Multi-Parameter Riesz Commutators

Brett D. Wick

Vanderbilt University
Department of Mathematics

Sixth Prairie Analysis Seminar
University of Kansas
October 14th, 2006

This is joint work with:

Stefanie Petermichl
University of Texas at Austin

Michael T. Lacey
Georgia Institute of Technology

Jill C. Pipher
Brown University

Hilbert and Riesz Transforms

- The Hilbert Transform is defined by

$$H(f)(x) := \frac{1}{\pi}\int_{\mathbb{R}} f(y)\frac{1}{x-y}dy = f * \left(\frac{1}{\pi y}\right)(x).$$

- Which can be viewed on the Fourier Transform side as:

$$\widehat{H(f)}(\xi) := -i\mathrm{sgn}(\xi)\hat{f}(\xi).$$

- The Riesz Transforms are the n-dimensional generalizations of the Hilbert Transform. For each $1 \leq j \leq n$ we have

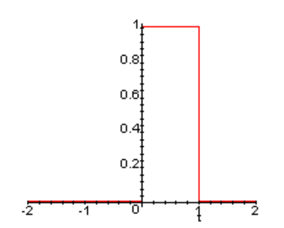
$$R_j(f)(x) := \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\int_{\mathbb{R}^n} f(y)\frac{x_j - y_j}{|x-y|^{n+1}}dy = f * \left(\frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\frac{y_j}{|y|^{n+1}}\right)(x).$$

- On the frequency side:

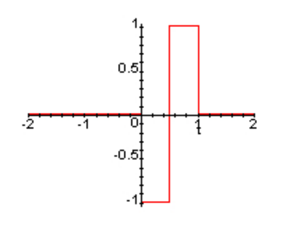
$$\widehat{R_j(f)}(\xi) = -i\frac{\xi_j}{|\xi|}\hat{f}(\xi).$$

A Wavelet Basis for $L^2(\mathbb{R}^n)$

- Let $h^1(x) := \mathbf{1}_{[0,1)}(x)$ and let $h^0(x) := -\mathbf{1}_{[0,1/2)}(x) + \mathbf{1}_{[1/2,1)}(x)$



$h^1(x)$



$h^0(x)$

- Let
$$\mathcal{D}_n := \{2^{-k}(j + [0,1)^n) : j \in \mathbb{Z}^n, k \in \mathbb{Z}\}$$
i.e., the usual dyadic grid in $\mathbb{R}^n$.

A Wavelet Basis for $L^2(\mathbb{R}^n)$

- Let $\mathrm{Tr}_y(f)(x) := f(x-y)$ and $\mathrm{Dil}_t(f)(x) := t^{-n/2} f(\frac{x}{t})$.
- Define

$$\mathrm{Sig}^n := \{\epsilon = (\epsilon_1, \ldots, \epsilon_n) : \epsilon_i \in \{0,1\}\} \setminus \{(1,\ldots,1)\}.$$

- For $Q \in \mathcal{D}_n$ and $\epsilon \in \mathrm{Sig}^n$ set

$$h_Q^{\epsilon}(x) := \prod_{j=1}^{n} \mathrm{Tr}_{c(Q)} \mathrm{Dil}_{|Q|} h^{\epsilon_j}(x_j).$$

- $\{h_Q^{\epsilon} : Q \in \mathcal{D}_n, \epsilon \in \mathrm{Sig}^n\}$ is the Haar wavelet basis for $L^2(\mathbb{R}^n)$.

The Space $BMO(\mathbb{R}^n)$

Definition

$$\|b\|_{BMO} := \sup_{Q} \frac{1}{|Q|} \int_{Q} |f(x) - f_Q|^2 dx$$

Theorem (C. Fefferman (1971))

The dual of $H^1(\mathbb{R}^n)$ is $BMO(\mathbb{R}^n)$, i.e., $\left(H^1(\mathbb{R}^n)\right)^* = BMO(\mathbb{R}^n)$.

Definition (Square Function Characterization)

A function is in (dyadic) $BMO(\mathbb{R}^n)$ if and only if for any (dyadic) cube Q' we have a constant C such that:

$$\frac{1}{|Q'|} \sum_{Q \subset Q'} \sum_{\epsilon \in \mathrm{Sig}^n} |\langle b, h_Q^{\epsilon} \rangle|^2 \leq C.$$

BMO and Riesz Transforms

For each $j = 1, \ldots, n$ define the following commutator operator on $L^2(\mathbb{R}^n)$:

$$[b, R_j](f)(x) := b(x)R_j(f)(x) - R_j(bf)(x).$$

Theorem (Coifman, Rochberg, and Weiss (1976))

Let $b \in BMO(\mathbb{R}^n)$, then for $j = 1, \ldots, n$

$$\|[b, R_j]\|_{2\to 2} \lesssim \|b\|_{BMO(\mathbb{R}^n)}.$$

If $\|[b, R_j]\|_{2\to 2} < +\infty$ for $j = 1, \ldots, n$, then

$$\|b\|_{BMO(\mathbb{R}^n)} \lesssim \max \|[b, R_j]\|_{2\to 2}.$$

Gives $BMO(\mathbb{R}^n)$ as a space of operators on $L^2(\mathbb{R}^n)$.

