# Composition of Haar Paraproducts

Brett D. Wick

Georgia Institute of Technology
School of Mathematics

Hilbert Function Spaces
Gargnano sul Garda
May 20 – 24, 2013

This talk is based on joint work with:

Eric T. Sawyer

McMaster University

Sandra Pott

Lund University

Maria Reguera Rodriguez

Universidad Autónoma de Barcelona

# Saraton's Conjecture

- $H^2(\mathbb{D})$, the standard Hardy space on $\mathbb{D}$.
- $\mathbb{P} : L^2(\mathbb{T}) \to H^2(\mathbb{D})$ be the orthogonal projection.
- A Toeplitz operator with symbol $\varphi$ is the following map from $H^2(\mathbb{D}) \to H^2(\mathbb{D})$:
$$T_\varphi(f) \equiv \mathbb{P}(\varphi f).$$
- An important question raised by Sarason is the following:

**Question (Sarason Question (Revised Version))**

*Obtain necessary and sufficient (testable (?)) conditions so that one can tell if $T_\varphi T_{\overline{\psi}}$ is bounded on $H^2(\mathbb{D})$ by evaluating these conditions.*

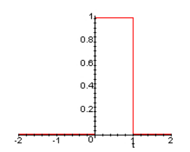
Possible to rephrase this question as one about the two-weight boundedness of the Hilbert transform. Deep work by Nazarov, Treil, Volberg, and then subsequent work by Lacey, Sawyer, Shen, Uriarte-Tuero allow for an answer in terms of the Hilbert transform.

# Haar Paraproducts

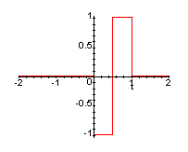
- $L^2 \equiv L^2(\mathbb{R})$;
- $\mathcal{D}$ is the standard grid of dyadic intervals on $\mathbb{R}$;
- Define the Haar function $h_I^0$ and averaging function $h_I^1$ by

$$h_I^0 \equiv h_I \equiv \frac{1}{\sqrt{|I|}}\left(-\mathbf{1}_{I_-} + \mathbf{1}_{I_+}\right) \quad I \in \mathcal{D}$$

$$h_I^1 \equiv \frac{1}{|I|}\mathbf{1}_I \quad I \in \mathcal{D}.$$




$h^1_{[0,1]}(x)$




$h^0_{[0,1]}(x)$

- $\{h_I\}_{I \in \mathcal{D}}$ is an orthonormal basis of $L^2$.

# Haar Paraproducts from Multiplication Operators

Given a function $b$ and $f$ it is possible to study their pointwise product by expanding in their Haar series:

$$\begin{aligned}
bf &= \left(\sum_{I\in\mathcal{D}} \langle b, h_I\rangle_{L^2} h_I\right)\left(\sum_{J\in\mathcal{D}} \langle f, h_J\rangle_{L^2} h_J\right)\\
&= \sum_{I,J\in\mathcal{D}} \langle b, h_I\rangle_{L^2} \langle f, h_J\rangle_{L^2} h_I h_J\\
&= \left(\sum_{I=J}+\sum_{I\subsetneq J}+\sum_{J\subsetneq I}\right) \langle b, h_I\rangle_{L^2} \langle f, h_J\rangle_{L^2} h_I h_J\\
&= \sum_{I\in\mathcal{D}} \langle b, h_I\rangle_{L^2} \langle f, h_I\rangle_{L^2} h_I^1 + \sum_{I\in\mathcal{D}} \langle b, h_I\rangle_{L^2} \left\langle f, h_I^1\right\rangle_{L^2} h_I\\
&\quad + \sum_{I\in\mathcal{D}} \left\langle b, h_I^1\right\rangle_{L^2} \langle f, h_I\rangle_{L^2} h_I.
\end{aligned}$$

# Haar Paraproducts

## Definition (Haar Paraproducts)

Given a symbol sequence $b = \{b_I\}_{I \in \mathcal{D}}$ and a pair $(\alpha, \beta) \in \{0,1\}^2$, define the *dyadic paraproduct* acting on a function $f$ by

$$\mathsf{P}_b^{(\alpha,\beta)} f \equiv \sum_{I \in \mathcal{D}} b_I \left\langle f, h_I^{\beta} \right\rangle_{L^2} h_I^{\alpha}.$$

The index $(\alpha, \beta)$ is referred to as the *type* of $\mathsf{P}_b^{(\alpha,\beta)}$.

