Arbitrage and Convexity in Discrete Financial Models

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Introduction

- Having taught Financial Mathematics from a disturbingly theorem-less textbook, I was motivated to find some actual Mathematics behind the jargon and heuristics.
- Much computational machinery has been applied to modeling financial markets. In this talk, I will survey some of the results that depend on linear algebra, cones, and convexity in $\mathbf{R}^n$.
- My goal is to give two versions of the "Fundamental Theorem on Asset Pricing," for the case of discrete financial models.

Assets and Portfolios

- An *asset* $a : T \times \Omega \to \mathbf{R}$ is a stochastic process, a time-varying random variable on a probability space Ω.
- $a(t, \omega)$ is the price of the asset at time t in state ω.
- T contains time $t = 0$, the *present*, and $a(0, \omega) \stackrel{\text{def}}{=} a(0)$ is independent of ω.
- A *riskless* asset is independent of ω at all times $t \in T$. All other assets are *risky*.
- A *portfolio* is a weighted sum of assets $\sum_i x_i a_i(t, \omega)$, usually written as the vector $\mathbf{x} = (x_i)$ of weights.

Discrete Financial Models

The simplest choices for T and Ω are the finite sets $T = \{0, 1\}$ and $\Omega = \{1, 2, \ldots, n\}$. Then calculations are performed using just pairs and vectors of prices:

- The *spot price* $a_i(0)$, of asset a_i, assumed constant in all states at time $t = 0$.
- The *payoff* $a_i(1, j)$, of asset a_i, at future time $t = 1$, in state $\omega = j$.

The payoff vector $\mathbf{a}_i = (a_i(1, 1), \ldots, a_i(1, j), \ldots, a_i(1, n))$ lists all the modeled future prices for the asset.

Market Matrices

Using $T = \{0, 1\}$ and $\Omega = \{1, 2, \ldots, n\}$, a *market* with m assets is modeled by $\mathbf{q}$ and A, namely:

- Vector $\mathbf{q} \stackrel{\text{def}}{=} (a_i(0))$ of spot prices, and
- Matrix of payoffs

$$A \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{1} \\ \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_m \end{pmatrix}, = \begin{pmatrix} 1 & \ldots & 1 \\ a_1(1,1) & \ldots & a_1(1,n) \\ \vdots & \ddots & \vdots \\ a_m(1,1) & \ldots & a_m(1,n) \end{pmatrix},$$

where $a_i(1, j)$ is the payoff of asset i in state j.

NOTE: The top row of A is the *numeraire*, also called *cash*, a unit of which has constant payoff 1 in all states $j = 1, \ldots, n$.

Spot Prices and Payoffs

In the discrete financial model $\mathbf{q}, A$, any portfolio $\sum_i x_i a_i(t, \omega)$ represented by the vector of weights $\mathbf{x}$ has

- spot price $\mathbf{x}^T \mathbf{q}$, and
- payoff vector $\mathbf{x}^T A$.

Note: For the linear algebra computations, payoff vectors will be row vectors while spot price vectors, portfolio weight vectors, and probability mass functions will be column vectors. Unfortunately, this is only one of the several conventions in use.

Arbitrage and Positivity

An *arbitrage* is a portfolio $\mathbf{x}$ that yields profit without risk.
There are various kinds of arbitrage, some deterministic and some probabilistic. They may be defined using componentwise positivity or nonnegativity.
For $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbf{R}^n$, and so on,

- Write $\mathbf{v} > \mathbf{0}$, and say that $\mathbf{v}$ is *positive*, if $(\forall j)\, v_j > 0$.
- Write $\mathbf{v} \geq \mathbf{0}$, and say that $\mathbf{v}$ is *nonnegative*, if $(\forall j)\, v_j \geq 0$.
- Write $\mathbf{v} > \mathbf{w}$ to mean $\mathbf{v} - \mathbf{w} > \mathbf{0}$.
- Write $\mathbf{v} \geq \mathbf{w}$ to mean $\mathbf{v} - \mathbf{w} \geq \mathbf{0}$.

Such positivity is a property of *orthants*, which are special cases of *convex cones*.

