Supplement 3: Manifolds

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PMF 2022: Dimensionality Reduction and Manifold Estimation

January 24, 2022

1 Partitions of Unity.

Recall that a topological space $(X, \mathcal{T})$ is *compact* if every open cover of X contains a finite subcover. This is a strong condition, but there are several related weaker conditions of interest.

1.1 Closures

Set $F \subset X$ is called *closed* iff its complement $F^c \stackrel{\text{def}}{=} X \setminus F$ is open.

The following are de Morgan's formulas. Index set I is arbitrary:

$$(\bigcap_{\alpha\in I} S_\alpha)^c = \bigcup_{\alpha\in I} S_\alpha^c, \qquad (\bigcup_{\alpha\in I} S_\alpha)^c = \bigcap_{\alpha\in I} S_\alpha^c.$$

Use these to deduce that

- Arbitrary intersections of closed sets are closed.
- Finite unions of closed sets are closed.

Define the *closure* $\bar{S}$ of a subset $S \subset X$ to be the smallest closed set that contains S, namely

$$\bar{S} = \bigcap_{\{F: F^c\in\mathcal{T},\ S\subset F\}} F$$

which is the intersection of all the closed sets F (whose complements are open sets $F^c \in \mathcal{T}$) that contain S.

Any closed subset F of a compact set K is compact: if $F \subset \cup_{\alpha\in I} G_\alpha$ is any open cover, then $K \subset F^c \cup [\cup_{\alpha\in I} G_\alpha]$ is an open cover of K, hence it has a finite subcover $F^c \cup G_1 \cup \cdots \cup G_N$, hence $F \subset G_1 \cup \cdots \cup G_N$ is a finite subcover of F.

Consequently, if K is compact and $S \subset K$ also has $\bar{S} \subset K$, then $\bar{S}$ is compact.

A compact subset K need not be closed. For example, let $X = \{a, b\}$ and $\mathcal{T} = \{\emptyset, \{a\}, \{a, b\}\}$. Then $\{a\}$ is compact but not closed, since its complement $\{a\}^c = \{b\} \notin \mathcal{T}$. (There are many other similarly contrived examples.)

However, any compact subset K of a Hausdorff space must be closed. Prove this by showing that its complement K^c is open, for which it suffices to show that each $x \in K^c$ belongs to some open set $G \subset K^c$. Find that G as follows:

- For each $y \in K$, find disjoint open G_y, H_y with $x \in G_y$ and $y \in H_y$. These exist by the Hausdorff property.
- Note that $K \subset \bigcup_{y \in K} H_y$ is an open cover.
- Since K is compact, there exists a finite subcover which may be denoted $K \subset H_1 \cup \cdots \cup H_N$.
- Let G_i be the open set around x corresponding to H_i, $i = 1, \ldots, N$.
- Put $G \stackrel{\text{def}}{=} G_1 \cap \cdots \cap G_N$. This finite intersection of open sets is open.
- Note that $x \in G$.
- $G \subset K^c$, since $(\forall i) G \cap H_i = \emptyset$ and $K \subset \cup H_i$.

Conclude that K^c is open, so K is closed.

1.2 Local Compactness

A topological space $(X, \mathcal{T})$ is *locally compact* if every point $x \in X$ has a compact neighborhood,

A d-dimensional manifold $\mathcal{M}$ is locally compact since it is a metric space that is locally homeomorphic to $\mathbf{E}^d$. Every $x \in \mathcal{M}$ belongs to some chart (G, ϕ) with open G, so there exists $\epsilon > 0$ such that $x \in B(x, 2\epsilon) \subset G$. But then

$$\bar{B}(x, \epsilon) \stackrel{\text{def}}{=} \{y \in \mathcal{M} : \mathrm{d}(x, y) \leq \epsilon\} \subset G,$$

and $\phi(\bar{B}(x, \epsilon)) \subset \phi(G) \subset \mathbf{E}^d$ is closed and bounded. Such sets are compact by the Heine-Borel theorem. Conclude that

$$\bar{B}(x, \epsilon)) = \phi^{-1}\left(\phi(\bar{B}(x, \epsilon))\right) \subset \mathcal{M}$$

is a compact neighborhood of x.

1.3 Paracompactness

Start with the notion of *refinement* of open covers: collection $\mathcal{G}' \subset \mathcal{T}$ is a refinement of $\mathcal{G} \subset \mathcal{T}$ iff

$$(\forall G \in \mathcal{G})(\exists G' \in \mathcal{G}') G' \subset G.$$

For example, in a metric space X, $\mathcal{G}' = \{B(x, \epsilon/2) : x \in X\}$ is a refinement of $\mathcal{G} = \{B(x, \epsilon) : x \in X\}$.

