In fact, for any $p \geq 1$, there is a so-called p-norm:

$$\|\mathbf{x}\|_p \stackrel{\text{def}}{=} \left(|x(1)|^p + \cdots + |x(N)|^p\right)^{\frac{1}{p}}. \tag{2.12}$$

The formula extends to the limit $p \to \infty$:

$$\|\mathbf{x}\|_\infty \stackrel{\text{def}}{=} \max\{|x(1)|, \ldots, |x(N)|\}. \tag{2.13}$$

But the choice of norm does not matter too much in a finite dimensional vector space. For example, the norms $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_\infty$ defined on $\mathbf{E}^N$ in terms of the standard basis vectors $\{\mathbf{e}_n\}$ satisfy the following system of inequalities:

$$\|\mathbf{x}\|_1 \leq \sqrt{N}\,\|\mathbf{x}\|_2 \qquad \|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1 \tag{2.14}$$

$$\|\mathbf{x}\|_1 \leq N\,\|\mathbf{x}\|_\infty \qquad \|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_1 \tag{2.15}$$

$$\|\mathbf{x}\|_2 \leq \sqrt{N}\|\mathbf{x}\|_\infty \qquad \|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_2 \tag{2.16}$$

In fact, all norms on an N-dimensional vector space satisfy similar inequalities:

Theorem 2.3 *Any two norms $\|\cdot\|_p$ and $\|\cdot\|_q$ for an N-dimensional vector space $\mathbf{X}$ are comparable. That is, there are positive numbers A, B, C, D such that for all $\mathbf{x} \in \mathbf{X}$,*

$$A\|\mathbf{x}\|_p \leq \|\mathbf{x}\|_q \leq B\|\mathbf{x}\|_p, \quad \textit{and} \quad C\|\mathbf{x}\|_q \leq \|\mathbf{x}\|_p \leq D\|\mathbf{x}\|_q.$$

Proof: Fixing a basis $\{\mathbf{b}_n : n = 1, \ldots, N\} \subset \mathbf{X}$, we will show that the norm $\|\cdot\|_\infty$ defined by Equation 2.13 is comparable to all others. Let $\|\cdot\|$ be any other norm on $\mathbf{X}$. Then by its sublinearity,

$$\|\mathbf{x}\| \leq \sum_{n=1}^{N} \|x(n)\mathbf{b}_n\| = \sum_{n=1}^{N} |x(n)|\,\|\mathbf{b}_n\| \leq \left(\max_{1\leq n\leq N} |x(n)|\right)\left(\sum_{n=1}^{N} \|\mathbf{b}_n\|\right). \tag{2.17}$$

So $\|\mathbf{x}\| \leq B\|\mathbf{x}\|_\infty$ for $B = \|\mathbf{b}_1\| + \cdots + \|\mathbf{b}_N\| > 0$. We know that $B > 0$ because $\|\cdot\|$ is nondegenerate and basis vectors are nonzero.

For the other inequality, suppose toward contradiction that there is no $A > 0$ such that $A\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|$ for every $\mathbf{x}$. Then there must be an infinite sequence of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots\} \subset \mathbf{X}$ with the property that $\|\mathbf{x}_k\|_\infty = 1$ for all $k = 1, 2, \ldots$, but $\|\mathbf{x}_k\| \to 0$ as $k \to \infty$. We may suppose without loss of generality that the expansion coefficients $(x_k(1), \ldots, x_k(N))$ of $\mathbf{x}_k = \sum_{n=1}^{N} x_k(n)\mathbf{b}_n$, $k = 1, 2, \ldots$ are real numbers and define a point in $\mathbf{R}^N$, since N complex scalars are the same as $2N$ real scalars. These real coordinates are at most 1 in absolute value, so $\{\mathbf{x}_k : k = 1, 2, \ldots\}$ gives an infinite sequence of points confined to the unit hypercube $[-1, 1]^N \subset \mathbf{R}^N$. Cutting this hypercube in half along each axis gives 2^N subcubes, at least one of which must contain infinitely many[1] of the points $\{\mathbf{x}_k\}$. For notational convenience, let us suppose without loss of generality that it is the subcube $[0, 1]^N$ of nonnegative

[1] How can we be sure that such a packed subcube exists? Why does this proof fail in infinite dimensional space?