Convexity and Cones

- A set $S \subset \mathbf{R}^n$ is *convex* iff

$$\mathbf{x}, \mathbf{y} \in S \implies (\forall \lambda \in [0,1])\ \lambda \mathbf{x} + (1-\lambda)\mathbf{y} \in S.$$

  Any subspace is convex.
- Set $S \subset \mathbf{R}^n$ is a *cone* iff

$$\mathbf{x} \in S \implies (\forall \lambda > 0) \lambda \mathbf{x} \in S.$$

  Any subspace is a cone.

Open, Pointless, and Closed Orthants

- The closed orthant of vectors with nonnegative coordinates,

$$K \stackrel{\text{def}}{=} \{\mathbf{y} \in \mathbf{R}^n : \mathbf{y} \geq \mathbf{0}\},$$

  is a closed convex cone.

- Remove the point $\mathbf{0}$ to get the *pointless* orthant

$$K \setminus \mathbf{0} = \{\mathbf{y} \in \mathbf{R}^n : \mathbf{y} \geq \mathbf{0},\ (\exists j)\, y_j > 0\}.$$

  This is also a convex cone but is neither open nor closed.

- The interior of K is an open convex cone:

$$K^o \stackrel{\text{def}}{=} \{\mathbf{y} \in \mathbf{R}^n : (\forall j)\, y_j > 0\} = \{\mathbf{y} \in \mathbf{R}^n : \mathbf{y} > \mathbf{0}\}.$$

Dual Cones

Let $S \subset \mathbf{R}^n$ be any set.

- The *dual cone* of S is

$$S' \stackrel{\text{def}}{=} \{\mathbf{x} \in \mathbf{R}^n : \ (\forall \mathbf{y} \in S)\, \mathbf{x}^T \mathbf{y} \geq 0\}.$$

- If S is a subspace, then $S' = S^\perp$ is its orthogonal complement.
- The *strict dual cone* of S is

$$S^* \stackrel{\text{def}}{=} \{\mathbf{x} \in \mathbf{R}^n : \ (\forall \mathbf{y} \in S)\, \mathbf{x}^T \mathbf{y} > 0\}.$$

- If $\mathbf{0} \in S$, then $S^* = \emptyset$. Thus if S is a subspace, then $S^* = \emptyset$.

For any set $S \subset \mathbf{R}^n$, both S' and S^* are convex cones.

Self-Duality and Double Duality

Some useful facts:

- $K' = K$, that is, the nonnegative orthant is a self-dual cone.
- $(K^o)' = K$ and $(K^o)^* = K \setminus \mathbf{0}$.
- $(K \setminus \mathbf{0})' = K$ and $(K \setminus \mathbf{0})^* = K^o$.
- $((K^o)^*)^* = K^o$, that is, the open positive orthant is its own strict double dual cone.

A more subtle fact:

Theorem (Double Dual Cone)

If Q is a closed convex cone, then $(Q')' = Q$.

Double Dual Cone Proof

First note that $Q \subset (Q')'$:

$$\mathbf{q} \in Q \implies (\forall \mathbf{z} \in Q')\, \mathbf{q}^T \mathbf{z} \geq 0 \implies \mathbf{q} \in (Q')'.$$

Now suppose toward contradiction that $\mathbf{b} \in (Q')'$ but $\mathbf{b} \notin Q$. Then there is a nonzero vector $\mathbf{x}$ and a constant γ defining a separating hyperplane by the function $f(\mathbf{y}) \stackrel{\text{def}}{=} \mathbf{x}^T \mathbf{y} - \gamma$ where

$$f(\mathbf{b}) < 0, \quad \text{but} \quad (\forall \mathbf{q} \in Q)\, f(\mathbf{q}) > 0.$$

Since Q is a closed cone it contains $\mathbf{0}$, so $f(\mathbf{0}) = -\gamma > 0$, so $\gamma < 0$. Also, fix $\mathbf{q} \in Q$ and let $\lambda \to \infty$ while noting that $\lambda \mathbf{q} \in Q$, so

$$\mathbf{x}^T \mathbf{q} = \lim_{\lambda \to \infty} \left(\mathbf{x}^T \mathbf{q} - \frac{\gamma}{\lambda} \right) = \lim_{\lambda \to \infty} \frac{1}{\lambda} f(\lambda \mathbf{q}) \geq 0.$$

Thus $\mathbf{x} \in Q'$. But then $\mathbf{b} \in (Q')'$ gives the contradiction $f(\mathbf{b}) = \mathbf{x}^T \mathbf{b} - \gamma \geq -\gamma > 0$. $\square$

Deterministic Arbitrages

These are portfolios $\mathbf{x}$, in a market A with spot prices $\mathbf{q}$, that do not depend on the probability mass function on Ω.

