# Supplement 2: Manifolds

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## 1 Tangent Vectors, Spaces, and Bundles.

### 1.1 $T_x\mathcal{M}$ is a vector space

Fix $x \in \mathcal{M}$. Check three properties.

#### 1.1.1 $T_x\mathcal{M}$ contains $\mathbf{0}$

The equivalence class of the constant curve $\gamma(t) \equiv x$, for which

$$\frac{d}{dt}[\phi \circ \gamma(t)]\Big|_{t=0} = \mathbf{0} \in \mathbf{E}^d,$$

for any differentiable chart $(G, \phi)$ with $x \in G$, is the zero vector in $T_x\mathcal{M}$.

#### 1.1.2 $v \in T_x\mathcal{M} \implies cv \in T_x\mathcal{M}$

Let $c \in \mathbf{R}$ be given. If $v \in T_x\mathcal{M}$ is represented by $\gamma$, then $\eta(t) = \gamma(ct)$ represents $cv$ since, for any differentiable chart $(G, \phi)$ with $x \in G$,

$$\frac{d}{dt}[\phi \circ \eta(t)]\Big|_{t=0} = \frac{d}{dt}[\phi \circ \gamma(ct)]\Big|_{t=0} = c\frac{d}{dt}[\phi \circ \gamma(t)]\Big|_{t=0},$$

by the chain rule. Note that $\eta$ is defined on the possibly smaller open interval $(-1/|c|, 1/|c|)$ rather than $(-1, 1)$. This technical problem may be overcome by using

$$\eta(t) = \gamma(\tan^{-1}(ct)) \tag{1}$$

whose domain is $(-\infty, \infty)$ and which gives the same chain rule result since

$$\frac{d}{dt}[\tan^{-1}(ct)]\Big|_{t=0} = c\frac{d}{dt}[\tan^{-1}(t)]\Big|_{t=0} = c\left[\frac{1}{1+t^2}\right]\Big|_{t=0} = c$$

Alternatively, the chain rule may be avoided by representing $cv$ with the curve $\eta(t) \stackrel{\text{def}}{=} \phi^{-1}(c\phi \circ \gamma(t))$, which may be different for each coordinate map $\phi$:

$$\frac{d}{dt}[\phi \circ \eta(t)]\Big|_{t=0} = \frac{d}{dt}\left[\phi \circ \phi^{-1}(c\phi \circ \gamma(t))\right]\Big|_{t=0} = c\frac{d}{dt}[\phi \circ \gamma(t)]\Big|_{t=0}.$$

### 1.1.3 $v, w \in T_x\mathcal{M} \implies v + w \in T_x\mathcal{M}$

Suppose that $v, w \in T_x\mathcal{M}$ are equivalence classes of differentiable curves through $x$, represented respectively by $\gamma, \eta$. Let $(G, \phi)$ be any chart with $x \in G$. Then it is possible to define a "sum" of $\gamma, \eta$ as

$$\xi(t) \stackrel{\text{def}}{=} \phi^{-1}(\phi \circ \gamma(t) + \phi \circ \eta(t)),$$

since $\phi \circ \gamma$ and $\phi \circ \eta$ both belong to the linear space $\mathbf{E}^d$.

WOLOG $\phi(x) = \mathbf{0} \in \mathbf{E}^d$, else replace it with the compatible map

$$\phi_x(z) \stackrel{\text{def}}{=} \phi(z) - \phi(x), \qquad z \in G, \tag{2}$$

which has the same domain and differentiability but satisfies $\phi_x(x) = \mathbf{0}$.

