Putnam Practice, 9 Nov 2012

Matrix Algebra Problems

I. DISCUSSION

Determinant:

- If A, B are $n \times n$ matrices, then $\det AB = \det A \det B$. Also $\det I = 1$, so $\det A^{-1} = 1/\det A$.

- Expansion by minors: if $A = (a_{ij})$, $i, j = 1, \ldots, n$, then for any $j \in \{1, \ldots, n\}$,

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij},$$

  and likewise for any $i \in \{1, \ldots, n\}$,

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij},$$

  where A_{ij}, called the *ij minor*, is the $(n-1) \times (n-1)$ submatrix of A obtained by deleting row i and column j.

- If $\{\lambda_i : i = 1, \ldots, n\}$ is the set of eigenvalues of A, with multiplicities, then $\det A = \prod_{i=1}^{n} \lambda_i$.

Trace: for $A = (a_{ij})$, $i, j = 1, \ldots, n$, define $\operatorname{tr} A = \sum_{i=1}^{n} a_{ii}$.

- If A, B are $n \times n$ matrices, then $\operatorname{tr} A + B = \operatorname{tr} A + \operatorname{tr} B$. Also $\operatorname{tr} I = n$ and $\operatorname{tr} 0 = 0$.

- Commutators: $\operatorname{tr} AB = \operatorname{tr} BA$, so $\operatorname{tr} AB - BA = 0$.

- If $\{\lambda_i : i = 1, \ldots, n\}$ is the set of eigenvalues of A, with multiplicities, then $\operatorname{tr} A = \sum_{i=1}^{n} \lambda_i$.

- If A, S are $n \times n$ matrices and S is invertible, then $\operatorname{tr} S^{-1} A S = \operatorname{tr} A$.

- If $A \neq 0$, then $\operatorname{tr} AA^T = \sum_{i,j} a_{ij}^2 > 0$.

Orthogonal matrix: an $n \times n$ matrix satisfying $AA^T = I$.

- If A is an orthogonal matrix, then so is A^T.

- The rows of an orthogonal $n \times n$ matrix form an orthonormal basis for $\mathbf{R}^n$.

- The columns of an orthogonal $n \times n$ matrix, which are the rows of its transpose, also form an orthonormal basis for $\mathbf{R}^n$.

Upper triangular matrix: an $n \times n$ matrix $A = (a_{ij})$ satisfying $i > j \implies a_{ij} = 0$. Lower triangular means $i < j \implies a_{ij} = 0$.

- If A and B are upper triangular, then so is AB. If lower triangular, then AB is lower triangular.
- If $A = (a_{ij})$ is upper triangular or lower triangular, then $\det A = \prod_i a_{ii}$.

Functions of a matrix:

- Polynomial $p(z) = c_0 + c_1 z + \cdots + c_d z^d$ of degree d defines $p(A) = c_0 I + c_1 A + \cdots + c_d A^d$.
- Characteristic polynomial: let $p(z) = \det(A - zI)$ for $n \times n$ matrix A and $n \times n$ identity matrix I. This is a polynomial of degree n in the complex variable z whose roots are the eigenvalues of A. Then $p(0) = \det A$, and the coefficient of z^{n-1} is $\operatorname{tr} A$.
- Cayley-Hamilton theorem: if $p(z) = \det(A - zI)$ is the characteristic polynomial for $n \times n$ matrix A, then $p(A) = 0$.
- Taylor series $f(z) = \sum_{k=0}^{\infty} c_k z^k$ may be evaluated on a matrix A as $f(A) = c_0 I + \sum_{k=1}^{\infty} c_k A^k$, whenever $\|A\|$ is less than the radius of convergence of f. Here $\|A\|$ is any matrix norm.
- $\exp A = I + A + \frac{1}{2}A^2 + \cdots + \frac{1}{n!}A^n + \cdots$ converges for any matrix A.

Diagonalization: an $n \times n$ matrix $A = (a_{ij})$ is diagonal if $i \neq j \implies a_{ij} = 0$.

- A is a diagonal matrices iff A is both upper triangular and lower triangular. Hence, if A and B are diagonal matrices, then so is AB
- If $A = (a_{ii})$ is diagonal, and f is any function, then $f(A) = (f(a_{ii}))$ is the diagonal matrix obtained by applying f to the elements a_{ii} of A.
- Spectral theorem: if $A = A^T$ is a symmetric matrix, then there is an orthogonal matrix S and a diagonal matrix D such that $A = SDS^T$. The columns of S are eigenvectors for A and the diagonal elements of D are eigenvalues for A.
- Simultaneous diagonalization: If A and B are symmetric matrices and $AB = BA$, then there is an orthogonal matrix S such that $S^T AS$ is diagonal and $S^T BS$ is diagonal.

II. PROBLEMS

Arranged easier to harder, roughly speaking.

