Arbitrage and Convexity in Discrete Financial Models

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Introduction

- Financial Mathematics contains actual theorems, but they are often buried in jargon and heuristics.
- Computational machinery applied to modeling financial markets includes linear algebra and convex optimization in $\mathbf{R}^n$.
- My goal is to give two versions of the “Fundamental Theorem on Asset Pricing,” for the case of discrete financial models.

Assets and Portfolios

- An *asset* $a : T \times \Omega \to \mathbf{R}$ is a stochastic process, a time-varying random variable on a probability space Ω.
- $a(t, \omega)$ is the price of the asset at time t in state ω.
- T contains time $t = 0$, the *present*, and $a(0, \omega) \stackrel{\text{def}}{=} a(0)$ is independent of ω.
- A *riskless* asset is independent of ω at all times $t \in T$. All other assets are *risky*.
- A *portfolio* is a weighted sum of assets $\sum_i x_i a_i(t, \omega)$, usually written as the vector $\mathbf{x} = (x_i)$ of weights.

Long and Short Positions

In portfolio $\mathbf{x}$ of assets with price $\sum_i x_i a_i(t, \omega)$,

- asset a_i is *held long* if $x_i > 0$;
- asset a_i is *sold short* if $x_i < 0$.

Assets sold short are borrowed and must be returned. One example is a bank loan. The cost of returning a_i at time t in state ω is a *liability*, priced by $-a_i(t, \omega)$.

Note: Both a_i and x_i can be any real number.

Arbitrage

Traditionally, an *arbitrage* is a mispriced asset that offers profit without risk.

Formal definition using stochastic processes: a *deterministic arbitrage* is an asset $a(t, \omega)$ that

- costs nothing or leaves a surplus at $t = 0$: $a(0) \leq 0$,
- never loses value: $(\forall t > 0) \ \mathrm{Pr}(\{\omega : a(t, \omega) < 0\}) = 0$,
- has a positive price in some states at some future time: $(\exists t > 0) \ \mathrm{Pr}(\{\omega : a(t, \omega) > 0\}) > 0$.

Remark. A weaker *expected arbitrage* is an asset a with $a(0) \leq 0$ but $\mathrm{E}(a(t)) > 0$ for some future time $t > 0$.
Any deterministic arbitrage is an expected arbitrage.

Discrete Financial Models

The simplest choices for T and Ω are the finite sets $T = \{0, 1\}$ and $\Omega = \{1, 2, \ldots, n\}$. Then calculations are performed using just pairs and vectors of prices:

- The *spot price* $a_i(0)$, of asset a_i, assumed constant in all states at time $t = 0$.
- The *payoff* $a_i(1, j)$, of asset a_i, at future time $t = 1$, in state $\omega = j$.

The payoff vector $\mathbf{a}_i = (a_i(1, 1), \ldots, a_i(1, j), \ldots, a_i(1, n))$ lists all the modeled future prices for the asset.

Market Matrices

Using $T = \{0, 1\}$ and $\Omega = \{1, 2, \ldots, n\}$, a *market* with m assets is modeled by $\mathbf{q}$ and A, namely:

- Vector $\mathbf{q} \stackrel{\text{def}}{=} (a_i(0))$ of spot prices, and
- Matrix of payoffs

$$A \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{1} \\ \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_m \end{pmatrix}, = \begin{pmatrix} 1 & \ldots & 1 \\ a_1(1,1) & \ldots & a_1(1,n) \\ \vdots & \ddots & \vdots \\ a_m(1,1) & \ldots & a_m(1,n) \end{pmatrix},$$

where $a_i(1, j)$ is the payoff of asset i in state j.

Note: The top row of A is the *numeraire*, also called *cash*, a unit of which has constant payoff 1 in all states $j = 1, \ldots, n$.

Spot Prices and Payoffs

In the discrete financial model $\mathbf{q}, A$, any portfolio $\sum_i x_i a_i(t, \omega)$ represented by the vector of weights $\mathbf{x}$ has

- spot price $\mathbf{x}^T \mathbf{q}$, and
- payoff vector $\mathbf{x}^T A$.