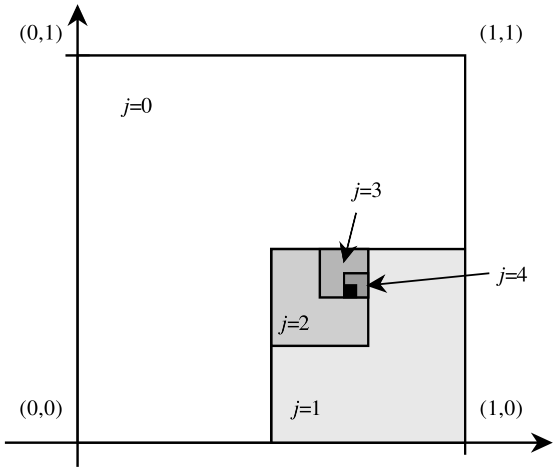

Figure 2.1: Nested subcubes at stages $j = 0, 1, 2, 3, 4$, to illustrate the two-dimensional case of Theorem 2.3.

coordinates. Call this the *stage* 0 *subcube*, and let $\mathbf{y}_0 \in \mathbf{R}^N$ be one of its points, say $\mathbf{x}_{k(0)}$.

We next define the *stage* j *subcube* for $j > 0$ by cutting the stage $j-1$ subcube in half along each axis and picking one of its 2^N subcubes, making sure that the chosen one contains infinitely many of the points $\{\mathbf{x}_k\}$. An example with $N = 2$ and $j = 0, 1, 2, 3, 4$ is depicted in Figure 2.1. We define $\mathbf{y}_j$ to be one of those points, say $\mathbf{x}_{k(j)}$. We can make the choice such that $k(0) < k(1) < k(2) < \cdots$. Now notice that the n^{th} coordinates of the points $\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2, \ldots$ form a Cauchy sequence, since the first j binary digits of $y_j(n), y_{j+1}(n), \ldots$ are all the same. By the completeness of the reals, Theorem 1.9, there is a unique real number $y(n) = \lim_{j\to\infty} y_j(n)$ for $n = 1, 2 \ldots, N$. Putting $\mathbf{x} \stackrel{\text{def}}{=} \sum_{n=1}^N y(n)\mathbf{b}_n \in \mathbf{X}$, we see that $\|\mathbf{x}\|_\infty = 1$ since $\|\mathbf{x}_k\|_\infty = 1$ for all k, including $k = k(j)$, and

$$\big|\, \|\mathbf{x}_{k(j)}\|_\infty - \|\mathbf{x}\|_\infty \,\big| \le \|\mathbf{x}_{k(j)} - \mathbf{x}\|_\infty = \max_{1\le n\le N} |y_j(n) - y(n)| \to 0,$$

as $j \to \infty$. Therefore $\mathbf{x} \ne \mathbf{0}$. On the other hand, we have the contradiction $\|\mathbf{x}\| = 0$, since $\|\mathbf{x}_{k(j)}\| \to 0$ as $j \to \infty$ by assumption, and

$$\big|\, \|\mathbf{x}_{k(j)}\| - \|\mathbf{x}\| \,\big| \le \|\mathbf{x}_{k(j)} - \mathbf{x}\| \le B\|\mathbf{x}_{k(j)} - \mathbf{x}\|_\infty = B \max_{1\le n\le N} |y_j(n) - y(n)| \to 0,$$

as $j \to \infty$, using Inequality 2.17. Hence $\|\cdot\|$ cannot be arbitrarily small relative to $\|\cdot\|_\infty$: there must be some $A > 0$ such that $A\|\mathbf{x}\|_\infty \le \|\mathbf{x}\|$ for every $\mathbf{x}$.