## Question (Discrete Sarason Question)

*For each choice of pairs $(\alpha, \beta), (\epsilon, \delta) \in \{0,1\}^2$, obtain necessary and sufficient conditions on symbols $b$ and $d$ so that*

$$\left\| \mathsf{P}_b^{(\alpha,\beta)} \circ \mathsf{P}_d^{(\epsilon,\delta)} \right\|_{L^2 \to L^2} < \infty.$$

# Internal Cancellations and Simple Characterizations

When there are internal zeros the behavior of $\mathsf{P}_b^{(\alpha,0)} \circ \mathsf{P}_d^{(0,\beta)}$reduces to the behavior of $P_a^{(\alpha,\beta)}$ for a special symbol $a$. For $f, g \in L^2$, let $f \otimes g : L^2 \to L^2$ be the map given by

$$f \otimes g(h) \equiv f \left\langle g, h \right\rangle_{L^2} .$$

Then:

$$\begin{aligned}
\mathsf{P}_b^{(\alpha,0)} \circ \mathsf{P}_d^{(0,\beta)} &= \left( \sum_{I \in \mathcal{D}} b_I h_I^{\alpha} \otimes h_I \right) \left( \sum_{J \in \mathcal{D}} d_J h_J \otimes h_J^{\beta} \right) \\
&= \sum_{I \in \mathcal{D}} b_I d_I \, h_I^{\alpha} \otimes h_I^{\beta} \\
&= P_{b \circ d}^{(\alpha,\beta)}.
\end{aligned}$$

Here $b \circ d$ is the Schur product of the symbols, i.e., $(b \circ d)_I = b_I d_I$.

# Norms and Induced Sequences

For a sequence $a = \{a_I\}_{I\in\mathcal{D}}$ define the following quantities:

$$\begin{aligned} \|a\|_{\ell^\infty} &\equiv \sup_{I\in\mathcal{D}} |a_I| ; \\ \|a\|_{CM} &\equiv \sqrt{\sup_{I\in\mathcal{D}} \frac{1}{|I|} \sum_{J\subset I} |a_J|^2} . \end{aligned}$$

Associate to $\{a_I\}_{I\in\mathcal{D}}$ two additional sequences indexed by $\mathcal{D}$:

$$\begin{aligned} E(a) &\equiv \left\{ \frac{1}{|I|} \sum_{J\subset I} a_J \right\}_{I\in\mathcal{D}} ; \\ \widehat{S}\,(a) &\equiv \left\{ \left\langle \sum_{J\in\mathcal{D}} a_J h_J^1, h_I \right\rangle_{L^2} \right\}_{I\in\mathcal{D}} = \left\{ \sum_{J\subsetneq I} a_J \widehat{h_J^1}\,(I) \right\}_{I\in\mathcal{D}} . \end{aligned}$$

# Classical Characterizations

**Theorem (Characterizations of Type $(0,0)$, $(0,1)$, and $(1,0)$)**

*The following characterizations are true:*

$$\begin{aligned}\left\|\mathsf{P}_a^{(0,0)}\right\|_{L^2\to L^2} &= \|a\|_{\ell^\infty}\,;\\ \left\|\mathsf{P}_a^{(0,1)}\right\|_{L^2\to L^2} &= \left\|\mathsf{P}_a^{(1,0)}\right\|_{L^2\to L^2} \approx \|a\|_{CM}\,.\end{aligned}$$

$$\mathsf{P}_a^{(1,1)} = \mathsf{P}_{\widehat{S}(a)}^{(1,0)} + \mathsf{P}_{\widehat{S}(a)}^{(0,1)} + \mathsf{P}_{E(a)}^{(0,0)}\,.$$

**Theorem (Characterization of Type $(1,1)$)**

*The operator norm* $\left\|\mathsf{P}_a^{(1,1)}\right\|_{L^2\to L^2}$ *of* $\mathsf{P}_a^{(1,1)}$ *on* $L^2$ *satisfies*

$$\left\|\mathsf{P}_a^{(1,1)}\right\|_{L^2\to L^2} \approx \left\|\widehat{S}(a)\right\|_{CM} + \|E(a)\|_{\ell^\infty}\,.$$