- *Type one arbitrage*, or *immediate arbitrage*, leaves a surplus as it is assembled but has nonnegative payoff in any future state:

  IA1: $\mathbf{x}^T\mathbf{q} < 0$.
  IA2: $\mathbf{x}^T A \geq \mathbf{0}$. Equivalently, $\mathbf{x}^T A \in K$.

- *Type two arbitrage*, or *arbitrage opportunity*, costs nothing to assemble and cannot lose value, but has a positive payoff in some future state:

  AO1: $\mathbf{x}^T\mathbf{q} \leq 0$.
  AO2: $\mathbf{x}^T A \geq \mathbf{0}$, and $(\exists j)\, \mathbf{x}^T A(j) > 0$. Equivalently, $\mathbf{x}^T A \in K \setminus \mathbf{0}$.

Arbitrage and Martingales

- An *arbitrage expectation*, which is not deterministic, costs nothing to assemble but has positive expected payoff:

AE1: $\mathbf{x}^T\mathbf{q} \leq 0$

AE2: $\mathbf{x}^T A\mathbf{y} > 0$, where $\mathbf{y}$ is the probability mass function on the states $1, \ldots, n$ in Ω.

- An asset $a(t, \omega)$ is a *martingale* stochastic process if

$$\mathbf{x}^T A\mathbf{y} = \mathrm{E}(a(t)) = \mathrm{E}(a(0)) = a(0) = \mathbf{x}^T\mathbf{q}.$$

Thus no arbitrage expectations can exist in any financial model that assumes assets are martingales.

Profitable Portfolios

Let A be a market matrix.

- A *profitable portfolio* $\mathbf{p}$ is one that has nonnegative payoff in all states: $\mathbf{p}^T A \geq \mathbf{0}$. Equivalently, $\mathbf{p}^T A \in K$.
- The set P of profitable portfolios is a dual cone: $P = (AK)'$.
- A *strictly profitable portfolio* $\mathbf{s}$ also has a positive payoff in some state: $(\exists j)\, \mathbf{s}^T A(j) > 0$. Equivalently, $\mathbf{s}^T A \in K \setminus \mathbf{0}$.
- Equivalently, $(\forall \mathbf{k} \in K^o)\ \mathbf{s}^T A\mathbf{k} > 0$.
- Equivalently, $\mathbf{s}^T A \in (K^o)^*$.
- The set S of strictly profitable portfolios is a strict dual cone: $S = (AK^o)^*$

The Usefulness of Cash

- A matrix of assets without a numeraire might have no profitable portfolios. For example,

$$A = \begin{pmatrix} -1 & 1 \end{pmatrix}$$

  is a one-asset matrix with two states.
- A market with a numeraire, or cash, as its zero row, allows the all-cash portfolio $\mathbf{x} = (1, 0, \ldots, 0)$. Since this satisfies $\mathbf{x}^T A(j) = 1$ for all j, it is both profitable and strictly profitable.
- More generally, if there is any riskless asset such that $(\forall \omega)\, a(1, \omega) = a(1) \neq 0$, then $P \neq \emptyset$ and $S \neq \emptyset$.

Arbitrage Axioms

An immediate arbitrage is an arbitrage opportunity is an arbitrage expectation:

$$\exists\ IA \implies \exists\ AO \implies \exists\ AE. \tag{1}$$

The universal desire for profit creates unlimited demand for arbitrages so it is assumed that if assets are freely traded, then prices will adjust instantly to consume any supply. This may be stated as an axiom:

Axiom 1 *There are no arbitrages.*

The chain of implications for no arbitrages is the reverse of (1):

$$\nexists\ AE \implies \nexists\ AO \implies \nexists\ IA. \tag{2}$$

No Arbitrages

The absence of arbitrages in a market may be stated using orthants:

Definition (No IA)