Note: For the linear algebra computations, payoff vectors will be row vectors while spot price vectors, portfolio weight vectors, and probability mass functions will be column vectors. Unfortunately, this is only one of the several conventions in use.

Next: characterize "arbitrage" using $\mathbf{x}$, $\mathbf{q}$, and A.

Convexity and Cones

Start with some geometric concepts:

- A set $S \subset \mathbf{R}^n$ is *convex* iff

$$\mathbf{x}, \mathbf{y} \in S \implies (\forall \lambda \in [0,1])\ \lambda \mathbf{x} + (1-\lambda)\mathbf{y} \in S.$$

  Any subspace is convex.

- Set $S \subset \mathbf{R}^n$ is a *cone* iff

$$\mathbf{x} \in S \implies (\forall \lambda > 0) \lambda \mathbf{x} \in S.$$

  Any subspace is a cone.

Positivity

In $\mathbf{R}^n$, use componentwise positivity or nonnegativity.

For $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbf{R}^n$, and so on,

- write $\mathbf{v} > \mathbf{0}$, and say that $\mathbf{v}$ is *positive*, if $(\forall j)\, v_j > 0$;
- write $\mathbf{v} \geq \mathbf{0}$, and say that $\mathbf{v}$ is *nonnegative*, if $(\forall j)\, v_j \geq 0$;
- write $\mathbf{v} > \mathbf{w}$ to mean $\mathbf{v} - \mathbf{w} > \mathbf{0}$;
- write $\mathbf{v} \geq \mathbf{w}$ to mean $\mathbf{v} - \mathbf{w} \geq \mathbf{0}$.

Componentwise positivity and nonnegativity defines *orthants*, which are special cases of *convex cones*.

Open, Pointless, and Closed Orthants

Three useful examples of orthants:

- The closed orthant of vectors with nonnegative coordinates,

$$K \stackrel{\text{def}}{=} \{\mathbf{y} \in \mathbf{R}^n : \mathbf{y} \geq \mathbf{0}\},$$

  is a closed convex cone.

- Remove the point $\mathbf{0}$ to get the *pointless* orthant

$$K \setminus \mathbf{0} = \{\mathbf{y} \in \mathbf{R}^n : \mathbf{y} \geq \mathbf{0},\ (\exists j)\, y_j > 0\}.$$

  This is also a convex cone but is neither open nor closed.

- The interior of K is an open convex cone:

$$K^o \stackrel{\text{def}}{=} \{\mathbf{y} \in \mathbf{R}^n : (\forall j)\, y_j > 0\} = \{\mathbf{y} \in \mathbf{R}^n : \mathbf{y} > \mathbf{0}\}.$$

Deterministic Arbitrages in the Discrete Model

These are portfolios $\mathbf{x}$, in a market A with spot prices $\mathbf{q}$, that offer riskless profit for every probability mass function on Ω.

- *Type one arbitrage*, or *immediate arbitrage*, leaves a surplus as it is assembled at time 0 but has nonnegative payoff in any state at future time 1:

  IA1: $\mathbf{x}^T\mathbf{q} < 0$.
  IA2: $\mathbf{x}^T A \geq \mathbf{0}$. Equivalently, $\mathbf{x}^T A \in K$.

- *Type two arbitrage*, or *arbitrage opportunity*, costs nothing to assemble and cannot lose value, but has a positive payoff in some future state:

  AO1: $\mathbf{x}^T\mathbf{q} \leq 0$.
  AO2: $\mathbf{x}^T A \geq \mathbf{0}$, and $(\exists j)\, \mathbf{x}^T A(j) > 0$. Equivalently, $\mathbf{x}^T A \in K \setminus \mathbf{0}$.

Arbitrage and Martingales in the Discrete Model

- An *arbitrage expectation*, which is not deterministic, costs nothing to assemble but has positive expected payoff:

  AE1: $\mathbf{x}^T\mathbf{q} \le 0$

  AE2: $\mathbf{x}^T A\mathbf{y} > 0$, where $\mathbf{y}$ is the probability mass function on the states $1, \ldots, n$ in Ω.

