LECTURE 10: THE WHITNEY EMBEDDING THEOREM

1. The Whitney embedding theorems

Let M be a smooth manifold of dimension m. A natural question is: which manifolds can be embedded into $\mathbb{R}^N$ as smooth submanifolds?

Theorem 1.1 (The Whitney embedding theorem: easiest version)**.** *Any compact manifold M can be embedded into $\mathbb{R}^N$ for sufficiently large N.*

Proof. Let $\{\varphi_i, U_i, V_i\}_{1\le i\le k}$ be a finite set of coordinate charts on M so that $\mathcal{U} = \{U_i \mid 1 \le i \le k\}$ is an open cover of M. Let $\{\rho_i \mid 1 \le i \le k\}$ be a partition of unity subordinate to $\mathcal{U}$. Let $\tilde{\varphi}_i(p) = \rho_i(p)\varphi_i(p)$, extended to 0 outside U. Define

$$\Phi : M \to \mathbb{R}^{k(m+1)}, \quad p \mapsto (\tilde{\varphi}_1(p), \cdots, \tilde{\varphi}_k(p), \rho_1(p), \cdots, \rho_k(p)).$$

We claim that Φ is an injective map. In fact, suppose $\Phi(p_1) = \Phi(p_2)$. Take an index i so that $\rho_i(p_1) = \rho_i(p_2) \neq 0$. Then $p_1, p_2 \in \mathrm{supp}(\rho_i) \subset U_i$. It follows that $\varphi_i(p_1) = \varphi_i(p_2)$. So we must have $p_1 = p_2$ since φ_i is bijective.

Next let's prove that Φ is an immersion. In fact, for any $X_p \in T_pM$,

$$\begin{aligned} d\Phi_p(X_p) = (&X_p(\rho_1)\varphi_1(p) + \rho_1(p)(d\varphi_1)_p(X_p), \cdots, \\ &X_p(\rho_k)\varphi_k(p) + \rho_k(p)(d\varphi_k)_p(X_p), X_p(\rho_1), \cdots, X_p(\rho_k)). \end{aligned}$$

It follows that if $d\Phi_p(X_p) = 0$, then $X_p(\rho_i) = 0$ for all i, and thus $\rho_i(p)(d\varphi_i)_p(X_p) = 0$ for all i. Pick an index i so that $\rho_i(p) \neq 0$. We see $(d\varphi_i)_p(X_p) = 0$. Since φ_i is a diffeomorphism, we conclude that $X_p = 0$. So $d\Phi$ is injective.

Since Φ is an injective immersion, and M is compact, Φ must be an embedding. $\square$

Theorem 1.2 (The Whitney embedding theorem: median version)**.** *Any compact manifold M of dimension m can be embedded into $\mathbb{R}^{2m+1}$ and immersed into $\mathbb{R}^{2m}$.*

Proof. Suppose we already have an embedding $\Phi : M \to \mathbb{R}^N$ with $N > 2m + 1$. We will show that we can produce an embedding of M in $\mathbb{R}^{N-1}$.

To do so, for any $[v] \in \mathbb{RP}^{N-1}$, we let

$$P_{[v]} = \{u \in \mathbb{R}^N \mid u \cdot v = 0\} \simeq \mathbb{R}^{N-1}$$

be the orthogonal complement of $[v]$ in $\mathbb{R}^N$. Let $\Psi_{[v]} : \mathbb{R}^N \to P_{[v]}$ be the orthogonal projection to this hyperplane. We claim that the set of $[v]$'s for which $\Phi_{[v]} = \Psi_{[v]} \circ \Phi$ is not an embedding has measure zero in $\mathbb{RP}^{N-1}$, hence it is possible to choose $[v]$ so that $\Phi_{[v]}$ is an embedding. Note that if $\Phi_{[v]}$ fails to be an embedding, we must have either $\Phi_{[v]}$ is not injective, or $\Phi_{[v]}$ is not an immersion.