A market A with prices $\mathbf{q}$ is *immediate arbitrage free* iff any profitable portfolio must have a nonnegative price:

$$\mathbf{x}^T A \in K \implies \mathbf{x}^T \mathbf{q} \geq 0.$$

Definition (No AO)

A market A with prices $\mathbf{q}$ is *arbitrage opportunity free* iff any strictly profitable portfolio must have a positive price:

$$\mathbf{x}^T A \in K \setminus \mathbf{0} \implies \mathbf{x}^T \mathbf{q} > 0.$$

Fundamental Theorem on Asset Pricing

In an arbitrage free market, the price vector $\mathbf{q}$ is a weighted average of the payoffs in the states of Ω:

Theorem (FT from No IA)

Market A with spot prices $\mathbf{q}$ is immediate arbitrage free if and only if there is a vector $\mathbf{k} \in K$ such that

$$\mathbf{q} = A\mathbf{k}.$$

Proof:
($\Longleftarrow$): Suppose that $\mathbf{k} \in K$ solves $\mathbf{q} = A\mathbf{k}$ and let $\mathbf{x}$ be a profitable portfolio. Then

$$\mathbf{x}^T\mathbf{q} = \mathbf{x}^T(A\mathbf{k}) = (\mathbf{x}^T A)\mathbf{k} \geq 0,$$

since $\mathbf{x}^T A \in K$. Thus, by definition, market A with prices $\mathbf{q}$ is immediate arbitrage free.

Proof (continued)

($\implies$): Suppose that market A with prices $\mathbf{q}$ is IA free. Then:

- AK, for nonnegative orthant K, is a closed convex cone.
- $P = (AK)'$, namely the set of all profitable portfolios for A is the dual cone of AK, since

$$\mathbf{x} \in P \iff \mathbf{x}^T A \in K \iff (\forall \mathbf{k} \in K)\mathbf{x}^T A\mathbf{k} \geq 0.$$

- $\mathbf{q} \in P'$, since $A, \mathbf{q}$ is immediate arbitrage free:

$$(\forall \mathbf{x} \in P)\, \mathbf{x}^T \mathbf{q} \geq 0.$$

Hence $\mathbf{q} \in ((AK)')' = AK$, since the double dual of a closed convex cone is itself.
Conclude that there is some $\mathbf{k} \in K$ such that $\mathbf{q} = A\mathbf{k}$. □

Another Proof via Farkas's Lemma

This result from 1902 has the Fundamental Theorem as an immediate consequence:

Theorem (Farkas's Lemma)

Suppose that $A \in \mathbf{R}^{m \times n}$ is a matrix and $\mathbf{b} \in \mathbf{R}^m$ is a vector. Then exactly one of the following must be true:

X: *There exists $\mathbf{x} \in \mathbf{R}^m$ such that $\mathbf{x}^T A \geq \mathbf{0}$ and $\mathbf{x}^T \mathbf{b} < 0$.*

Y: *There exists $\mathbf{y} \in \mathbf{R}^n$ such that $A\mathbf{y} = \mathbf{b}$ and $\mathbf{y} \geq \mathbf{0}$.*

To prove the Fundamental Theorem, let matrix A and spot price vector $\mathbf{b}$ be a discrete financial model. If $A, \mathbf{b}$ is immediate arbitrage free, then Condition X cannot be true. By Condition Y, there is a vector $\mathbf{y} \in K$ such that $\mathbf{b} = A\mathbf{y}$. □

Proof of Farkas's Lemma

First observe that X and Y cannot both hold, for then

$$\mathbf{x}^T A\mathbf{y} = \mathbf{x}^T(A\mathbf{y}) = \mathbf{x}^T\mathbf{b} < 0,$$

while also $\mathbf{x}^T A\mathbf{y} = (\mathbf{x}^T A)\mathbf{y} \geq 0$, since both $(\mathbf{x}^T A) \geq \mathbf{0}$ and $\mathbf{y} \geq \mathbf{0}$.
Evidently, Condition Y holds if and only if

$$\mathbf{b} \in Q = AK \stackrel{\text{def}}{=} \{A\mathbf{k} : \mathbf{k} \in K\},$$

so if Y fails to hold it must be that $\mathbf{b} \notin Q$.
But Q is a nonempty closed convex cone. Thus there exists a nonzero vector $\mathbf{x} \in \mathbf{R}^m$ and a constant $\gamma \in \mathbf{R}$ defining a separating hyperplane function

$$f : \mathbf{R}^m \to \mathbf{R}, \quad f(\mathbf{y}) \stackrel{\text{def}}{=} \mathbf{x}^T\mathbf{y} - \gamma,$$

such that $f(\mathbf{b}) < 0$ but $f(\mathbf{q}) > 0$ for every $\mathbf{q} \in Q$.