- A stochastic process $a(t, \omega)$ is a *martingale* if

$$t > s \implies \mathrm{E}(a(t)|a(s)) = a(s).$$

For the discrete financial model, put $t > s = 0$ to get

$$\mathbf{x}^T A\mathbf{y} = \mathrm{E}(a(t)|a(0)) = \mathrm{E}(a(0)) = a(0) = \mathbf{x}^T\mathbf{q}.$$

So no arbitrage expectation, and thus no deterministic arbitrage, can exist if assets are martingales.

No-Arbitrage Axioms

An immediate arbitrage is an arbitrage opportunity is an arbitrage expectation:

$$\exists\ IA \implies \exists\ AO \implies \exists\ AE. \tag{1}$$

The universal desire for profit creates unlimited demand for arbitrages so it is assumed that if assets are freely traded, then prices will adjust instantly to consume any supply. This may be stated as an axiom:

Axiom 1 *There are no arbitrages.* □

The chain of implications for no arbitrages is the reverse of (1):

$$\nexists\ AE \implies \nexists\ AO \implies \nexists\ IA. \tag{2}$$

Profitable Portfolios

These are sets of portfolios, in a discrete financial model, that contain all possible arbitrages for market matrix A.

- A *profitable portfolio* $\mathbf{p}$ is one that has nonnegative payoff in all states: $\mathbf{p}^T A \geq \mathbf{0}$.
- Equivalently, $\mathbf{p}^T A \in K$.
- Equivalently, $(\forall \mathbf{k} \in K)\ \mathbf{p}^T A\mathbf{k} \geq 0$.
- A *strictly profitable portfolio* $\mathbf{s}$ is profitable and also has a positive payoff in some state: $(\exists j)\, \mathbf{s}^T A(j) > 0$.
- Equivalently, $\mathbf{s}^T A \in K \setminus \mathbf{0}$.
- Equivalently, $(\forall \mathbf{k} \in K^o)\ \mathbf{s}^T A\mathbf{k} > 0$.

The Usefulness of Cash

- A matrix of assets without a numeraire might have no strictly profitable portfolios. For example, the one-asset market matrix with two states

  $$A = \begin{pmatrix} -1 & 1 \end{pmatrix}$$

  satisfies $xA = (x, -x)$ so only $x = 0$ is profitable, and no $x \in \mathbf{R}$ is strictly profitable.
- A market with a numeraire, or cash, as its zeroth row, has an all-cash portfolio $\mathbf{x} = (1, 0, \ldots, 0)$ that satisfies $\mathbf{x}^T A(j) = 1$ for all j. This $\mathbf{x}$ is both profitable and strictly profitable.
- More generally, if there is any riskless asset such that $(\forall\omega)\, a(1,\omega) = a(1) \neq 0$, then there will exist nontrivial profitable and strictly profitable portfolios.

Henceforth, assume that A contains a riskless asset.

No Arbitrages, Geometrically

The absence of arbitrages in a market may now be stated using orthants and profitable portfolios:

Definition (No IA)

Market A with prices $\mathbf{q}$ is *immediate arbitrage free* iff any profitable portfolio must have a nonnegative price:

$$\mathbf{x}^T A \in K \implies \mathbf{x}^T \mathbf{q} \geq 0.$$

Definition (No AO)

Market A with prices $\mathbf{q}$ is *arbitrage opportunity free* iff any strictly profitable portfolio must have a positive price:

$$\mathbf{x}^T A \in K \setminus \mathbf{0} \implies \mathbf{x}^T \mathbf{q} > 0.$$

Dual Cones

Arbitrages may be characterized using convex cones and their duals. To start, define these for any set $S \subset \mathbf{R}^n$.