First let's consider $[v]$'s so that $\Phi_{[v]}$ is not injective. Then one can find $p_1 \neq p_2$ so that $\Phi_{[v]}(p_1) = \Phi_{[v]}(p_2)$, i.e. $0 \neq \Phi(p_1) - \Phi(p_2)$ lies in the line $[v]$. In other words, $[v] = [\Phi(p_1) - \Phi(p_2)]$. So $[v]$ must lie in the image of the map

$$\alpha : (M \times M) \setminus \Delta_M \to \mathbb{RP}^{N-1}, \quad (p_1, p_2) \mapsto [\Phi(p_1) - \Phi(p_2)],$$

where $\Delta_M = \{(p,p) \mid p \in M\}$ is the "diagonal" in $M \times M$. Since $(M \times M) \setminus \Delta_M$ is of dimension $2m < N-1$, Sard's theorem implies that the image of α is of measure zero in $\mathbb{RP}^{N-1}$.

Next let's consider $[v]$'s so that $\Phi_{[v]}$ is not an immersion. Then there exists some $p \in M$ and some $0 \neq X_p \in T_pM$ so that $(d\Phi_{[v]})_p(X_p) = 0$, i.e. $(d\Psi_{[v]})_{\Phi(p)}(d\Phi)_p(X_p) = 0$. Since $\Psi_{[v]}$ is linear, $d\Psi_{[v]} = \Psi_{[v]}$. It follows that $0 \neq (d\Phi)_p(X_p)$ is in $[v]$, i.e. $[v] = [(d\Phi)_p(X_p)]$. In other words, $[v]$ lies in the image of

$$\beta : TM \setminus \{0\} \to \mathbb{RP}^{N-1}, \quad (p, X_p) \mapsto [(d\Phi)_p(X_p)],$$

where $TM \setminus \{0\} = \{(p, X_p) \mid X_p \neq 0\}$ is an open submanifold of TM. Again since TM has dimension $2m < N-1$, by Sard's theorem, the image of β is of measure zero in $\mathbb{RP}^{N-1}$.

To see that M can be immersed into $\mathbb{R}^{2m}$, we first embed M into $\mathbb{R}^{2m+1}$, then repeat the last step, with the modification that we choose $X_p \in T_pM$ so that $|X_p| = 1$. $\square$

Theorem 1.3 (The Whitney embedding theorem: regular form)**.** *Any smooth manifold of dimension m can be immersed into $\mathbb{R}^{2m}$ and embedded into $\mathbb{R}^{2m+1}$.*

Proof. c.f. Lee's book. $\square$

Theorem 1.4 (The Whitney embedding theorem: strongest version)**.** *Any smooth manifold of dimension m can be immersed into $\mathbb{R}^{2m-1}$ and embedded into $\mathbb{R}^{2m}$.*

Remark. Well, there exists ever stronger results! e.g.

- Any compact orientable surface embeds to $\mathbb{R}^3$.
- For $m \neq 2^k$, any smooth m-manifold embeds to $\mathbb{R}^{2m-1}$. (But if $m = 2^k$, $\mathbb{RP}^m$ cannot be embedded into $\mathbb{R}^{2m-1}$).
- Any smooth m manifold can be immersed into $\mathbb{R}^{2m-a(m)}$, where $a(m)$ is the number of 1's that appear in the binary expansion of m.