# Alternate Interpretations: Testing Conditions

It is easy to see for paraproducts of type $(0,0)$ that:

$$\begin{aligned} \left\| \mathsf{P}_a^{(0,0)} \right\|_{L^2 \to L^2} &= \|a\|_{\ell^\infty} \\ &= \sup_{I \in \mathcal{D}} \left\| \mathsf{P}_a^{(0,0)} h_I \right\|_{L^2}. \end{aligned}$$

Moreover,

$$\begin{aligned} \left\| \mathsf{P}_a^{(1,0)} \right\|_{L^2 \to L^2} &= \left\| \mathsf{P}_a^{(0,1)} \right\|_{L^2 \to L^2} \\ &\approx \|a\|_{CM} \\ &\approx \sup_{I \in \mathcal{D}} \left\| \mathsf{P}_a^{(0,1)} h_I \right\|_{L^2}. \end{aligned}$$

These observations suggest seeking a characterization for the other compositions in terms of testing conditions on classes of functions.

# Characterization of Type $(0,1,1,0)$

For a sequence $a$, and interval $I \in \mathcal{D}$ let $\mathsf{Q}_I a \equiv \sum_{J \subset I} a_J h_J$.

**Theorem (E. Sawyer, S. Pott, M. Reguera-Rodriguez, BDW)**

*The composition* $\mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(1,0)}$ *is bounded on* $L^2$ *if and only if both*

$$\begin{aligned} \left\| \mathsf{Q}_I \mathsf{P}_b^{(0,1)} \mathsf{P}_d^{(1,0)} \left( \mathsf{Q}_I \overline{d} \right) \right\|_{L^2}^2 &\leq C_1^2 \left\| \mathsf{Q}_I d \right\|_{L^2}^2 \quad \forall I \in \mathcal{D}; \\ \left\| \mathsf{Q}_I \mathsf{P}_d^{(0,1)} \mathsf{P}_b^{(1,0)} \left( \mathsf{Q}_I \overline{b} \right) \right\|_{L^2}^2 &\leq C_2^2 \left\| \mathsf{Q}_I b \right\|_{L^2}^2 \quad \forall I \in \mathcal{D}. \end{aligned}$$

*Moreover, the norm of* $\mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(1,0)}$ *on* $L^2$ *satisfies*

$$\left\| \mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(1,0)} \right\|_{L^2 \to L^2} \approx C_1 + C_2$$

*where* $C_1$ *and* $C_2$ *are the best constants appearing above.*

# Rephrasing the Testing Conditions

We want to rephrase the testing conditions on $\mathsf{Q}_I\overline{d}$ and $\mathsf{Q}_I\overline{b}$:

$$\begin{aligned} \left\| \mathsf{Q}_I \mathsf{P}_b^{(0,1)} \mathsf{P}_d^{(1,0)} \left( \mathsf{Q}_I \overline{d} \right) \right\|_{L^2}^2 &\leq C_1^2 \left\| \mathsf{Q}_I d \right\|_{L^2}^2 \quad \forall I \in \mathcal{D}; \\ \left\| \mathsf{Q}_I \mathsf{P}_d^{(0,1)} \mathsf{P}_b^{(1,0)} \left( \mathsf{Q}_I \overline{b} \right) \right\|_{L^2}^2 &\leq C_2^2 \left\| \mathsf{Q}_I b \right\|_{L^2}^2 \quad \forall I \in \mathcal{D}. \end{aligned}$$

It isn't hard to see that these are equivalent to the following inequalities on the sequences:

$$\begin{aligned} \sum_{J\subset I} |b_J|^2 \frac{1}{|J|^2} \left( \sum_{L\subset J} |d_L|^2 \right)^2 &\leq C_1^2 \sum_{L\subset I} |d_L|^2 \quad \forall I \in \mathcal{D}; \\ \sum_{J\subset I} |d_J|^2 \frac{1}{|J|^2} \left( \sum_{L\subset J} |b_L|^2 \right)^2 &\leq C_2^2 \sum_{L\subset I} |b_L|^2 \quad \forall I \in \mathcal{D}. \end{aligned}$$

