Superposition and Orthogonality from Polynomials to Wavelets

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A Problem and Its Solution

Problem. Given any arbitrary function $f(x)$, find an accurate and easy-to-compute formula that approximates it.

Taylor's Theorem (c.1712) If $f(x)$ is smooth and x is confined to a bounded interval, then for any desired accuracy there is a polynomial

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

that approximates f to that accuracy on the interval.

Note that this is a *superposition* of the simple functions $1, x, x^2, \ldots$ with *weights* $a_0, a_1, a_2, \ldots$.

A Problem Requiring a More General Solution

The weights are computed from derivatives of the function f, for which we use calculus. But what if

- the function is not differentiable?
- the derivatives exists but are expensive to compute?
- the function is known only approximately?

Idea. Use approximate values of the weights that can be computed without differentiation.

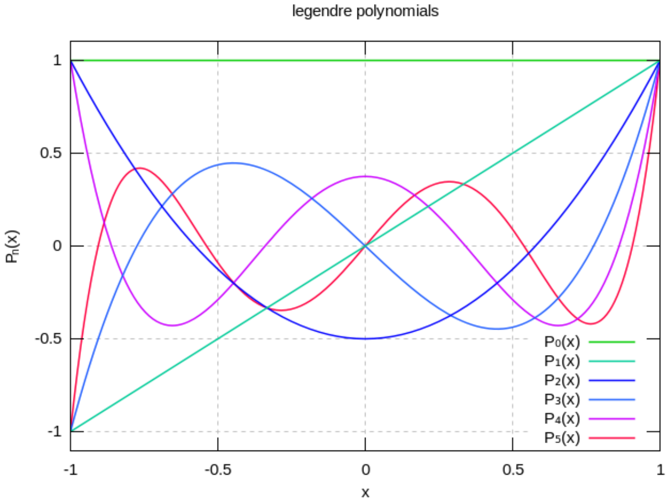
Two Great Mathematicians, Pure and Applied

Adrien-Marie Legendre (1752–1833) and
Jean-Baptiste Joseph Fourier (1768–1830)
Watercolor by Julien-Leopold Boilly, c.1820.

Adrien-Marie Legendre's Polynomials

$$
\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= \frac{1}{2}(3x^2 - 1) \\
P_3(x) &= \frac{1}{2}(5x^3 - 3x) \\
P_4(x) &= \frac{1}{8}(35x^4 - 30x^2 + 3) \\
P_5(x) &= \frac{1}{8}(63x^5 - 70x^3 + 15x) \\
&\vdots \\
\text{Recursion: } P_{n+1}(x) &= \frac{2n+1}{n+1} x P_n(x) - \frac{n}{n+1} P_{n-1}
\end{aligned}
$$

Graph of the First Six Legendre Polynomials



Legendre polynomials P_0 through P_5 plotted on their domain.

Legendre's Construction

Theorem

Any polynomial $p = p(x)$ may be written as a sum of Legendre polynomials, multiplied by weights $\{c_n : n = 0, 1, \dots\}$ specific to p:

$$p(x) = c_0 P_0(x) + c_1 P_1(x) + c_2 P_2(x) + c_3 P_3(x) + \cdots$$

Examples:

$$x^2 = \frac{1}{3} P_0(x) + \frac{2}{3} P_2(x) \qquad x^4 = \frac{1}{5} P_0(x) + \frac{4}{7} P_2(x) + \frac{8}{35} P_4(x)$$

$$x^3 = \frac{3}{5} P_1(x) + \frac{2}{5} P_3(x) \qquad x^5 = \frac{3}{7} P_1(x) + \frac{4}{9} P_3(x) + \frac{8}{63} P_5(x)$$

