properties: Each entry is one of 12 the note classes, modulo octave. Each row and each column contains each note class precisely once. All entries are be obtained from the top row, called the *original row,* or *prime row,* as follows. The leftmost column is the *inversion* of the top row, that is, the interval (modulo octave) from the top left note class to the n^{th} entry in the left column is the opposite of the interval from the top left note class to the n^{th} entry in the top row. The subsequent rows are *transpositions* of the top row; they are are filled in by starting with the left entry that has been provided above and transposing the first row, so the the intervals from entry 1 to entry m in the n^{th} row is the same as the interval from entry 1 to entry m in the first row.

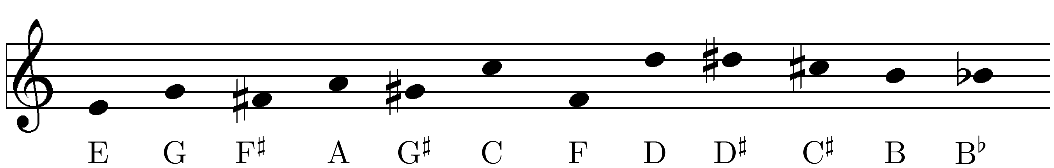
When we are finished, the columns will be transpositions of the inversion of the original row, or, equivalently, inversions of the various transpostions of the original row. The reason for this outcome is fairly obvious, but we will see precisely why this happens when we make the connection with modular arithmetic in the next chapter.

The number of possible original rows is

$$12! = 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1\,,$$

a number huge enough to make the possibilities seemingly endless.

As an example, consider this sequence of 12 notes:

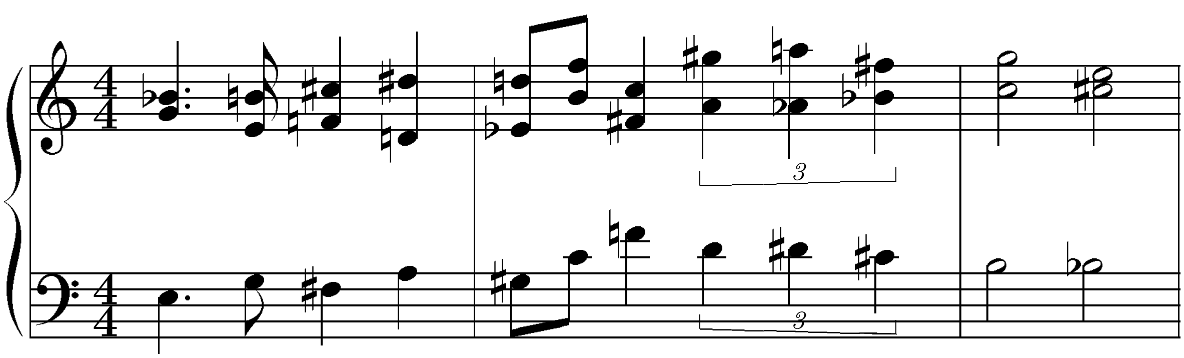


The spelling of notes in twelve-tone music often consists of a mixture of sharps and flats in no apparent pattern. Observe in the above that the sharp is used four times and the flat one time. Since each of the 12 note classes appears precisely once, this sequence qualifies as an original (top) row, which generates the row chart below.

E	G	F$^\sharp$	A	G$^\sharp$	C	F	D	D$^\sharp$	C$^\sharp$	B	B$^\flat$
C$^\sharp$	E	D$^\sharp$	F$^\sharp$	F	A	D	B	C	B$^\flat$	G$^\sharp$	G
D	F	E	G	F$^\sharp$	B$^\flat$	D$^\sharp$	C	C$^\sharp$	B	A	G$^\sharp$
B	D	C$^\sharp$	E	D$^\sharp$	G	C	A	B$^\flat$	G$^\sharp$	F$^\sharp$	F
C	D$^\sharp$	D	F	E	G$^\sharp$	C$^\sharp$	B$^\flat$	B	A	G	F$^\sharp$
G$^\sharp$	B	B$^\flat$	C$^\sharp$	C	E	A	F$^\sharp$	G	F	D$^\sharp$	D
D$^\sharp$	F$^\sharp$	F	G$^\sharp$	G	B	E	C$^\sharp$	D	C	B$^\flat$	A
F$^\sharp$	A	G$^\sharp$	B	B$^\flat$	D	G	E	F	D$^\sharp$	C$^\sharp$	C
F	G$^\sharp$	G	B$^\flat$	A	C$^\sharp$	F$^\sharp$	D$^\sharp$	E	D	C	B
G	B$^\flat$	A	C	B	D$^\sharp$	G$^\sharp$	F	F$^\sharp$	E	D	C$^\sharp$
A	C	B	D	C$^\sharp$	F	B$^\flat$	G	G$^\sharp$	F$^\sharp$	E	D$^\sharp$
B$^\flat$	C$^\sharp$	C	D$^\sharp$	D	F$^\sharp$	B	G$^\sharp$	A	G	F	E