# Characterization of Type $(0,1,0,0)$

## Theorem (E. Sawyer, S. Pott, M. Reguera-Rodriguez, BDW)

*The composition* $\mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(0,0)}$ *is bounded on* $L^2$ *if and only if both*

$$\begin{aligned} |d_I|^2 \left\| \mathsf{P}_b^{(0,1)} h_I \right\|_{L^2}^2 &\leq C_1^2 \quad \forall I \in \mathcal{D}; \\ \left\| \mathsf{Q}_I \mathsf{P}_d^{(0,0)} \mathsf{P}_b^{(1,0)} \mathsf{Q}_I \overline{b} \right\|_{L^2}^2 &\leq C_2^2 \left\| \mathsf{Q}_I b \right\|_{L^2}^2 \quad \forall I \in \mathcal{D}. \end{aligned}$$

*Moreover, the norm of* $\mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(0,0)}$ *on* $L^2$ *satisfies*

$$\left\| \mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(0,0)} \right\|_{L^2 \to L^2} \approx C_1 + C_2$$

*where* $C_1$ *and* $C_2$ *are the best constants appearing above.*

# Rephrasing Testing Conditions

Again, it is possible to recast the conditions:

$$\begin{aligned} |d_I|^2 \left\|\mathsf{P}_b^{(0,1)} h_I\right\|_{L^2}^2 &\leq C_1^2 \quad \forall I \in \mathcal{D}; \\ \left\|\mathsf{Q}_I \mathsf{P}_d^{(0,0)} \mathsf{P}_b^{(1,0)} \mathsf{Q}_I \overline{b}\right\|_{L^2}^2 &\leq C_2^2 \left\|\mathsf{Q}_I b\right\|_{L^2}^2 \quad \forall I \in \mathcal{D} \end{aligned}$$

as expressions depending only on the sequences. In particular, these are equivalent to the following inequalities:

$$\begin{aligned} \frac{|d_I|^2}{|I|} \sum_{L \subsetneq I} |b_L|^2 &\leq C_1^2 \quad \forall I \in \mathcal{D}; \\ \sum_{J \subset I} \frac{|d_J|^2}{|J|} \left( \sum_{K \subset J_+} |b_K|^2 - \sum_{K \subset J_-} |b_K|^2 \right)^2 &\leq C_2^2 \sum_{L \subset I} |b_L|^2 \quad \forall I \in \mathcal{D}. \end{aligned}$$

# Preliminaries

For $I \in \mathcal{D}$ set

$$T(I) \equiv I \times \left[\frac{|I|}{2}, |I|\right] \qquad \text{(Carleson Tile);}$$
$$Q(I) \equiv I \times [0, |I|] = \bigcup_{J \subset I} T(J) \qquad \text{(Carleson Square).}$$

- The dyadic lattice $\mathcal{D}$ is in correspondence with the Carleson Tiles.
- Let $\mathcal{H}$ denote the upper half plane $\mathbb{C}_+$: $\mathcal{H} = \bigcup_{I \in \mathcal{D}} T(I)$.
- For a non-negative function $\sigma$ let $L^2(\mathcal{H};\sigma)$ denote the functions that are square integrable with respect to $\sigma\, dA$, i.e,

$$\|f\|^2_{L^2(\mathcal{H};\sigma)} \equiv \int_{\mathcal{H}} |f(z)|^2 \, \sigma(z)\, dA(z) < \infty.$$

  When $\sigma \equiv 1$, $L^2(\mathcal{H};1) \equiv L^2(\mathcal{H})$.
- For $f \in L^2(\mathcal{H})$, let $\widetilde{f} \equiv \frac{f}{\|f\|_{L^2(\mathcal{H})}}$ denote the normalized function.

# Functions Constant on Tiles

Let $L_c^2(\mathcal{H}) \subset L^2(\mathcal{H})$ be the subspace of functions which are constant on tiles. Namely, $f : \mathcal{D} \to \mathbb{C}$

$$f = \sum_{I \in \mathcal{D}} f_I \mathbf{1}_{T(I)}.$$

Then

$$\begin{aligned} L_c^2(\mathcal{H}) &\equiv \left\{ f : \mathcal{D} \to \mathbb{C} : \sum_{I \in \mathcal{D}} |f(I)|^2 |I|^2 < \infty \right\}; \\ \|f\|^2_{L_c^2(\mathcal{H})} &\equiv \frac{1}{2} \sum_{I \in \mathcal{D}} |f(I)|^2 |I|^2 . \end{aligned}$$

Easy to show:

$$\left\{ \widetilde{\mathbf{1}}_{T(I)} \right\}_{I \in \mathcal{D}} \text{ is an orthonormal basis of } L_c^2(\mathcal{H});$$

$$\left\{ \widetilde{\mathbf{1}}_{Q(I)} \right\}_{I \in \mathcal{D}} \text{ is an Riesz basis of } L_c^2(\mathcal{H}).$$