$$\vdots$$

Application of Legendre's Construction

Taylor's polynomial for function $f(x)$ can be written as

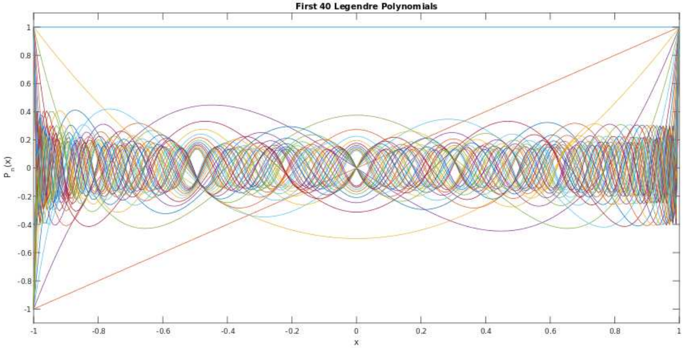
$$p(x) = b_0 P_0(x) + b_1 P_1(x) + \cdots + b_n P_n(x),$$

where the weights are given by integrals, rather than derivatives:

$$b_k = \left(k + \frac{1}{2}\right) \int_{-1}^{1} f(x) P_k(x)\, dx, \qquad k = 0, 1, \ldots, n.$$

(This may look just as hard, but in fact integrals are easy to approximate accurately from just a few values of $f(x)$.)

Graph of the First Forty Legendre Polynomials



Legendre polynomials P_0 through P_{39} plotted on their domain.

(Notice that the number of zero-crossings increases with the degree of the polynomial. Thus degree has some resemblance to the frequency in sine and cosine functions.)

Fourier's Construction

Theorem

Any function $f = f(t)$ may be written as a sum of sines and cosines, multiplied by numbers $\{a_n, b_n\}$ specific to f:

$$\begin{aligned} f(t) &= a_0 + a_1\cos(t) + a_2\cos(2t) + a_3\cos(3t) + \cdots \\ &\qquad + b_1\sin(t) + b_2\sin(2t) + b_3\sin(2t) + \cdots \end{aligned}$$

Fourier's weights are also given by integrals:

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx, \qquad k = 1, 2, \ldots \\ b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx, \qquad k = 1, 2, \ldots \end{aligned}$$

Key Ideas

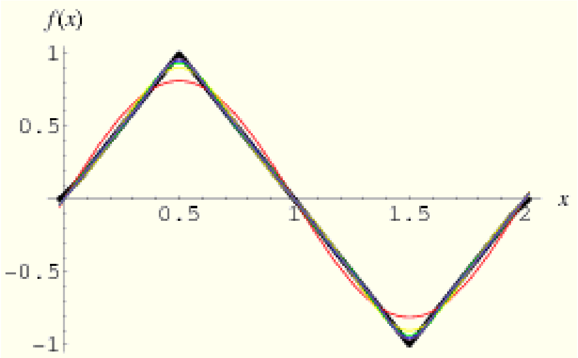
- Simple building blocks: fixed polynomials, or sines and cosines.
- Simple data encoding: one number for each building block.
- Complete and efficient: each function has a unique encoding.

The ingenious choice of *orthogonal* building blocks, like Legendre's polynomials or Fourier's sines and cosines, makes it possible to compute the weights by integration.

Application to Sound and Image Compression

- Audio recordings and images are functions.
- Functions are made of simple building blocks.
- Our senses are imperfect, so approximations suffice.
- Approximations are cheaper.

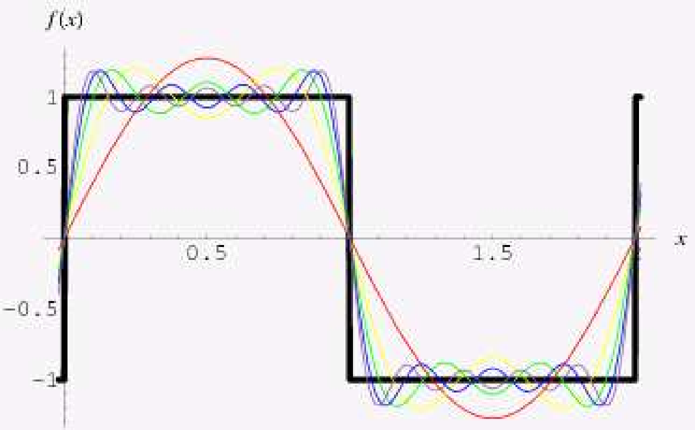
Example of Fourier's Construction (Good)



Adding up just sines with $b_n \sim 1/n^2$ to get a sawtooth.