Since homeomorphism $\phi$ is an open map, there exists $\epsilon > 0$ such that $B(\mathbf{0}, 2\epsilon) \subset \phi(G)$. Since $\gamma, \eta$ are continuous, there exists $\delta > 0$ such that the small open interval $I = (-\delta, \delta) \subset (-1, 1)$ satisfies

$$\gamma(I) \subset G,\ \phi(\gamma(I)) \subset B(\mathbf{0}, \epsilon); \quad \eta(I) \subset G,\ \phi(\eta(I)) \subset B(\mathbf{0}, \epsilon);$$

Then the domain of $\xi$ includes $I$, since

$$t \in (-\delta, \delta) \implies \phi \circ \gamma(t) + \phi \circ \eta(t) \in B(\mathbf{0}, 2\epsilon) \subset \phi(G).$$

Finally, reparametrize $\xi$ so that its domain includes $(-1, 1)$, again using the arctangent function:

$$\tilde{\xi}(t) \stackrel{\text{def}}{=} \xi\left(\delta \tan^{-1}(t/\delta)\right). \tag{3}$$

Then by the chain rule,

$$\frac{d}{dt}[\phi \circ \tilde{\xi}(t)]\Big|_{t=0} = \frac{d}{dt}[\phi \circ \xi(t)]\Big|_{t=0},$$

and thus

$$\begin{aligned}
\frac{d}{dt}\left[\phi \circ \tilde{\xi}(t)\right]\Big|_{t=0} &= \frac{d}{dt}\left[\phi \circ \phi^{-1}(\phi \circ \gamma(t) + \phi \circ \eta(t))\right]\Big|_{t=0} \\
&= \frac{d}{dt}\left[\phi \circ \gamma(t) + \phi \circ \eta(t)\right]\Big|_{t=0} \\
&= \frac{d}{dt}\left[\phi \circ \gamma(t)\right]\Big|_{t=0} + \frac{d}{dt}\left[\phi \circ \eta(t)\right]\Big|_{t=0}.
\end{aligned}$$

Conclude that $\xi$ represents the equivalence class of $v + w$.

## 1.2 $d\phi(x)$ is a linear homeomorphism from $T_x\mathcal{M}$ onto $\mathbf{E}^d$.

### 1.2.1 $d\phi(x)$ is linear.

Let $u, v \in T_x\mathcal{M}$ be given, represented by curves $\gamma, \eta$ through $x$, and suppose $c \in \mathbf{R}$ is also given.

$T_x\mathcal{M}$ is a vector space, so $u+cv \in T_x\mathcal{M}$ has a representative curve $\xi$ through $x$ which, except for domain adjustments, may be written as

$$\xi(t) = \phi^{-1}(\phi \circ \gamma(t) + c\phi \circ \eta(t)).$$

(Adjustments like $\phi \leftarrow \phi - \phi(x)$ and $t \leftarrow \delta \tan^{-1}(t/\delta)$ from Equations 2 and 3 would result in $\phi(x) = \mathbf{0}$ and domain $-1 < t < 1$ for all curves, without loss.) Then by definition,

$$\begin{aligned} d\phi(x)(u+cv) &= d_\xi \phi(x) \;=\; \frac{d}{dt}\left[\phi \circ \xi(t)\right]\Big|_{t=0} \\ &= \frac{d}{dt}\left[\phi \circ \phi^{-1}(\phi \circ \gamma(t) + c\phi \circ \eta(t))\right]\Big|_{t=0} \\ &= \frac{d}{dt}\left[\phi \circ \gamma(t)\right]\Big|_{t=0} + c\frac{d}{dt}\left[\phi \circ \eta(t)\right]\Big|_{t=0} \\ &= d_\gamma \phi(x) + c d_\eta \phi(x) \;=\; d\phi(x)(u) + c d\phi(x)(v). \end{aligned}$$

Conclude that $d\phi(x)$ is linear.

### 1.2.2 $d\phi(x)$ is surjective.

Let $\mathbf{e} = \{\mathbf{e}_1, \ldots, \mathbf{e}_d\}$ be the standard basis of $\mathbf{E}^d$. Fix $k$ and parametrize a curve $\gamma$ through $x \in \mathcal{M}$ with

$$\gamma(t) \stackrel{\text{def}}{=} \phi^{-1}(\phi(x) + t\mathbf{e}_k), \quad -1 < t < 1. \tag{4}$$