Product Spaces

- We are concerned with product spaces:

$$\mathbb{R}^{\vec{n}} = \mathbb{R}^{n_1} \otimes \cdots \otimes \mathbb{R}^{n_t} = \otimes_{s=1}^{t} \mathbb{R}^{n_s}$$

- $\mathcal{D}^{\vec{n}} := \otimes_{s=1}^{t} \mathcal{D}_{n_s}$ is the tensor product of the usual dyadic grids in $\mathbb{R}^{n_s}$. Any $R \in \mathcal{D}^{\vec{n}}$ is of the form

$$R = Q_1 \otimes \cdots \otimes Q_t$$

with each Q_s a dyadic cube in $\mathbb{R}^{n_s}$.
Also, let $\mathrm{Sig}^{\vec{n}} := \{\vec{\epsilon} = (\epsilon_1, \ldots, \epsilon_t) : \epsilon_s \in \mathrm{Sig}^{n_s}\}$

Tensor Product Wavelet Basis in $L^2(\otimes_{s=1}^t \mathbb{R}^{n_s})$

- Take the Haar wavelet basis described earlier in $\mathbb{R}^{n_s}$, i.e.,

$$\{h_{Q_s}^{\epsilon_s} : Q_s \in \mathcal{D}_{n_s}, \epsilon_s \in \mathrm{Sig}^{n_s}\}$$

For each $R \in \mathcal{D}^{\vec{n}}$ and $\vec{\epsilon} \in \mathrm{Sig}^{\vec{n}}$ define the following function:

$$h_R^{\vec{\epsilon}}(x_1, \ldots, x_t) := \prod_{s=1}^{t} h_{Q_s}^{\epsilon_s}(x_s)$$

- $\{h_R^{\vec{\epsilon}} : R \in \mathcal{D}^{\vec{n}}, \vec{\epsilon} \in \mathrm{Sig}^{\vec{n}}\}$ is a wavelet basis for $L^2(\otimes_{s=1}^t \mathbb{R}^{n_s})$.

Product $BMO(\otimes_{s=1}^{t}\mathbb{R}^{n_s})$

A Reasonable Guess:

Product BMO?

A function is in $BMO(\otimes_{s=1}^{t}\mathbb{R}^{n_s})$ if and only if for any rectangle S in $\otimes_{s=1}^{t}\mathbb{R}^{n_s}$ there exists a constant C such that:

$$\frac{1}{|S|}\sum_{R\subset S}\sum_{\vec{\epsilon}\in \mathrm{Sig}^{\vec{n}}}|\langle b, h_R^{\vec{\epsilon}}\rangle|^2 \leq C$$

THIS IS WRONG!!!

Defines a space called "Rectangular" BMO, which is larger than product $BMO(\otimes_{s=1}^{t}\mathbb{R}^{n_s})$. (Counter-example do to Carleson).
Instead of rectangles, one must use arbitrary open sets in $\otimes_{s=1}^{t}\mathbb{R}^{n_s}$.

Product $BMO(\otimes_{s=1}^{t}\mathbb{R}^{n_s})$

Correct Definition:

Definition (Product BMO)

A function b is in $BMO(\otimes_{s=1}^{t}\mathbb{R}^{n_s})$ if and only if for any **open** set U in $\otimes_{s=1}^{t}\mathbb{R}^{n_s}$ with finite measure there exists a constant C such that:

$$\frac{1}{|U|}\sum_{R\subset U}\sum_{\vec{\epsilon}\in \mathrm{Sig}^{\vec{n}}} |\langle b, h_R^{\vec{\epsilon}}\rangle|^2 \leq C.$$

How do you check on every open set?

Theorem (S.-Y.A. Chang, R. Fefferman (1980))

The dual of product $H^1(\otimes_{s=1}^{t}\mathbb{R}^{n_s})$ is product $BMO(\otimes_{s=1}^{t}\mathbb{R}^{n_s})$, i.e., $\left(H^1(\otimes_{s=1}^{t}\mathbb{R}^{n_s})\right)^* = BMO(\otimes_{s=1}^{t}\mathbb{R}^{n_s})$.