Compression: just three terms b_1, b_3, b_5 give the green curve.

Example of Fourier's Construction (Not So Good)



Adding up just sines with $b_n \sim 1/n$ to get a square wave.

Gibbs' phenomenon: overshoots never go away.

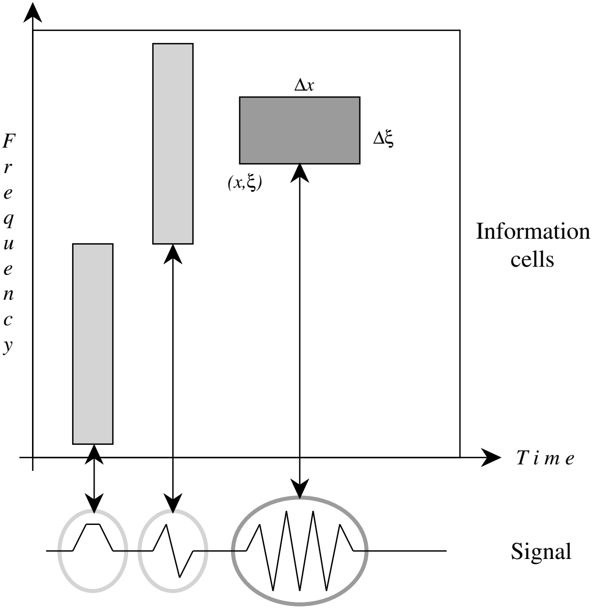
Problems with Fourier's Construction

- Infinitely many numbers $\{a_n, b_n\}$ are needed to represent a given function f, and some simple functions require very many for a good approximation.
- Sines and cosines are not localized, so that any error in a weight appears as error everywhere.
- Even if the function f is continuous, its Fourier series may not converge.

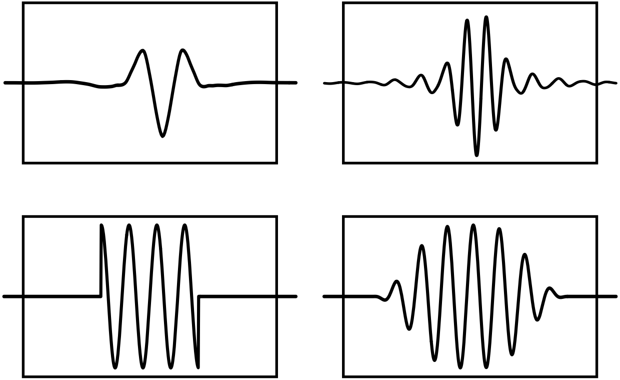
Two More Great Mathematicians

Alfréd Haar (1885–1933) and Ingrid Daubechies (1954–)

Time and Frequency Content Analyzed Together



Waveforms Localized in Time and Frequency



History B.D. [Before Daubechies]

- Fourier bases (1822, Paris)
- Haar bases (1910, *Math. Annalen*)
- Gabor functions (1946, *J. IEE*)
- Balian-Low theorem (1981, *CRAS*)
- Wilson bases (1987, Cornell)

Ingrid Daubechies' Construction

Theorem

Any function $f = f(t)$ may be written as a sum of wavelets $w_{jk}(t) \stackrel{\text{def}}{=} w(2^j t + k)$, multiplied by numbers c_{jk} specific to f:

$$f(t) = \sum_{j \in \mathbf{Z}} \sum_{k \in \mathbf{Z}} c_{jk} w_{jk}(t),$$

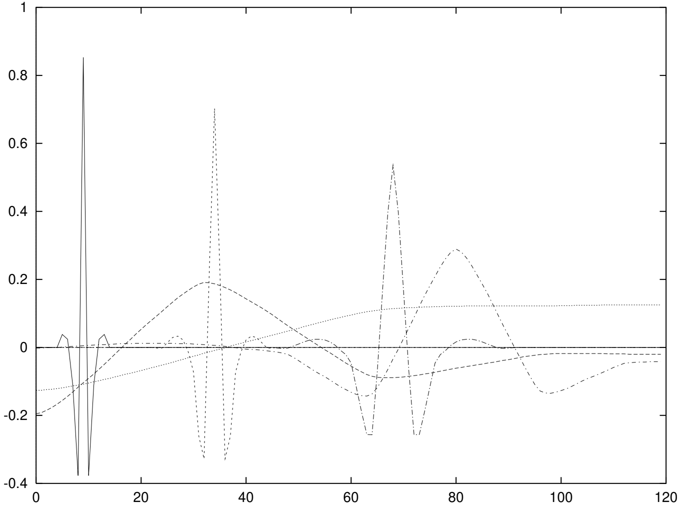
and the mother wavelet $w = w(t)$ can be chosen with these three properties:

Smoothness: *w and its first d derivatives $w', w'', \dots, w^{(d)}$ are continuous functions.*

Compact support: *$w(t)$ is zero at all $|t| > 5d$.*

Orthogonality: *The set $\{w_{jk} : j, k \in \mathbf{Z}\}$ is an orthonormal basis.*

Some Nice Wavelets



Six dilations and translations, on an interval, of a particular mother wavelet (9,7-biorthogonal symmetric).

History A.D. [After Daubechies]

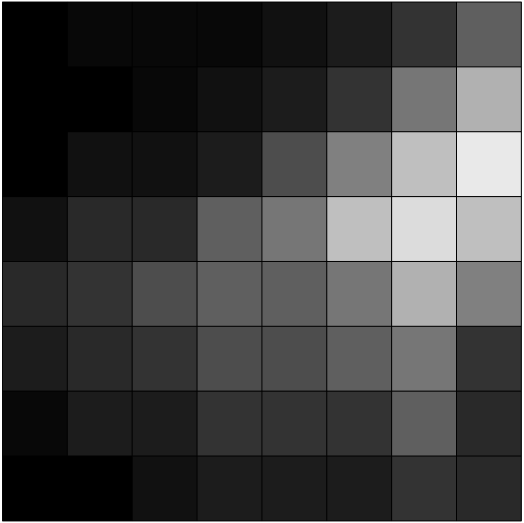
- Lapped orthogonal transforms (1990, *IEEE ASSP*)
- Biorthogonal wavelets, wavelet packets (1992, *IEEE IT*)
- WSQ fingerprint standard (1993, FBI)
- Wavelets on spheres (1995, *ACM*)
- The lifting implementation (1996, *ACHA*; 1998, *JFAA*;)
- JPEG-2000 compression (1999)

Example Images

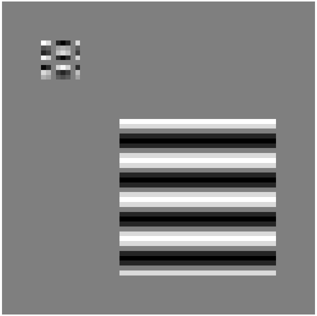
http://lenna.org/

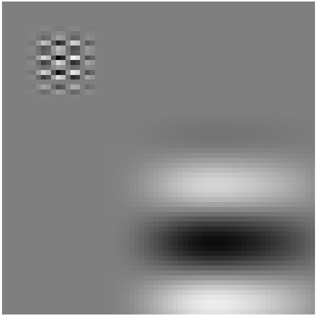
https://fbibiospecs.fbi.gov/certifications-1/wsq