2. Examples

Example. (**Embedding of $\mathbb{T}^2 = S^1 \times S^1$ into $\mathbb{R}^3$:**)

Recall that

$$\mathbb{T}^2 = S^1 \times S^1 = \{(x^1, x^2, x^3, x^4) \mid (x^1)^2 + (x^2)^2 = 1, (x^3)^2 + (x^4)^2 = 1\}.$$

Then

$$f : \mathbb{T}^2 \to \mathbb{R}^3, \quad (x^1, x^2, x^3, x^4) \mapsto (x^1(2+x^3), x^2(2+x^3), x^4).$$

is an embedding from T^2 to $\mathbb{R}^3$. In fact, one can decompose f as $f = g \circ \iota$, where $\iota : \mathbb{T}^2 \hookrightarrow \mathbb{R}^4$ is the standard embedding, and

$$g : \mathbb{R}^4 \to \mathbb{R}^3, \quad (x^1, x^2, x^3, x^4) \mapsto (x^1(2+x^3), x^2(2+x^3), x^4).$$

So $\mathrm{Im}(d\iota)_p$ at $p = (x^1, x^2, x^3, x^4)$ is spanned by $X_p = \langle x^2, -x^1, 0, 0\rangle$ and $Y_p = \langle 0, 0, x^4, -x^3\rangle$, and

$$dg = \begin{pmatrix} 2+x^3 & 0 & x^1 & 0 \\ 0 & 2+x^3 & x^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now it is easy to check that at any $p = (x^1, x^2, x^3, x^4) \in \mathbb{T}^2$,

$$dg_p(X_p) = \langle x^2(2+x^3), -x^1(2+x^3), 0\rangle$$
$$dg_p(Y_p) = \langle x^1x^4, x^2x^4, -x^3\rangle$$

are linearly independent.

Example. (**Embedding of $\mathbb{RP}^2$ into $\mathbb{R}^4$**:)

Consider the map

$$f : S^2/\sim = \mathbb{RP}^2 \to \mathbb{R}^4, \quad [(x^1, x^2, x^3)] \mapsto ((x^1)^2 - (x^2)^2, x^1x^2, x^1x^3, x^2x^3).$$

Obviously this is well-defined. It is not hard to check that f is injective.

To prove that f is immersion is a little it more complicated. Since the map

$$(x^1, x^2, x^3) \mapsto [(x^1, x^2, x^3)]$$

is a local diffeomorphism from S^2 to $\mathbb{RP}^2$, it is enough to check that the map

$$g : S^2 \to \mathbb{R}^4, \quad (x^1, x^2, x^3) \mapsto ((x^1)^2 - (x^2)^2, x^1x^2, x^1x^3, x^2x^3).$$

is an immersion. To prove this, we note that from the decomposition

$$g : S^2 \xhookrightarrow{\iota} \mathbb{R}^3 \xrightarrow{h} \mathbb{R}^4$$

we have $\mathrm{Im}(dg) = dh(\mathrm{Im}(d\iota))$, where

$$h : \mathbb{R}^3 \to \mathbb{R}^4, \quad (x^1, x^2, x^3) \mapsto ((x^1)^2 - (x^2)^2, x^1x^2, x^1x^3, x^2x^3).$$

Obviously $\mathrm{Im}(d\iota)$ is spanned by any two of the vectors $X_p = \langle x^2, -x^1, 0\rangle$, $Y_p = \langle x^3, 0, -x^1\rangle$ and $Z_p = \langle 0, x^3, -x^2\rangle$ at the point $p = (x^1, x^2, x^3) \in S^2$, and

$$dh = \begin{pmatrix} 2x^1 & -2x^2 & 0 \\ x^2 & x^1 & 0 \\ x^3 & 0 & x^1 \\ 0 & x^3 & x^2 \end{pmatrix}.$$

It follows that

$$dh(X_p) = \langle 4x^1x^2, (x^2)^2 - (x^1)^2, x^2x^3, -x^1x^3\rangle,$$
$$dh(Y_p) = \langle 2x^1x^3, x^2x^3, (x^3)^2 - (x^1)^2, -x^1x^2\rangle,$$
$$dh(Z_p) = \langle -2x^2x^3, x^1x^3, -x^1x^2, (x^3)^2 - (x^2)^2\rangle.$$

Now one can check that at any $p = (x^1, x^2, x^3) \in S^2$, two of these vectors are linearly independent.