# The Gram Matrix of $\mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(1,0)}$

Let $\mathfrak{G}_{\mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(1,0)}} = [G_{I,J}]_{I,J\in\mathcal{D}}$ be the Gram matrix of the operator $\mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(1,0)}$ relative to the Haar basis $\{h_I\}_{I\in\mathcal{D}}$. A simple computation show that it has entries:

$$\begin{aligned} G_{I,J} &= \left\langle \mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(1,0)} h_J, h_I \right\rangle_{L^2} = \left\langle \mathsf{P}_d^{(1,0)} h_J, \mathsf{P}_b^{(1,0)} h_I \right\rangle_{L^2} \\ &= \left\langle d_J\, h_J^1, b_I\, h_I^1 \right\rangle_{L^2} \\ &= \overline{b_I} d_J \frac{|I\cap J|}{|I|\,|J|} = \begin{cases} \overline{b_I} d_J \frac{1}{|I|} & \text{if} \quad J\subset I \\ \overline{b_I} d_J \frac{1}{|J|} & \text{if} \quad I\subset J \\ 0 & \text{if} \quad I\cap J=\emptyset. \end{cases} \end{aligned}$$

**Idea:** Construct $\mathsf{T}_{b,d}^{(0,1,1,0)} : L_c^2(\mathcal{H}) \to L_c^2(\mathcal{H})$ that has the same Gram matrix as $\mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(1,0)}$, but with respect to the basis $\left\{\widetilde{\mathbf{1}}_{T(I)}\right\}_{I\in\mathcal{D}}$.

# The Operator $\mathsf{T}_{b,d}^{(0,1,1,0)}$ and it Gram Matrix

For $\lambda \in \mathbb{R}$ and $a = \{a_I\}_{I\in\mathcal{D}}$ the multiplication operator $\mathcal{M}_a^\lambda$ is defined on basis elements $\widetilde{\mathbf{1}}_{T(K)}$ by

$$\mathcal{M}_a^\lambda \widetilde{\mathbf{1}}_{T(K)} \equiv a_K \left|K\right|^\lambda \widetilde{\mathbf{1}}_{T(K)}.$$

Define an operator $\mathsf{T}_{b,d}^{(0,1,1,0)}$ on $L_c^2\left(\mathcal{H}\right)$ by

$$\mathsf{T}_{b,d}^{(0,1,1,0)} \equiv \mathcal{M}_{\overline{b}}^0 \left( \sum_{K\in\mathcal{D}} \widetilde{\mathbf{1}}_{T(K)} \otimes \widetilde{\mathbf{1}}_{Q(K)} \right) \mathcal{M}_d^{-1}.$$

Then the Gram matrix $\mathfrak{G}_{\mathsf{T}_{b,d}^{(0,1,1,0)}} = \left[G_{I,J}\right]_{I,J\in\mathcal{D}}$ of $\mathsf{T}_{b,d}^{(0,1,1,0)}$ relative to the basis $\left\{\widetilde{\mathbf{1}}_{T(I)}\right\}_{I\in\mathcal{D}}$ has entries

$$\begin{aligned} G_{I,J} &= \left\langle \mathsf{T}_{b,d}^{(0,1,1,0)} \widetilde{\mathbf{1}}_{T(J)}, \widetilde{\mathbf{1}}_{T(I)} \right\rangle_{L^2(\mathcal{H})} \\ &= \overline{b_I} d_J \sqrt{2} \frac{\left|Q\left(I\right) \cap T\left(J\right)\right|}{\left|I\right| \left|J\right|^2} = \frac{1}{\sqrt{2}} \begin{cases} \overline{b_I} d_J \frac{1}{|I|} & \text{if} \quad J \subset I \\ 0 & \text{if} \quad J \not\subset I. \end{cases} \end{aligned}$$

# Connecting the Problem to a Two Weight Inequality

Up to an absolute constant, $\mathfrak{G}_{\mathsf{T}_{b,d}^{(0,1,1,0)}}$ matches $\mathfrak{G}_{\mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(1,0)}}$ in the lower triangle where $J \subset I$. So,

$$\left\| \mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(1,0)} \right\|_{L^2 \to L^2} \approx \left\| \mathsf{T}_{b,d}^{(0,1,1,0)} \right\|_{L^2(\mathcal{H}) \to L^2(\mathcal{H})} + \left\| \mathsf{T}_{d,b}^{(0,1,1,0)} \right\|_{L^2(\mathcal{H}) \to L^2(\mathcal{H})} .$$