Close Up of Correlated Pixels

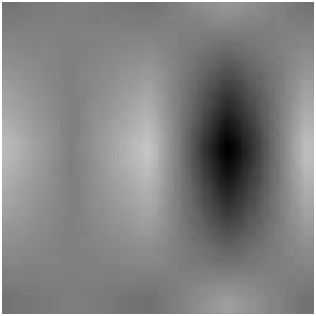


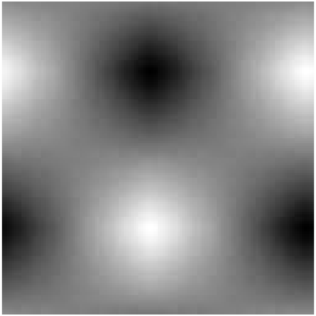
Two-Dimensional Waveforms I



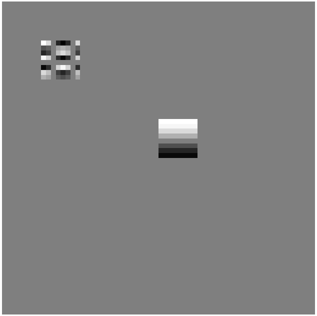


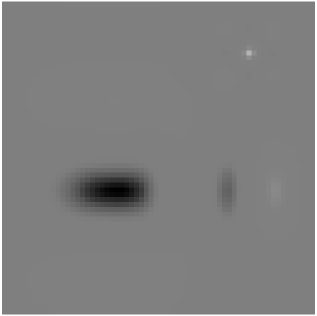
Two-Dimensional Waveforms II



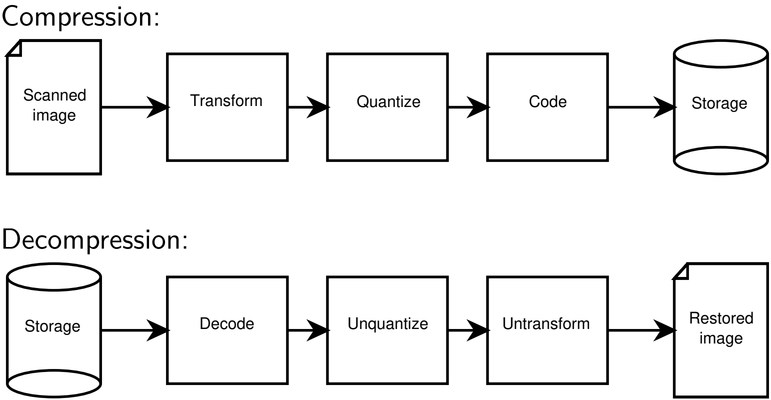


Two-Dimensional Waveforms III: JPEG vs. JPEG-2000





Transform Coding Image Compression



Parts Description

Compression:

Transform: convert pixels to amplitudes;

Quantize: round off the amplitudes to small numbers;

Code: remove redundancy from the small number sequence.

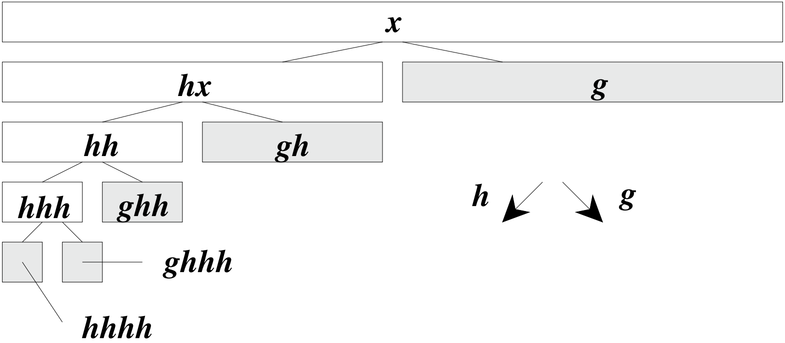
Decompression:

Decode: expand to recover the small number sequence;

Unquantize: insert an amplitude for each small number;

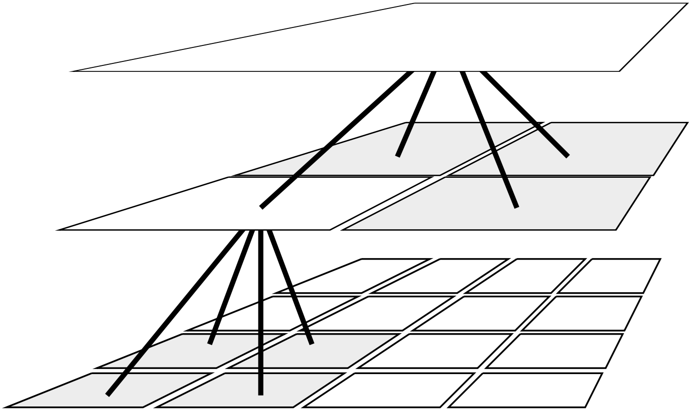
Untransform: recover pixels from approximate amplitudes.