(If necessary to stay within $\phi(G)$ for all $-1 < t < 1$, replace $t \leftarrow \delta \tan^{-1}(t/\delta)$ using small enough $\delta > 0$.) Then the directional derivative of $\phi$ along $\gamma$ is

$$d_\gamma \phi(x) = \frac{d}{dt}\left[\phi \circ \gamma(t)\right]\Big|_{t=0} = \frac{d}{dt}\left[\phi(x) + t\mathbf{e}_k\right]\Big|_{t=0} = \mathbf{e}_k.$$

Thus $\gamma$ represents a tangent vector $v_k$ for which $d\phi(x)(v_k) = d_\gamma\phi(x) = \mathbf{e}_k$. Repeating the Eq.4 construction for all $k \in \{1, \ldots, d\}$ gives distinct tangent vectors $\{v_1, \ldots, v_d\} \subset T_x\mathcal{M}$ with

$$d\phi(x)(v_k) = \mathbf{e}_k, \quad k = 1, \ldots, d.$$

Now suppose that $\mathbf{p} \in \mathbf{E}^d$ is given. Write $\mathbf{p} = \sum_k a_k \mathbf{e}_k$, using the basis for $\mathbf{E}^d$. Since $T_x\mathcal{M}$ is a vector space, it contains the linear combination $w \stackrel{\text{def}}{=} \sum_k a_k v_k$. Applying the linearity of $d\phi(x)$, compute

$$d\phi(x)(w) = d\phi(x)\left(\sum_k a_k v_k\right) = \sum_k a_k d\phi(x)(v_k) = \sum_k a_k \mathbf{e}_k = \mathbf{p}.$$

Hence $w \in T_x\mathcal{M}$ is a preimage of $\mathbf{p}$. Conclude that $d\phi(x)$ is surjective.

### 1.2.3 $d\phi(x)$ is injective.

Since $d\phi(x)$ is linear, it suffices to show that its nullspace is just $\{0\}$.

So suppose that $u \in T_x\mathcal{M}$ satisfies $d\phi(x)(u) = \mathbf{0}$. Let $\gamma$ be a curve through $x$ that represent $u$. Then by definition,

$$\mathbf{0} = d\phi(x)(u) = d_\gamma\phi(x) = \frac{d}{dt}[\phi \circ \gamma(t)]\Big|_{t=0}.$$

Now let $(H, \psi)$ be any chart in the maximal differentiable atlas for $\mathcal{M}$ such that $x \in G \cap H$. Let $\tau = \psi \circ \phi^{-1}$ be the differentiable transition function on $\phi(G \cap H)$, where $\tau : \mathbf{E}^d \to \mathbf{E}^d$. Then $D\tau(\mathbf{p}) : \mathbf{E}^d \to \mathbf{E}^d$ is a $d \times d$ matrix for any $\mathbf{p} \in \phi(G \cap H)$, and the chain rule may be used to evaluate:

$$\begin{aligned}
\frac{d}{dt}[\psi \circ \gamma(t)]\Big|_{t=0} &= \frac{d}{dt}[\psi \circ \phi^{-1} \circ \phi \circ \gamma(t)]\Big|_{t=0} = \frac{d}{dt}[\tau \circ \phi \circ \gamma(t)]\Big|_{t=0} \\
&= D\tau(\phi(\gamma(0)))\frac{d}{dt}[\phi \circ \gamma(t)]\Big|_{t=0} = D\tau(\phi(x))\mathbf{0} = \mathbf{0},
\end{aligned}$$

since $\gamma(0) = x$ and $\phi(x) \in \phi(G \cap H)$. Hence $u$ is the equivalence class of curves through $x$ that give the zero vector as the directional derivative for every chart, namely the zero tangent vector.

### 1.2.4 Remarks on higher derivatives

Finding $d\phi(x)$ consumes one derivative, which is all that is assumed to exist for a differentiable manifold. To define higher order derivatives, the atlas of charts on $\mathcal{M}$ must contain coordinate functions with $K > 1$ derivatives (for $C^K$ manifolds) or even infinitely many derivatives (for $C^\infty$, or *smooth* manifolds).