$BMO(\otimes_{s=1}^{t}\mathbb{R}^{n_s})$ and Iterated Commutators

- Additional cancellation is present in the multi-parameter setting and this can still be studied via commutators.
- We need iterated (nested) commutators:
  Let R_{s,j_s} denote the j_s th Riesz transform taken in the s parameter variable.
  For $s = 1, \ldots, t$ and for $1 \leq j_s \leq n_s$ we consider the following iterated (nested) commutators on $L^2(\otimes_{s=1}^{t}\mathbb{R}^{n_s})$:

$$[\cdots[b, R_{1,j_1}], R_{2,j_2}], \cdots], R_{t,j_t}](f)(x)$$

2 Parameter Iterated Commutator in $\mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2}$

For $s = 1, 2$ and $1 \leq j_s \leq n_s$ the iterated commutator is:

$$\begin{aligned}[[b, R_{1,j_1}], R_{2,j_2}](f)(x) \quad := \quad & b(x)R_{1,j_1}R_{2,j_2}(f)(x) - R_{1,j_1}(b)(x)R_{2,j_2}(f)(x) \\ & -R_{2,j_2}(b)(x)R_{1,j_1}(f)(x) + R_{1,j_1}R_{2,j_2}(bf)(x)\end{aligned}$$

$BMO(\otimes_{s=1}^{t}\mathbb{R})$ as an Operator Space

Theorem (C. Sadosky and S. Ferguson (2001))

Let $b \in BMO(\otimes_{s=1}^{t}\mathbb{R})$, *then*

$$\|[\cdots[b, H_1], H_2], \cdots], H_t]]\|_{2\to 2} \lesssim \|b\|_{BMO(\otimes_{s=1}^{t}\mathbb{R})}.$$

Theorem (M. Lacey and S. Ferguson (2002), M. Lacey and E. Terwilleger (2004))

If $\|[\cdots[b, H_1], H_2], \cdots], H_t]\|_{2\to 2} < +\infty$, *then*

$$\|b\|_{BMO(\otimes_{s=1}^{t}\mathbb{R})} \lesssim \|[\cdots[b, H_1], H_2], \cdots], H_t]\|_{2\to 2}.$$

Restatement of Nehari's Theorem for little Hankels on the polydisc.
KEY POINT: Provides a useful characterization of $BMO(\otimes_{s=1}^{t}\mathbb{R})$.

Main Result

It is possible to generalize the Coifman, Rochberg, Weiss result to the product setting, and the Ferguson Lacey, Lacey, Terwilleger results to more general Euclidean spaces:

Theorem (S. Petermichl, J. Pipher, M. Lacey, BW)

Let $b \in BMO(\otimes_{s=1}^{t}\mathbb{R}^{n_s})$, then for $s = 1, \ldots, t$, and all $1 \leq j_s \leq n_s$

$$\|[\cdots[b, R_{1,j_1}], R_{2,j_2}], \cdots], R_{t,j_t}]\|_{2\to 2} \lesssim \|b\|_{BMO(\otimes_{s=1}^{t}\mathbb{R}^{n_s})}.$$

If $\|[\cdots[b, R_{1,j_1}], R_{2,j_2}], \cdots], R_{t,j_t}]\|_{2\to 2} < +\infty$ for all $s = 1, \ldots, t$ and all $1 \leq j_s \leq n_s$, then

$$\|b\|_{BMO(\otimes_{s=1}^{t}\mathbb{R}^{n_s})} \lesssim \max \|[\cdots[b, R_{1,j_1}], R_{2,j_2}], \cdots], R_{t,j_t}]\|_{2\to 2}.$$

The Upper Bound

- Main Idea: Express the commutator as a sum of paraproducts.
- Use the multi-parameter paraproducts of Muscalu, Pipher, Tao and Thiele.

$$\mathrm{B}(f_1, f_2) := \sum_{R \in \mathcal{D}^{\vec{n}}} \frac{\langle f_1 \varphi_{1,R} \rangle}{|R|^{1/2}} \langle f_2, \varphi_{2,R} \rangle \, \varphi_{3,R}.$$

  Key Point:

$$\mathrm{B} : BMO \times L^p \to L^p$$

- Relatively straightforward, though technical computations and estimates give the result.

The Lower Bound

- Again rely upon paraproducts.
- Define a space reduced BMO, which plays the role of rectangle BMO. This space is "related" to product BMO via Journé's Lemma.
- If the commutators are bounded, then we have an initial weak lower bound in terms of reduced BMO. We want to boot-strap this lower bound to a lower bound in terms of product BMO.
- There are difficulties:
  - The approach used in Lacey-Ferguson and Lacey-Terwilleger depends upon the relationship between the Hilbert transform and projections.
  - We need to do something similar in the Hilbert transform case. To accomplish this we perform a reduction to deal with "nice" multipliers.
  - With this reduction it is possible to implement the general scheme established in the papers Lacey-Ferguson and Lacey-Terwilleger.

Other Problems Considered

- The theorem also implies a weak factorization result for the product Hardy space $H^1(\otimes_{s=1}^t \mathbb{R}^{n_s})$ in terms of L^2 functions and Riesz transforms.
- Commutators in One-Parameter have connections to Div-Curl Lemmas.

  Let E be a divergence free vector field, and B be a curl free vector field, then

  $$E \cdot B \in H^1(\mathbb{R}^n)$$

  Our theorem implies a new Div-Curl Lemma, but one which allows divergence/curl free vector fields in each variable separately.
- Connections with Hankel/Toeplitz operators on weighted Bergman spaces in several complex variables, and the mapping properties of little Hankels on different Hardy spaces.