Wavelet Transform: Multiresolution Signal Splitting



Split signal x into averages hx and details gx.
Replace $x \leftarrow hx$ and repeat

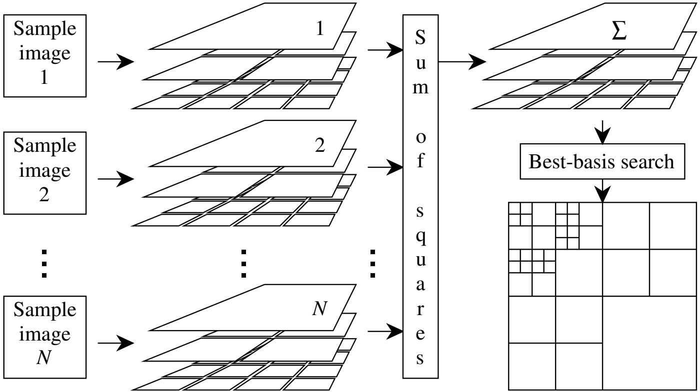
Multiresolution Image Splitting



Picture (at top) becomes thumbnail (at bottom left) plus two layers of saved details (highlighted).

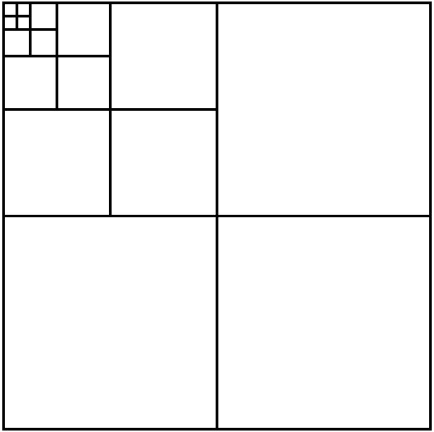
Storage of Multiresolution Image Data

Custom Compression Algorithms



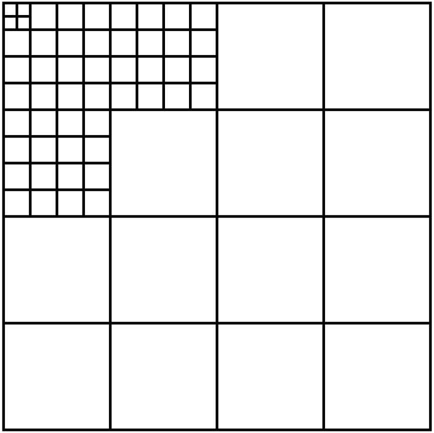
Training algorithm for a custom transform coding image compression algorithm.

Good Bases for Images I



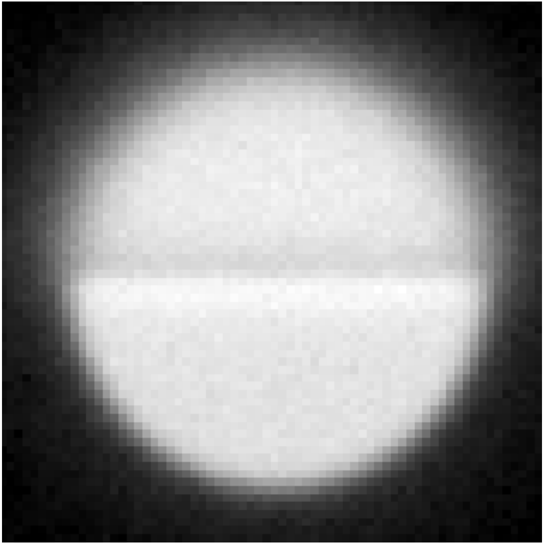
Five-level wavelet basis, used in JPEG-2000.