- The *dual cone* of S is

$$S' \stackrel{\text{def}}{=} \{\mathbf{x} \in \mathbf{R}^n : \ (\forall \mathbf{y} \in S)\, \mathbf{x}^T \mathbf{y} \geq 0\}.$$

- If S is a subspace, then $S' = S^{\perp}$ is its orthogonal complement.
- The *strict dual cone* of S is

$$S^* \stackrel{\text{def}}{=} \{\mathbf{x} \in \mathbf{R}^n : \ (\forall \mathbf{y} \in S)\, \mathbf{x}^T \mathbf{y} > 0\}.$$

- If $\mathbf{0} \in S$, then $S^* = \emptyset$. Thus if S is a subspace, then $S^* = \emptyset$.

Remark. For any set $S \subset \mathbf{R}^n$, both S' and S^* are convex cones.

Self-Duality and Double Duality

Some useful facts:

- $K' = K$, that is, the nonnegative orthant is a self-dual cone.
- It follows that $(K')' = (K)' = K$, that is, the double dual of the closed nonnegative orthant is itself.
- $(K^o)' = K$ and $(K^o)^* = K \setminus \mathbf{0}$.
- $(K \setminus \mathbf{0})' = K$ and $(K \setminus \mathbf{0})^* = K^o$.
- It follows that $((K^o)^*)^* = (K \setminus \mathbf{0})^* = K^o$, that is, the open positive orthant is its own strict double dual cone.

We will see that this reflexivity of double duals holds for convex cones in general.

Profitable Portfolios as Dual Cones

Lemma

For market matrix A,

- *The set P of profitable portfolios is a dual cone: $P = (AK)'$.*
- *The set S of strictly profitable portfolios is a strict dual cone: $S = (AK^o)^*$.*

Here $AK \stackrel{\text{def}}{=} \{A\mathbf{k} : \mathbf{k} \in K\}$ and $AK^o \stackrel{\text{def}}{=} \{A\mathbf{k} : \mathbf{k} \in K^o\}$.

Proof.

By the previous characterizations,

$$\begin{aligned} \mathbf{p} \in P &\iff (\forall k \in K)\mathbf{p}^T A\mathbf{k} \geq 0 \iff \mathbf{p} \in (AK)' \\ \mathbf{s} \in S &\iff (\forall k \in K^o)\mathbf{s}^T A\mathbf{k} > 0 \iff \mathbf{s} \in (AK^o)^* \end{aligned}$$

using associativity $(\mathbf{x}^T A)\mathbf{k} = \mathbf{x}^T (A\mathbf{k})$ and the definitions. □

Fundamental Theorem on Asset Pricing

In an arbitrage free market, the price vector $\mathbf{q}$ is a weighted average of the payoffs in the states of Ω:

Theorem (FT from No IA)

Market A with spot prices $\mathbf{q}$ is immediate arbitrage free if and only if there is a vector $\mathbf{k} \in K$ such that

$$\mathbf{q} = A\mathbf{k}.$$

Proof:
($\Longleftarrow$): Suppose that $\mathbf{k} \in K$ solves $\mathbf{q} = A\mathbf{k}$ and let $\mathbf{x}$ be a profitable portfolio. Then

$$\mathbf{x}^T\mathbf{q} = \mathbf{x}^T(A\mathbf{k}) = (\mathbf{x}^T A)\mathbf{k} \geq 0,$$

since $\mathbf{x}^T A \in K$. Thus, by definition, market A with prices $\mathbf{q}$ is immediate arbitrage free.

Proof (continued)

$(\implies)$: Suppose that market A with prices $\mathbf{q}$ is IA free. Then:

- AK, for nonnegative orthant K, is a closed convex cone.
- $P = (AK)'$, namely the set of all profitable portfolios for A is the dual cone of AK, by the lemma.
- $\mathbf{q} \in P'$, since $A, \mathbf{q}$ is immediate arbitrage free:

$$(\forall \mathbf{x} \in P)\, \mathbf{x}^T \mathbf{q} \geq 0.$$

Hence $\mathbf{q} \in ((AK)')'$.
But in fact $((AK)')' = AK$, since AK is a closed convex cone.
Then $\mathbf{q} \in ((AK)')' = AK$, so there is some $\mathbf{k} \in K$ such that $\mathbf{q} = A\mathbf{k}$. $\square$

It remains to prove the Double Dual Cone Theorem.