Farkas Proof, Part II

Now $\mathbf{0} \in Q$, since $\mathbf{0} \in K$, so $f(\mathbf{0}) = \mathbf{x}^T\mathbf{0} - \gamma = -\gamma > 0$, and therefore $\gamma < 0$. But then

$$f(\mathbf{b}) = \mathbf{x}^T\mathbf{b} - \gamma < 0 \implies \mathbf{x}^T\mathbf{b} < \gamma < 0.$$

On the other hand, $f(\mathbf{q}) > 0$ implies only that $\mathbf{x}^T\mathbf{q} > \gamma$. But since Q is a cone, any $\mathbf{q} \in Q$ and any $\lambda > 0$ result in $\lambda\mathbf{q} \in Q$, so

$$(\forall\lambda > 0)\, f(\lambda\mathbf{q}) = \lambda\mathbf{x}^T\mathbf{q} - \gamma > 0 \implies (\forall\lambda > 0)\, \mathbf{x}^T\mathbf{q} > \gamma/\lambda,$$

and this can only be true for negative γ if $\mathbf{x}^T\mathbf{q} \geq 0$ for all $\mathbf{q} \in Q$. Writing $\mathbf{q} = A\mathbf{k}$ gives

$$(\forall\mathbf{k} \in K)\, \mathbf{x}^T A\mathbf{k} \geq 0,$$

so $\mathbf{x}^T A$ is in the dual cone of K. But K is self-dual, so $\mathbf{x}^T A \geq \mathbf{0}$. Conclude that Condition X holds. □

Hyperplane Separation

Farkas's Lemma, the Double Dual Cone Lemma, and thus the Fundamental Theorem on Asset Pricing all follow from a purely geometric fact about closed convex sets:

Theorem (Hyperplane Separation)

Suppose that $Q \subset \mathbf{R}^m$ *is a nonempty closed convex set and* $\mathbf{b} \in \mathbf{R}^m$ *is a point not in* Q*. Then there exist a nonzero vector* $\mathbf{x} \in \mathbf{R}^m$ *and a constant* $\gamma \in \mathbf{R}$ *defining a hyperplane as the zeros of the function*

$$f(\mathbf{y}) \stackrel{\text{def}}{=} \mathbf{x}^T \mathbf{y} - \gamma,$$

such that $f(\mathbf{b}) < 0$ *but* $f(\mathbf{q}) > 0$ *for every* $\mathbf{q} \in Q$*.*

Proof I: Construct a hyperplane

Define $s : \mathbf{R}^m \to \mathbf{R}$ by $s(\mathbf{y}) \stackrel{\text{def}}{=} \|\mathbf{y} - \mathbf{b}\|^2$, continuous and differentiable with gradient

$$\nabla s(\mathbf{y}) = 2(\mathbf{y} - \mathbf{b}) \in \mathbf{R}^m.$$

It achieves its minimum at a nearest point $\mathbf{q}_0 \in Q$ to $\mathbf{b}$. Put $f(\mathbf{y}) \stackrel{\text{def}}{=} \mathbf{x}^T\mathbf{y} - \gamma$ for

$$\mathbf{x} = \mathbf{q}_0 - \mathbf{b}, \quad \gamma = \frac{\|\mathbf{q}_0\|^2 - \|\mathbf{b}\|^2}{2}.$$

Hyperplane $\{\mathbf{y} : f(\mathbf{y}) = 0\}$ is normal to $\mathbf{q}_0 - \mathbf{b}$ and passes through the midpoint between $\mathbf{b}$ and $\mathbf{q}_0$.
It remains to show that f separates $\mathbf{b}$ from Q.