Good Bases for Images II



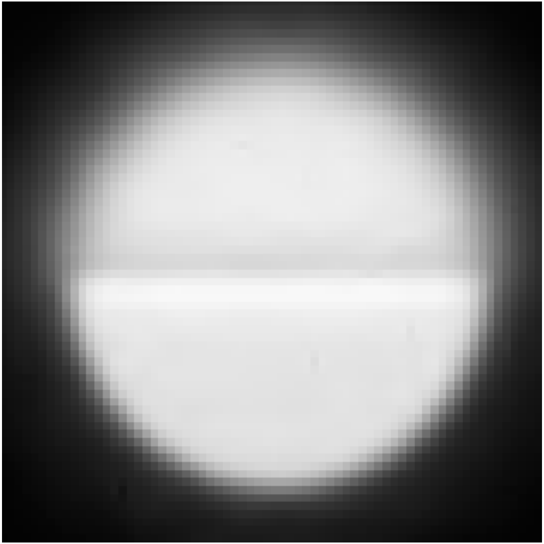
Five-level wavelet packet basis, used in WSQ.

Compression Sometimes Improves Things



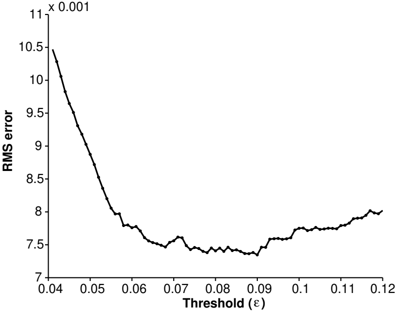
Rough Radiation Dose Approximation in 2D:
4 M particle simulation

...By Eliminating the Rough Errors



Improved Approximation in 2D:
Compressed 4 M particle simulation

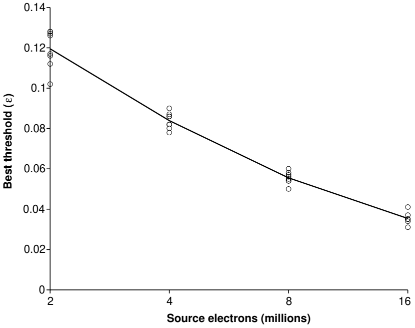
...If the Right Amount of Compression is Done



Deasy et al., Fig. 3

Reduction in RMS error by a rough approximation compressed toward a smooth target, by wavelet threshold.

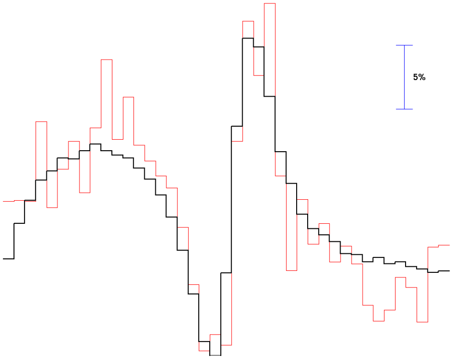
...Which, Fortunately, is Easy to Find.



Deasy, et al., Fig. 4

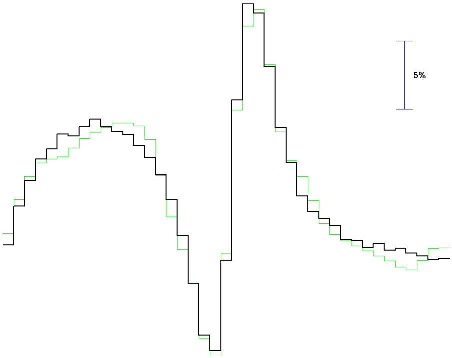
Best wavelet thresholds for compression from a rough approximation.

Example: Rough Radiation Dose Approximation – 1D



4 M particle simulation — 1D cross-section, close up.

Example: Compressed Approximation –1D



Compressed 4 M particle simulation — 1D cross-section, close up.

Some Notable Works

- Ingrid Daubechies. "Orthonormal Bases of Compactly Supported Wavelets." *Comm. Pure Appl. Math.* 41(1988),909–996.
- Ingrid Daubechies. *Ten Lectures on Wavelets*. CBMS-NSF Regional Conference Series in Applied Mathematics. SIAM Press, Philadelphia, 1992.
- Albert Cohen, Ingrid Daubechies and Jean-Christophe Feauveau. "Biorthogonal Bases of Compactly Supported Wavelets" *Comm. Pure Appl. Math.* 45(1992),485–500.
- Ingrid Daubechies and Wim Sweldens. "Factoring Wavelet Transforms into Lifting Steps." *Fourier Anal. Appl.* 4:3(1998),245–267.