Double Dual of a Closed Convex Cone

Theorem (Double Dual Cone)

If Q is a closed convex cone, then $(Q')' = Q$.

Proof: First note that $Q \subset (Q')'$:

$$\mathbf{q} \in Q \implies (\forall \mathbf{z} \in Q')\, \mathbf{q}^T \mathbf{z} \geq 0 \implies \mathbf{q} \in (Q')'.$$

Now suppose toward contradiction that $\mathbf{b} \in (Q')'$ but $\mathbf{b} \notin Q$. Then there is a nonzero vector $\mathbf{x}$ and a constant γ defining a separating hyperplane by the function $f(\mathbf{y}) \stackrel{\text{def}}{=} \mathbf{x}^T \mathbf{y} - \gamma$ where

$$f(\mathbf{b}) < 0, \quad \text{but} \quad (\forall \mathbf{q} \in Q)\, f(\mathbf{q}) > 0.$$

Since Q is a closed cone it contains $\mathbf{0}$, so $f(\mathbf{0}) = -\gamma > 0$, so $\gamma < 0$.

Double Dual Cone Proof, Continued

Also, fix $\mathbf{q} \in Q$ and let $\lambda \to \infty$ while noting that $\lambda\mathbf{q} \in Q$, so

$$\mathbf{x}^T\mathbf{q} = \lim_{\lambda\to\infty}\left(\mathbf{x}^T\mathbf{q} - \frac{\gamma}{\lambda}\right) = \lim_{\lambda\to\infty}\frac{1}{\lambda}f(\lambda\mathbf{q}) \geq 0.$$

Thus $\mathbf{x} \in Q'$. But then $\mathbf{b} \in (Q')'$ gives the contradiction $f(\mathbf{b}) = \mathbf{x}^T\mathbf{b} - \gamma \geq -\gamma > 0$. $\square$

We are still not done!
Now we need to prove the Hyperplane Separation Theorem.
The most general version is a consequence of the Hahn-Banach Theorem, but we give a proof below using only Calculus methods.

Another Proof via Farkas's Lemma

This result from 1902 has FT from No IA as a corollary:

Theorem (Farkas's Lemma)

Suppose that $A \in \mathbf{R}^{m \times n}$ is a matrix and $\mathbf{b} \in \mathbf{R}^m$ is a vector. Then exactly one of the following must be true:

X: *There exists $\mathbf{x} \in \mathbf{R}^m$ such that $\mathbf{x}^T A \geq \mathbf{0}$ and $\mathbf{x}^T \mathbf{b} < 0$.*

Y: *There exists $\mathbf{y} \in \mathbf{R}^n$ such that $A\mathbf{y} = \mathbf{b}$ and $\mathbf{y} \geq \mathbf{0}$.*

(*To prove FT from No IA:*
Let A be the market matrix and $\mathbf{b}$ the spot price vector from a discrete financial model.
If $A, \mathbf{b}$ is immediate arbitrage free, then Condition X cannot be true. By Condition Y, there is a vector $\mathbf{y} \in K$ such that $\mathbf{b} = A\mathbf{y}$.)

Proof of Farkas's Lemma

Proof: First observe that X and Y cannot both hold, for then

$$\mathbf{x}^T A\mathbf{y} = \mathbf{x}^T(A\mathbf{y}) = \mathbf{x}^T\mathbf{b} < 0,$$

while also $\mathbf{x}^T A\mathbf{y} = (\mathbf{x}^T A)\mathbf{y} \geq 0$, since both $(\mathbf{x}^T A) \geq \mathbf{0}$ and $\mathbf{y} \geq \mathbf{0}$. Evidently, Condition Y holds if and only if

$$\mathbf{b} \in Q \stackrel{\text{def}}{=} AK = \{A\mathbf{k} : \mathbf{k} \in K\},$$

so if Y fails to hold it must be that $\mathbf{b} \notin Q$.
But Q is a nonempty closed convex cone. Thus there exists a nonzero vector $\mathbf{x} \in \mathbf{R}^m$ and a constant $\gamma \in \mathbf{R}$ defining a separating hyperplane function

$$f : \mathbf{R}^m \to \mathbf{R}, \quad f(\mathbf{y}) \stackrel{\text{def}}{=} \mathbf{x}^T\mathbf{y} - \gamma,$$

such that $f(\mathbf{b}) < 0$ but $f(\mathbf{q}) > 0$ for every $\mathbf{q} \in Q$.