The inequality we wish to characterize is

$$\left\| \mathcal{M}_b^0 \mathsf{U} \mathcal{M}_d^{-1} f \right\|_{L_c^2(\mathcal{H})} = \left\| \mathsf{T}_{b,d}^{(0,1,1,0)} f \right\|_{L_c^2(\mathcal{H})} \lesssim \|f\|_{L_c^2(\mathcal{H})} .$$

Define $\mathsf{U}$ on $L_c^2(\mathcal{H})$, where

$$\mathsf{U} \equiv \sum_{K \in \mathcal{D}} \widetilde{\mathbf{1}}_{T(K)} \otimes \widetilde{\mathbf{1}}_{Q(K)} .$$

For appropriate choice of weights $\sigma$ and $w$ on $\mathcal{H}$ the desired estimate is simply:

$$\| \mathsf{U}(\sigma k) \|_{L_c^2(\mathcal{H};w)} \lesssim \|k\|_{L_c^2(\mathcal{H};\sigma)} .$$

# A Two Weight Theorem for Positive Operators

**Theorem (S. Pott, E. Sawyer, M. Reguera-Rodriguez, BDW)**

*Let $w$ and $\sigma$ be non-negative weights on $\mathcal{H}$. Then*

$$\mathsf{U}(\sigma\cdot) : L^2(\mathcal{H};\sigma) \to L^2(\mathcal{H};w)$$

*is bounded if and only if the following testing condition holds:*

$$\left\| \mathbf{1}_{Q(I)} \mathsf{U}\left(\sigma \mathbf{1}_{Q(I)}\right) \right\|_{L^2(\mathcal{H};w)}^2 \leq C_0^2 \left\| \mathbf{1}_{Q(I)} \right\|_{L^2(\mathcal{H};\sigma)}^2 .$$

- The proof of this Theorem is a translation of Sawyer's proof strategy for two weight inequalities for positive operators.
- Choosing $w \equiv \sum_{I\in\mathcal{D}} |b_I|^2 \mathbf{1}_{T(I)}$ and $\sigma \equiv \sum_{I\in\mathcal{D}} \frac{|d_I|^2}{|I|^2} \mathbf{1}_{T(I)}$ (and unraveling the definitions) gives the forward testing condition.
- Appropriate choice of $w$ and $\sigma$ will then provide the backward testing condition when studying $T_{d,b}^{(0,1,1,0)}$.

# The Gram Matrix of $\mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(0,0)}$

Let $\mathfrak{G}_{\mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(0,0)}} = [G_{I,J}]_{I,J\in\mathcal{D}}$ be the Gram matrix of the operator $\mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(0,0)}$ relative to the Haar basis $\{h_I\}_{I\in\mathcal{D}}$. A simple computation shows its entries are:

$$
\begin{aligned}
G_{I,J} &= \left\langle \mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(0,0)} h_J, h_I \right\rangle_{L^2} = \left\langle \mathsf{P}_d^{(0,0)} h_J, \mathsf{P}_b^{(1,0)} h_I \right\rangle_{L^2} \\
&= \left\langle d_J h_J, b_I h_I^1 \right\rangle_{L^2} \\
&= \overline{b_I} d_J \widehat{h_I^1}(J) = \begin{cases} \overline{b_I} d_J \frac{-1}{\sqrt{|J|}} & \text{if} \quad I \subset J_- \\ \overline{b_I} d_J \frac{1}{\sqrt{|J|}} & \text{if} \quad I \subset J_+ \\ 0 & \text{if} \quad J \subset I \text{ or } I \cap J = \emptyset. \end{cases}
\end{aligned}
$$

**Idea:** Construct $\mathsf{T}_{b,d}^{(0,1,0,0)} : L_c^2(\mathcal{H}) \to L_c^2(\mathcal{H})$ that has the same Gram matrix as $\mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(0,0)}$, but with respect to the basis $\left\{ \widetilde{\mathbf{1}}_{T(I)} \right\}_{I\in\mathcal{D}}$.