Proof II: $f(\mathbf{b}) < 0$

Compute $f(\mathbf{b}) = \mathbf{q}_0^T\mathbf{b} - \frac{\|\mathbf{q}_0\|^2 + \|\mathbf{b}\|^2}{2}$. The Cauchy-Schwartz inequality and the arithmetic-geometric mean inequality together imply

$$\mathbf{q}_0^T\mathbf{b} \leq \|\mathbf{q}_0\|\|\mathbf{b}\| \leq \frac{\|\mathbf{q}_0\|^2 + \|\mathbf{b}\|^2}{2},$$

with equality only if $\mathbf{q}_0 = \mathbf{b}$. Conclude that $f(\mathbf{b}) < 0$.

Proof III: $f(\mathbf{q}) > 0$

Take any $\mathbf{q} \in Q$ and suppose toward contradiction that $f(\mathbf{q}) \leq 0$. Then

$$(\mathbf{q}_0 - \mathbf{b})^T \mathbf{q} \;\leq\; \frac{\|\mathbf{q}_0\|^2 - \|\mathbf{b}\|^2}{2},$$

so $\nabla s(\mathbf{q}_0)^T(\mathbf{q} - \mathbf{q}_0) \leq -\|\mathbf{q}_0 - \mathbf{b}\|^2 \; < 0$. Hence there is some small $\lambda \in (0, 1)$ for which

$$s(\mathbf{q}_0 + \lambda[\mathbf{q} - \mathbf{q}_0]) < s(\mathbf{q}_0).$$

But Q is convex, so $\mathbf{q}_0 + \lambda[\mathbf{q} - \mathbf{q}_0] = (1 - \lambda)\mathbf{q}_0 + \lambda\mathbf{q} \in Q$, and this contradicts the extremal property of $\mathbf{q}_0$.
Conclude that $f(\mathbf{q}) > 0$. □

Fundamental Theorem on Asset Pricing II

In an arbitrage opportunity free market, the weight vector is strictly positive:

Theorem (FT from No AO)

Market A with spot prices $\mathbf{q}$ is arbitrage opportunity free if and only if there is a vector $\mathbf{k} \in K^o$ such that

$$\mathbf{q} = A\mathbf{k}.$$

Proof:
($\Longleftarrow$): Suppose that $\mathbf{k} \in K^o$ solves $\mathbf{q} = A\mathbf{k}$ and let $\mathbf{x}$ be a strictly profitable portfolio. Then

$$\mathbf{x}^T\mathbf{q} = \mathbf{x}^T(A\mathbf{k}) = (\mathbf{x}^T A)\mathbf{k} > 0,$$

since $\mathbf{x}^T A \in K \setminus \mathbf{0}$. Thus, by definition, market A with prices $\mathbf{q}$ is arbitrage opportunity free.

Proof (continued)

($\Longrightarrow$): Suppose that market A with prices $\mathbf{q}$ is AO free. Then:

- AK^o, for open positive orthant K^o, is a convex cone.
- $S = (AK^o)^*$, namely the set of all strictly profitable portfolios for A is the strict dual cone of AK^o, since

$$\mathbf{x} \in S \iff \mathbf{x}^T A \in K \setminus \mathbf{0} \iff (\forall \mathbf{k} \in K^o)\mathbf{x}^T A\mathbf{k} > 0.$$

- $\mathbf{q} \in S^*$, since $A, \mathbf{q}$ is arbitrage opportunity free:

$$(\forall \mathbf{x} \in S)\, \mathbf{x}^T \mathbf{q} > 0.$$

Hence $\mathbf{q} \in ((AK^o)^*)^* = AK^o$.
Conclude that there is some $\mathbf{k} \in K^o$ such that $\mathbf{q} = A\mathbf{k}$. $\square$

Strict Duals and Weak Separation

The last step in the FT from No AO Theorem relies on:

Theorem (Strict Double Dual Cone)

If Q is an open convex cone in $\mathbf{R}^n$, then $(Q^*)^* = Q$. □

That in turn follows from:

Theorem (Weak Hyperplane Separation)

If $C, D \subset \mathbf{R}^n$ are disjoint convex sets, then there is a nonzero vector $\mathbf{x} \in \mathbf{R}^n$ and a constant $\gamma \in \mathbf{R}$ defining a hyperplane as the zeros of $f(\mathbf{y}) \stackrel{\text{def}}{=} \mathbf{x}^T \mathbf{y} - \gamma$, satisfying

$$(\forall \mathbf{c} \in C)\, f(\mathbf{c}) \le 0 \quad \textit{and} \quad (\forall \mathbf{d} \in D)\, f(\mathbf{d}) \ge 0 \quad \square$$

Application to Derivative Pricing

Suppose that payoff matrix A with spot price vector $\mathbf{q}$ corresponds to an arbitrage free market.