Farkas Proof, Continued

Now $\mathbf{0} \in Q$, since $\mathbf{0} \in K$, so $f(\mathbf{0}) = \mathbf{x}^T\mathbf{0} - \gamma = -\gamma > 0$, and therefore $\gamma < 0$. But then

$$f(\mathbf{b}) = \mathbf{x}^T\mathbf{b} - \gamma < 0 \implies \mathbf{x}^T\mathbf{b} < \gamma < 0.$$

On the other hand, $f(\mathbf{q}) > 0$ implies only that $\mathbf{x}^T\mathbf{q} > \gamma$. But since Q is a cone, any $\mathbf{q} \in Q$ and any $\lambda > 0$ result in $\lambda\mathbf{q} \in Q$, so

$$(\forall\lambda > 0)\, f(\lambda\mathbf{q}) = \lambda\mathbf{x}^T\mathbf{q} - \gamma > 0 \implies (\forall\lambda > 0)\, \mathbf{x}^T\mathbf{q} > \gamma/\lambda,$$

and this can only be true for negative γ if $\mathbf{x}^T\mathbf{q} \geq 0$ for all $\mathbf{q} \in Q$. Writing $\mathbf{q} = A\mathbf{k}$ gives

$$(\forall\mathbf{k} \in K)\, \mathbf{x}^T A\mathbf{k} \geq 0,$$

so $\mathbf{x}^T A$ is in the dual cone of K. But K is self-dual, so $\mathbf{x}^T A \geq \mathbf{0}$. Conclude that Condition X holds. $\square$

Hyperplane Separation

Farkas's Lemma, the Double Dual Cone Lemma, and thus the Fundamental Theorem on Asset Pricing all follow from a purely geometric fact about closed convex sets:

Theorem (Hyperplane Separation)

Suppose that $Q \subset \mathbf{R}^m$ is a nonempty closed convex set and $\mathbf{b} \in \mathbf{R}^m$ is a point not in Q. Then there exist a nonzero vector $\mathbf{x} \in \mathbf{R}^m$ and a constant $\gamma \in \mathbf{R}$ defining a hyperplane as the zeros of the function

$$f(\mathbf{y}) \stackrel{\text{def}}{=} \mathbf{x}^T\mathbf{y} - \gamma,$$

such that $f(\mathbf{b}) < 0$ but $f(\mathbf{q}) > 0$ for every $\mathbf{q} \in Q$.

Proof I: Construct a hyperplane

Define $s : \mathbf{R}^m \to \mathbf{R}$ by $s(\mathbf{y}) \stackrel{\text{def}}{=} \|\mathbf{y} - \mathbf{b}\|^2$, continuous and differentiable with gradient

$$\nabla s(\mathbf{y}) = 2(\mathbf{y} - \mathbf{b}) \in \mathbf{R}^m.$$

It achieves its minimum at a nearest point $\mathbf{q}_0 \in Q$ to $\mathbf{b}$. Put $f(\mathbf{y}) \stackrel{\text{def}}{=} \mathbf{x}^T \mathbf{y} - \gamma$ for

$$\mathbf{x} = \mathbf{q}_0 - \mathbf{b}, \quad \gamma = \frac{\|\mathbf{q}_0\|^2 - \|\mathbf{b}\|^2}{2}.$$

Hyperplane $\{\mathbf{y} : f(\mathbf{y}) = 0\}$ is normal to $\mathbf{q}_0 - \mathbf{b}$ and passes through the midpoint between $\mathbf{b}$ and $\mathbf{q}_0$.
It remains to show that f separates $\mathbf{b}$ from Q.