The goal of twelve-tone composing is to create a musical composition which uses the sequences of notes found in the rows and/or columns, or by taking their *retrogrades,* which reverse the order of the sequences. The retrogrades are obtained by reading the rows from right to left or the columns from bottom to top.

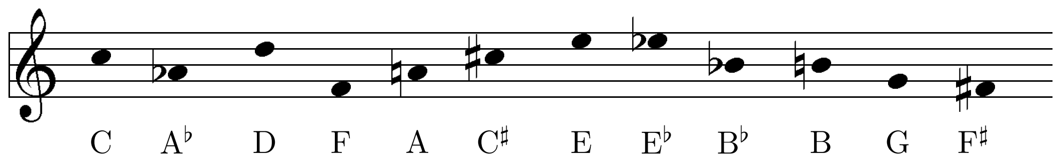
This example, from [1], is based on the row chart above.



Note that the sequence of notes in the bass clef is the original row, the top line in the treble clef is the retrograde of the original row, and the bottom line in the treble clef is the second column of the row chart, which is a transposition of the inversion. The sequences from the row chart appear are applied horizontally in the music notes creating a clashing effect of dissonant chords.

Since there is little feeling of tonal center, twelve-tone music is often written in the key of C. The spelling may be different from what appears in the row chart (Observe the A$^\flat$, rather then G$^\sharp$, in the above example.), and they may change during the compsition.

Sometimes the notes from a sequence are assembled in vertical fashion. Consider the following original row.



The following example, again from [1], uses this row by using groups of notes vertically.

References

1. David Cope, *New Music composition*, Schirmer Books, 1977.

Exercises

(1) Which interval in the given n-scale best approximates the given keyboard inteval? Express the interval in n-chromatic units.

(a) 19-scale, major third (b) 48-scale, tritone

(c) 37-scale, step (d) 7-scale, down a sixth

(2) Express the following interval ratios in terms of n-chromatic units, for the given n. Round off to 2 digits to the right of the decimal.

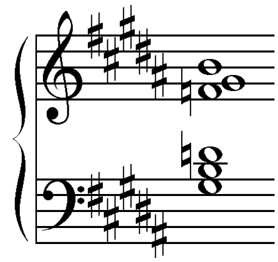
(a) ratio $\frac{5}{2}$; $n = 19$
(b) ratio 3 ; $n = 8$
(c) ratio 0.85 ; $n = 13$
(d) ratio 2π ; $n = 4$ (i.e., the chromatic scale of minor thirds)

(3) Identify these chords by root note with suffix (e.g., B m^7) <u>and</u> root scale note with suffix (e.g., V^7). Indicate the chromatic n-scale's best approximation of the chord by giving, for each note in the chord, it's interval in (whole) n-chromatic units from the bottom note.

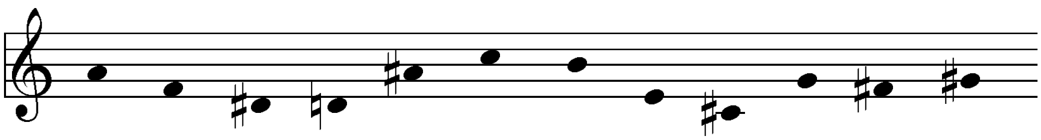
(a)  $n = 17$

(b)  $n = 21$

(c)  $n = 9$

(d)  $n = 4$

(4) Create a twelve-tone row chart having this sequence as its original row:



Write a short composition (say, ≤ 3 bars) which uses only the retrograde of original row's inversion (i.e., the left column of the chart read from bottom to top), incorporating some harmonic material.