# The Operator $\mathsf{T}_{b,d}^{(0,1,0,0)}$

Now consider the operator $\mathsf{T}_{b,d}^{(0,1,0,0)}$ defined by

$$\mathsf{T}_{b,d}^{(0,1,0,0)} \equiv \mathcal{M}_{\overline{b}}^{-1} \left( \sum_{K\in\mathcal{D}} \widetilde{\mathbf{1}}_{Q_{\pm}(K)} \otimes \widetilde{\mathbf{1}}_{T(K)} \right) \mathcal{M}_{d}^{\frac{1}{2}}.$$

Here

$$\mathbf{1}_{Q_{\pm}(K)} \equiv -\sum_{L\subset K_-} \mathbf{1}_{T(L)} + \sum_{L\subset K_+} \mathbf{1}_{T(L)}.$$

A straightforward computation shows

$$\begin{aligned} \left\| \mathbf{1}_{Q_{\pm}(K)} \right\|_{L^2(\mathcal{H})} &= \frac{|K|}{2}; \\ \mathcal{M}_a^{\lambda} \mathbf{1}_{Q_{\pm}(K)} &= -\sum_{L\subset K_-} a_L \left|L\right|^{\lambda} \mathbf{1}_{T(L)} + \sum_{L\subset K_+} a_L \left|L\right|^{\lambda} \mathbf{1}_{T(L)}. \end{aligned}$$

# The Gram Matrix for the Operator $\mathsf{T}_{b,d}^{(0,1,0,0)}$

The Gram matrix $\mathfrak{G}_{\mathsf{T}_{b,d}^{(0,1,0,0)}} = [G_{I,J}]_{I,J\in\mathcal{D}}$ of $\mathsf{T}_{b,d}^{(0,1,0,0)}$ relative to the basis $\left\{\widetilde{\mathbf{1}}_{T(I)}\right\}_{I\in\mathcal{D}}$ then has entries given by

$$
\begin{aligned}
G_{I,J} &= \left\langle \mathsf{T}_{b,d}^{(0,1,0,0)} \widetilde{\mathbf{1}}_{T(J)}, \widetilde{\mathbf{1}}_{T(I)} \right\rangle_{L^2(\mathcal{H})} \\
&= \sqrt{2} \begin{cases} -\overline{b_I} d_J \left|J\right|^{-\frac{1}{2}} & \text{if} \quad I \subset J_- \\ \overline{b_I} d_J \left|J\right|^{-\frac{1}{2}} & \text{if} \quad I \subset J_+ \\ 0 & \text{if} \quad J \subset I \text{ or } I \cap J = \emptyset. \end{cases}
\end{aligned}
$$

Thus, up to an absolute constant, $\mathfrak{G}_{\mathsf{T}_{b,d}^{(0,1,0,0)}} = \mathfrak{G}_{\mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(0,0)}}$ , and so

$$
\left\| \mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(0,0)} \right\|_{L^2 \to L^2} \approx \left\| \mathsf{T}_{b,d}^{(0,1,0,0)} \right\|_{L^2(\mathcal{H}) \to L^2(\mathcal{H})} .
$$

# Connecting to a Two Weight Inequality

The inequality we wish to characterize is:

$$\left\| \mathcal{M}_{\overline{b}}^{-1} \mathsf{U} \mathcal{M}_{d}^{\frac{1}{2}} f \right\|_{L^2_c(\mathcal{H})} = \left\| \mathsf{T}_{b,d}^{(0,1,0,0)} f \right\|_{L^2_c(\mathcal{H})} \lesssim \|f\|_{L^2_c(\mathcal{H})} .$$

Where the operator $\mathsf{U}$ on $L^2(\mathcal{H})$ is defined by

$$\mathsf{U} \equiv \sum_{K\in\mathcal{D}} \widetilde{\mathbf{1}}_{Q_\pm(K)} \otimes \widetilde{\mathbf{1}}_{T(K)}.$$

One sees that the inequality to be characterized is equivalent to:

$$\|\mathsf{U}(\mu g)\|_{L^2_c(\mathcal{H};\nu)} \lesssim \|g\|_{L^2_c(\mathcal{H};\mu)} ,$$

where the weights $\mu$ and $\nu$ are given by

$$\begin{aligned} \nu &\equiv \sum_{I\in\mathcal{D}} |b_I|^2 |I|^{-2} \mathbf{1}_{T(I)} \\ \mu &\equiv \sum_{I\in\mathcal{D}} |d_I|^{-2} |I|^{-1} \mathbf{1}_{T(I)}. \end{aligned}$$