Proof II: $f(\mathbf{b}) < 0$

Compute $f(\mathbf{b}) = \mathbf{q}_0^T \mathbf{b} - \frac{\|\mathbf{q}_0\|^2 + \|\mathbf{b}\|^2}{2}$. The Cauchy-Schwartz inequality and the arithmetic-geometric mean inequality together imply

$$\mathbf{q}_0^T \mathbf{b} \leq \|\mathbf{q}_0\| \|\mathbf{b}\| \leq \frac{\|\mathbf{q}_0\|^2 + \|\mathbf{b}\|^2}{2},$$

with equality only if $\mathbf{q}_0 = \mathbf{b}$. Conclude that $f(\mathbf{b}) < 0$.

Proof III: $f(\mathbf{q}) > 0$

Take any $\mathbf{q} \in Q$ and suppose toward contradiction that $f(\mathbf{q}) \leq 0$. Then

$$(\mathbf{q}_0 - \mathbf{b})^T \mathbf{q} \;\leq\; \frac{\|\mathbf{q}_0\|^2 - \|\mathbf{b}\|^2}{2},$$

so $\nabla s(\mathbf{q}_0)^T (\mathbf{q} - \mathbf{q}_0) \leq -\|\mathbf{q}_0 - \mathbf{b}\|^2 \; < 0$. Hence there is some small $\lambda \in (0, 1)$ for which

$$s(\mathbf{q}_0 + \lambda[\mathbf{q} - \mathbf{q}_0]) < s(\mathbf{q}_0).$$

But Q is convex, so $\mathbf{q}_0 + \lambda[\mathbf{q} - \mathbf{q}_0] = (1 - \lambda)\mathbf{q}_0 + \lambda\mathbf{q} \in Q$, and this contradicts the extremal property of $\mathbf{q}_0$.
Conclude that $f(\mathbf{q}) > 0$. $\square$

Fundamental Theorem on Asset Pricing II

Under stronger hypotheses, the weight vector is strictly positive:

Theorem (FT from No AO)

Market matrix A with numeraire and spot prices $\mathbf{q}$ is arbitrage opportunity free if and only if there is a vector $\mathbf{k} \in K^o$ such that

$$\mathbf{q} = A\mathbf{k}.$$

Proof:
$(\Longleftarrow)$: Suppose that $\mathbf{k} \in K^o$ solves $\mathbf{q} = A\mathbf{k}$ and let $\mathbf{x}$ be a strictly profitable portfolio. Then

$$\mathbf{x}^T\mathbf{q} = \mathbf{x}^T(A\mathbf{k}) = (\mathbf{x}^T A)\mathbf{k} > 0,$$

since $\mathbf{x}^T A \in K \setminus \mathbf{0}$. Thus, by definition, market A with prices $\mathbf{q}$ is arbitrage opportunity free.

Proof (continued)

($\implies$): Suppose that market A with prices $\mathbf{q}$ is AO free. Then:

- AK^o, for open positive orthant K^o, is an open convex cone in the column space of A (by the Open Mapping Theorem).
- $S = (AK^o)^*$, namely the set of all strictly profitable portfolios for A, is the strict dual cone of AK^o, by previous lemma.
- $\mathbf{q} \in S^*$, since $A, \mathbf{q}$ is arbitrage opportunity free:

$$(\forall \mathbf{x} \in S)\, \mathbf{x}^T \mathbf{q} > 0.$$

Hence $\mathbf{q} \in ((AK^o)^*)^*$.
But $((AK^o)^*)^* = AK^o$, by the Strict Double Dual Cone Theorem.
Conclude that there is some $\mathbf{k} \in K^o$ such that $\mathbf{q} = A\mathbf{k}$. $\square$

The proof that $((AK^o)^*)^* = AK^o$ is left as an exercise.

Application to Derivative Pricing

Suppose that payoff matrix A with spot price vector $\mathbf{q}$ corresponds to an arbitrage free market.