- The vector $\mathbf{k}$, which is nonzero if $\mathbf{q} \neq \mathbf{0}$, is called a *risk neutral probability mass function*, when normalized to have unit sum.
- Any derivative asset with future payoff vector $\mathbf{d}$ has a risk neutral spot price $\mathbf{d}^T\mathbf{k}$.

Derivative assets are often *contingent claims*.

Contingent Claims

These are contracts to pay or collect some amount depending on the price of *underlying assets*. Examples are:

Call Option to buy an asset for a stated *strike price* at or before a stated *expiry time*.

Put Option to sell an asset for a strike price at or before expiry.

Swap Exchange one sequence of payments for another with different terms.

Future Agreement to buy an asset for a stated strike price at a future date.

Hedges

- Financial institutions that sell contingent claims seek to *hedge*, or replicate them, with a portfolio of other assets that equals or exceeds the cost of the contingent claim in all modeled states Ω.
- If $\mathbf{c}$ is the cost vector of the contingent claim over Ω, namely the liability of the financial institution that sold it, then a hedge portfolio $\mathbf{h}$ over a market A must satisfy

$$\mathbf{h}^T A \geq \mathbf{c}.$$

- At spot prices $\mathbf{q}$, the cost of the hedge portfolio is $\mathbf{h}^T\mathbf{q}$.

Complete Markets

- Market A is *complete* if any contingent claim can be hedged, namely if the row space of A is all of $\mathbf{R}^n$.
- Since the row space is dependent on the discrete financial model, this cannot be guaranteed without additional assumptions.
- *Binomial models*, where $n = 2$ and $m = 1$ so that A is a 2×2 matrix

$$A = \begin{pmatrix} a_0(1,1) & a_0(1,2) \\ a_1(1,1) & a_1(1,2) \end{pmatrix}, \qquad \begin{matrix} a_0(1,1) = a_0(1,2), \\ a_1(1,1) \neq a_1(1,2) \end{matrix}$$

with a numeraire (or other riskless asset) $a_0 \neq \mathbf{0}$, and a single risky asset a_1, are always complete, so there is a unique hedge for any contingent claim on the underlying asset a_1.

Incomplete Markets

In the general case, when the market is *incomplete*, the seller of a contingent claim $\mathbf{c}$ constructs a hedge portfolio $\mathbf{h}$ by solving

$$\textit{Minimize } \mathbf{h}^T\mathbf{q} \textit{ subject to } \mathbf{h}^T A \geq \mathbf{c}.$$

Conversely, the buyer of the contingent claim $\mathbf{c}$ compares its price to the alternative portfolio $\mathbf{k}$ solving

$$\textit{Maximize } \mathbf{k}^T\mathbf{q} \textit{ subject to } \mathbf{k}^T A \leq \mathbf{c}.$$

These are both convex optimization problems solvable by *linear programming*.

Bid-Ask Spread

If market A with prices $\mathbf{q}$ is arbitrage free, then any profitable portfolio $\mathbf{x}$ must have a nonnegative price:

$$\mathbf{x}^T A \geq \mathbf{0} \implies \mathbf{x}^T \mathbf{q} \geq 0.$$

Let $\mathbf{x} = \mathbf{h} - \mathbf{k}$ be the difference of the portfolios solving the hedge optimization problems. Then

$$\mathbf{x}^T A = \mathbf{h}^T A - \mathbf{k}^T A \geq \mathbf{c} - \mathbf{c} = \mathbf{0},$$

so we may conclude that $\mathbf{h}^T \mathbf{q} \geq \mathbf{k}^T \mathbf{q}$. The nonempty interval

$$[\mathbf{k}^T \mathbf{q}, \mathbf{h}^T \mathbf{q}]$$

is the *no-arbitrage bid-ask spread* for the contingent claim $\mathbf{c}$.

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