## Theorem (S. Pott, E. Sawyer, M. Reguera-Rodriguez, BDW)

*Suppose that* $\mu$ *and* $\nu$ *are positive measures on* $\mathcal{H}$ *that are constant on tiles, i.e.,* $\mu \equiv \sum_{I\in\mathcal{D}} \mu_I \mathbf{1}_{T(I)}$, $\nu \equiv \sum_{I\in\mathcal{D}} \nu_I \mathbf{1}_{T(I)}$. *Then*

$$\mathsf{U}(\mu\cdot) : L^2_c(\mathcal{H};\mu) \to L^2_c(\mathcal{H};\nu)$$

*if and only if both*

$$\begin{aligned} \left\| \mathsf{U}\left(\mu \mathbf{1}_{T(I)}\right) \right\|_{L^2_c(\mathcal{H};\nu)} &\leq C_1 \left\| \mathbf{1}_{T(I)} \right\|_{L^2_c(\mathcal{H};\mu)} = \sqrt{\mu\left(T\left(I\right)\right)}, \\ \left\| \mathbf{1}_{Q(I)} \mathsf{U}^*\left(\nu \mathbf{1}_{Q(I)}\right) \right\|_{L^2_c(\mathcal{H};\mu)} &\leq C_2 \left\| \mathbf{1}_{Q(I)} \right\|_{L^2_c(\mathcal{H};\nu)} = \sqrt{\nu\left(Q\left(I\right)\right)}, \end{aligned}$$

*hold for all* $I \in \mathcal{D}$. *Moreover, we have that*

$$\|\mathsf{U}\|_{L^2_c(\mathcal{H};\mu)\to L^2_c(\mathcal{H};\nu)} \approx C_1 + C_2$$

*where* $C_1$ *and* $C_2$ *are the best constants appearing above.*

# An Application: Linear Bound for Hilbert Transform

- For a weight $w$, i.e., a positive locally integrable function on $\mathbb{R}$, let $L^2(w) \equiv L^2(\mathbb{R}; w)$.
- A weight belongs to $A_2$ if: $[w]_{A_2} \equiv \sup_I \langle w \rangle_I \langle w^{-1} \rangle_I < +\infty$.
- The Hilbert transform is the operator: $H(f)(x) \equiv \text{p.v.} \int_{\mathbb{R}} \frac{f(y)}{y-x} dy$.

**Theorem (Petermichl)**

*Let $w \in A_2$. Then $\|H\|_{L^2(w)\to L^2(w)} \lesssim [w]_{A_2}$, and the linear growth is optimal.*

- $\|T\|_{L^2(w)\to L^2(w)} = \left\| M_{w^{\frac{1}{2}}} T M_{w^{-\frac{1}{2}}} \right\|_{L^2 \to L^2}$;
- $H$ is the average of dyadic shifts Ш;
- $M_{w^{\frac{1}{2}}} Ш M_{w^{-\frac{1}{2}}}$ can be written as a sum of nine compositions of paraproducts; Some of which are amenable to the Theorems above.
- However, each term can be shown to have norm no worse than $[w]_{A_2}$.

# An Open Question

Unfortunately, the methods described do not appear to work to handle type $(0,1,0,1)$ compositions. However, the following question is of interest:

## Question

*For each $I \in \mathcal{D}$ determine function $F_I, B_I \in L^2$ of norm $1$ such that $\mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(0,1)}$ is bounded on $L^2$ if and only if*

$$\begin{aligned} \left\| \mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(0,1)} F_I \right\|_{L^2} &\leq C_1 \quad \forall I \in \mathcal{D}; \\ \left\| \mathsf{P}_d^{(1,0)} \circ \mathsf{P}_b^{(1,0)} B_I \right\|_{L^2} &\leq C_2 \quad \forall I \in \mathcal{D}. \end{aligned}$$

*Moreover, we will have*

$$\left\| \mathsf{P}_b^{(0,1)} \circ \mathsf{P}_d^{(0,1)} \right\|_{L^2 \to L^2} \approx C_1 + C_2.$$


(Modified from the Original Dr. Fun Comic)

Thanks to Nicola, John, Marco, Stefan, and Maura for Organizing the Meeting!

Thank You!