Write $\mathbf{q} = A\mathbf{k}$ by the Fundamental Theorem.

- The vector $\mathbf{k}$, which is nonzero if $\mathbf{q} \neq \mathbf{0}$, is called a *risk neutral probability mass function*, when normalized to have unit sum.
- Any derivative asset with future payoff vector $\mathbf{d}$ has a risk neutral spot price $\mathbf{d}^T\mathbf{k}$.

Derivative assets are often *contingent claims*.

Contingent Claims

These are contracts to pay or collect some amount depending on the price of *underlying assets*. Examples are:

Call Option to buy an asset for a stated or computed *strike price* at or before a stated *expiry time*.

Put Option to sell an asset for a strike price at or before expiry.

Swap Obligation to exchange one sequence of payments for another with different terms.

Forward Obligation to buy or sell an asset for a stated strike price at a future date.

Future Forward contract backed by cash in a supervised Margin Account.

Hedges

Traditionally, in gambling, a hedge for a bet is another bet that limits potential loss but also limits potential profit.

- Financial institutions that sell contingent claims seek to *hedge*, or replicate them, with a portfolio of other assets whose value equals or exceeds the cost of the contingent claim in all modeled states Ω.
- If $\mathbf{c}$ is the cost vector of the contingent claim over Ω, namely the liability of the financial institution that sold it, then a hedge portfolio $\mathbf{h}$ over a market A must satisfy

$$\mathbf{h}^T A \geq \mathbf{c}.$$

- At spot prices $\mathbf{q}$, the cost of the hedge portfolio is $\mathbf{h}^T \mathbf{q}$.

Complete Markets

- Market A is *complete* if any contingent claim can be hedged, namely if the row space of A is all of $\mathbf{R}^n$.
- Since the row space is dependent on the discrete financial model, this cannot be guaranteed without additional assumptions.
- *Binomial models*, where $n = 2$ and $m = 1$ so that A is a 2×2 matrix

$$A = \begin{pmatrix} a_0(1,1) & a_0(1,2) \\ a_1(1,1) & a_1(1,2) \end{pmatrix}, \qquad \begin{array}{l} a_0(1,1) = a_0(1,2), \\ a_1(1,1) \neq a_1(1,2). \end{array}$$

  Such A, with a numeraire (or other riskless asset) $a_0 \neq \mathbf{0}$ and a single risky asset a_1, are always complete, so there is a unique hedge for any contingent claim on the underlying a_1.

Incomplete Markets

In the general case, when the market is *incomplete*, the seller of a contingent claim $\mathbf{c}$ constructs a hedge portfolio $\mathbf{h}$ by solving

Minimize $\mathbf{h}^T\mathbf{q}$ *subject to* $\mathbf{h}^T A \geq \mathbf{c}$.

Conversely, the buyer of the contingent claim $\mathbf{c}$ compares its price to the alternative portfolio $\mathbf{k}$ solving

Maximize $\mathbf{k}^T\mathbf{q}$ *subject to* $\mathbf{k}^T A \leq \mathbf{c}$.

These are both convex optimization problems solvable by *linear programming*.

Bid-Ask Spread

If market A with prices $\mathbf{q}$ is arbitrage free, then any profitable portfolio $\mathbf{x}$ must have a nonnegative price:

$$\mathbf{x}^T A \geq \mathbf{0} \implies \mathbf{x}^T \mathbf{q} \geq 0.$$

Let $\mathbf{x} = \mathbf{h} - \mathbf{k}$ be the difference of the portfolios solving the hedge optimization problems. Then

$$\mathbf{x}^T A = \mathbf{h}^T A - \mathbf{k}^T A \geq \mathbf{c} - \mathbf{c} = \mathbf{0},$$

so we may conclude that $\mathbf{h}^T \mathbf{q} \geq \mathbf{k}^T \mathbf{q}$. The nonempty interval

$$[\mathbf{k}^T \mathbf{q}, \mathbf{h}^T \mathbf{q}]$$

is the *no-arbitrage bid-ask spread* for the contingent claim $\mathbf{c}$.

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