1991,A–2 Let $\mathbf{A}$ and $\mathbf{B}$ be different $n \times n$ matrices with real entries. If $\mathbf{A}^3 = \mathbf{B}^3$ and $\mathbf{A}^2\mathbf{B} = \mathbf{B}^2\mathbf{A}$, can $\mathbf{A}^2 + \mathbf{B}^2$ be invertible?

1990,B–3 Let S be a set of 2×2 integer matrices whose entries a_{ij} (1) are all squares of integers and, (2) satisfy $a_{ij} \leq 200$. Show that if S has more than 50387 ($= 15^4 - 15^2 - 15 + 2$) elements, then it has two elements that commute.

1986, A–4 A *transversal* of an $n \times n$ matrix A consists of n entries of A, no two in the same row or column. Let $f(n)$ be the number of $n \times n$ matrices A satisfying the following two conditions:

(a) Each entry $\alpha_{i,j}$ of A is in the set $\{-1, 0, 1\}$.

(b) The sum of the n entries of a transversal is the same for all transversals of A.

An example of such a matrix A is

$$A = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Determine with proof a formula for $f(n)$ of the form

$$f(n) = a_1 b_1^n + a_2 b_2^n + a_3 b_3^n + a_4,$$

where the a_i's and b_i's are rational numbers.

1994,A–4 Let A and B be 2×2 matrices with integer entries such that $A, A+B, A+2B, A+3B$, and $A+4B$ are all invertible matrices whose inverses have integer entries. Show that $A+5B$ is invertible and that its inverse has integer entries.

1994,B–4 For $n \geq 1$, let d_n be the greatest common divisor of the entries of $A^n - I$, where

$$A = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Show that $\lim_{n\to\infty} d_n = \infty$.

1990,A–5 If $\mathbf{A}$ and $\mathbf{B}$ are square matrices of the same size such that $\mathbf{ABAB} = \mathbf{0}$, does it follow that $\mathbf{BABA} = \mathbf{0}$?

1992,B–5 Let D_n denote the value of the $(n-1) \times (n-1)$ determinant

$$\begin{bmatrix} 3 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 4 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 5 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 6 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & n+1 \end{bmatrix}.$$

Is the set $\left\{\frac{D_n}{n!}\right\}_{n\geq 2}$ bounded?

1992,B–6 Let $\mathcal{M}$ be a set of real $n \times n$ matrices such that

(i) $I \in \mathcal{M}$, where I is the $n \times n$ identity matrix;

(ii) if $A \in \mathcal{M}$ and $B \in \mathcal{M}$, then either $AB \in \mathcal{M}$ or $-AB \in \mathcal{M}$, but not both;

(iii) if $A \in \mathcal{M}$ and $B \in \mathcal{M}$, then either $AB = BA$ or $AB = -BA$;

(iv) if $A \in \mathcal{M}$ and $A \neq I$, there is at least one $B \in \mathcal{M}$ such that $AB = -BA$.

Prove that $\mathcal{M}$ contains at most n^2 matrices.

1987, B–5 Let O_n be the n-dimensional vector $(0, 0, \cdots, 0)$. Let M be a $2n \times n$ matrix of complex numbers such that whenever $(z_1, z_2, \ldots, z_{2n})M = O_n$, with complex z_i, not all zero, then at least one of the z_i is not real. Prove that for arbitrary real numbers $r_1, r_2, \ldots, r_{2n}$, there are complex numbers $w_1, w_2, \ldots, w_n$ such that

$$\mathrm{re}\left[M\begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}\right] = \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix}.$$

(Note: if C is a matrix of complex numbers, $\mathrm{re}(C)$ is the matrix whose entries are the real parts of the entries of C.)

1988, B–5 For positive integers n, let M_n be the $2n+1$ by $2n+1$ skew-symmetric matrix for which each entry in the first n subdiagonals below the main diagonal is 1 and each of the remaining entries below the main diagonal is -1. Find, with proof, the rank of M_n. (According to one definition, the rank of a matrix is the largest k such that there is a $k \times k$ submatrix with nonzero determinant.)

1988, A–6 If a linear transformation A on an n-dimensional vector space has $n+1$ eigenvectors such that any n of them are linearly independent, does it follow that A is a scalar multiple of the identity? Prove your answer.

1986, B–6 Suppose A, B, C, D are $n \times n$ matrices with entries in a field F, satisfying the conditions that AB^T and CD^T are symmetric and $AD^T - BC^T = I$. Here I is the $n \times n$ identity matrix, and if M is an $n \times n$ matrix, M^T is its transpose. Prove that $A^T D - C^T B = I$.

1985, B–6 Let G be a finite set of real $n \times n$ matrices $\{M_i\}$, $1 \leq i \leq r$, which form a group under matrix multiplication. Suppose that $\sum_{i=1}^r \operatorname{tr}(M_i) = 0$, where $\operatorname{tr}(A)$ denotes the trace of the matrix A. Prove that $\sum_{i=1}^r M_i$ is the $n \times n$ zero matrix.