To finish the proof, given any two norms $\|\cdot\|_p$ and $\|\cdot\|_q$ we find $A_p, B_p, A_q, B_q > 0$ such that

$$A_p\|\mathbf{x}\|_\infty \le \|\mathbf{x}\|_p \le B_p\|\mathbf{x}\|_\infty, \quad \text{and} \quad A_q\|\mathbf{x}\|_\infty \le \|\mathbf{x}\|_q \le B_q\|\mathbf{x}\|_\infty.$$

But then $A = A_q/B_p$, $B = B_q/A_p$, $C = A_p/B_q$, and $D = B_p/A_q$ satisfy the theorem. □

If $\mathbf{X}$ is an N-dimensional vector space and $\mathbf{B} = \{\mathbf{b}_n : n = 1, \ldots, N\} \subset \mathbf{X}$ is a basis, then every $\mathbf{x} = \sum_n c(n)\mathbf{b}_n \in \mathbf{X}$ corresponds to one and only one $\mathbf{c} = (c(1), \ldots, c(N)) \in \mathbf{E}^N$:

$$\mathbf{X} \stackrel{\mathbf{B}}{\longleftrightarrow} \mathbf{E}^N. \tag{2.18}$$

Having the freedom to choose $\mathbf{B}$ can sometimes simplify calculations in $\mathbf{X}$, which in practice must be done by mapping vectors $\mathbf{x}$ to points in $\mathbf{E}^N$.

Infinite dimensions

We will only consider a simple kind of infinite dimensional vector space $\mathbf{X}$, one which has a norm and a basis $\mathbf{B} = \{\mathbf{b}_n : n = 1, 2, \ldots\} \subset \mathbf{X}$ satisfying the following:

Schauder Basis Axioms

Linear independence: Any finite subset of the vectors in $\mathbf{B}$ is linearly independent.

Completeness: Each $\mathbf{x} \in \mathbf{X}$ has a $\mathbf{B}$-expansion $\mathbf{x} = \sum_n c(n)\mathbf{b}_n$.

Unique representation: If $\mathbf{x} = \sum_n c(n)\mathbf{b}_n$ and $\mathbf{x} = \sum_n c'(n)\mathbf{b}_n$, then $c(n) = c'(n)$ for all n.

Completeness in an infinite dimensional vector space $\mathbf{X}$ is interpreted to mean that for every $\mathbf{x} \in \mathbf{X}$, there is a sequence $\{c(n) : n = 1, 2, \ldots\}$ of expansion coefficients such that

$$\lim_{N\to\infty} \left\| \mathbf{x} - \sum_{n=1}^{N} c(n)\mathbf{b}_n \right\| = 0.$$

Alternatively, a set of vectors $\mathbf{B} \subset \mathbf{X}$ is called *dense* in $\mathbf{X}$ if, for every fixed vector $\mathbf{x} \in \mathbf{X}$ and $\epsilon > 0$, there is a finite linear combination $\mathbf{x}_\epsilon$ of vectors in $\mathbf{B}$ satisfying $\|\mathbf{x} - \mathbf{x}_\epsilon\| < \epsilon$. It is clear that a Schauder basis $\mathbf{B}$ is dense, but a set which is not a basis may also be dense.

We may extend the definition of linear span to arbitrary subsets of arbitrary vector spaces $\mathbf{X}$: for any $\mathbf{B} \subset \mathbf{X}$,

$$\operatorname{span}\mathbf{B} \stackrel{\text{def}}{=} \left\{ \sum_{n=1}^{N} a(n)\mathbf{b}_n : N \geq 0; \mathbf{b}_n \in \mathbf{B}, a(n) \text{ scalar, all } 1 \leq n \leq N. \right\}. \tag{2.19}$$

It is easy to verify from the vector space axioms that $\operatorname{span}\mathbf{B} \subset \mathbf{X}$. This definition agrees with Equation 2.5 for finite sets $\mathbf{B}$, and yields all *finite* linear combinations of elements of $\mathbf{B}$ in the general case. Notice that $\operatorname{span}\mathbf{X} = \mathbf{X}$.

Different norms for an infinite dimensional $\mathbf{X}$ need not be comparable, so a fixed norm must be chosen before the Schauder bases axioms can be verified. Also, the ordering of Schauder basis vectors is important in the infinite dimensional case since the same vectors taken in another order may lose the completeness property. If a