Next, say that an open cover $\{G_\alpha : \alpha \in I\}$ is *locally finite* iff every $x \in X$ is contained in some neighborhood U_x that intersects only finitely many sets in the cover. Namely,

$$(\forall x \in X)(\exists U_x \in \mathcal{T})\,(x \in U_x \text{ and } \{\alpha \in I : U_x \cap G_\alpha \neq \emptyset\} \text{ is finite})\,.$$

Finally, say that $(X, \mathcal{T})$ is *paracompact* iff every open cover has a locally finite refinement.

Remark. If there is a locally finite refinement $\mathcal{G}' = \{G'_\beta : \beta \in J\}$ of $\mathcal{G} = \{G_\alpha : \alpha \in I\}$, then there is an identically-indexed locally finite refinement $\mathcal{H} = \{H_\alpha : \alpha \in I\}$ of $\mathcal{G}$ such that $H_\alpha \subset G_\alpha$ for each $\alpha \in I$. It may be constructed by choosing a single $\alpha = i(\beta) \in I$ for each $\beta \in J$ such that $G'_\beta \subset G_\alpha$, and then putting

$$H_\alpha = \bigcup_{\{\beta \in J:\ i(\beta) = \alpha\}} G'_\beta .$$

It is clear that $H_\alpha \subset G_\alpha$. But also, any neighborhood that intersects only finitely many G'_β can intersect at most finitely many H_α. (Since i is single-valued, there will be more intersecting β indices than α indices.) Thus $\mathcal{H}$ is a locally finite refinement of $\mathcal{G}$.

Theorem. *A locally compact second countable Hausdorff space is paracompact.*

Proof. See Proposition A1.6 in `PartOfUnity_LocFiniteRefinements.pdf` on the class website. □

Remark. A differentiable manifold is locally compact (because it is finite dimensional), second countable (by definition), and Hausdorff (because it is a metric space). Hence any differentiable manifold is paracompact.

Remark. Any compact differentiable manifold is obviously paracompact since any finite cover is locally finite.

1.4 Bump functions

Start by constructing a continuously differentiable nonnegative function $b : \mathbf{E}^d \to \mathbf{R}$ satisfying

- $|x| \geq 2 \implies b(x) = 0$. Namely, b is supported in $\bar{B}(\mathbf{0}, 2)$, the closed ball of radius 2 centered at $\mathbf{0}$.
- $|x| \leq 1 \implies b(x) > 0$. Thus b is strictly positive in $B(\mathbf{0}, 1)$.

Examples among elementary functions include

$$b(x) = \begin{cases} \frac{1}{2}[1 + \cos(\pi|x|/2)], & |x| \leq 2, \\ 0, & |x| > 2 \end{cases}$$

which has one continuous derivative, and

$$b(x) = \begin{cases} \exp[-1/(4-|x|^2)], & |x| < 2, \\ 0, & |x| \geq 2 \end{cases}$$

which has infinitely many continuous derivatives, as may be shown by induction and l'Hôpital's rule.

1.5 Bump Functions on a Manifold

Suppose that $(\mathcal{M}, \mathcal{T}, \mathcal{A})$ is a d-dimensional differentiable manifold. This is a paracompact topological space, so every open cover has a locally finite refinement.

Given a chart (G, ϕ) and fixed $x \in G$, it may be assumed WOLOG that

- $\phi(x) = \mathbf{0} \in \mathbf{E}^d$, else use $\phi_x(z) \stackrel{\text{def}}{=} \phi(z) - \phi(x)$.
- $\phi(G) \subset B(\mathbf{0}, 2)$, else use $\phi_\epsilon(z) \stackrel{\text{def}}{=} \epsilon\phi(z)$ with sufficiently small $\epsilon > 0$.

Then, using a bump function $b : \mathbf{E}^d \to \mathbf{R}$ from the previous section, define $b_G : \mathcal{M} \to \mathbf{R}$ by

$$b_G(x) \stackrel{\text{def}}{=} \begin{cases} b \circ \phi(x), & x \in G \\ 0, & x \in \mathcal{M} \setminus G. \end{cases}$$

Each such function is differentiable on $\mathcal{M}$.

Now let $\mathcal{M} \subset \{G_\alpha : \alpha \in I\}$ be a locally finite cover. Define $g : \mathcal{M} \to \mathbf{R}$ by

$$g(x) \stackrel{\text{def}}{=} \sum_{\alpha \in I} b_{G_\alpha}(x), \qquad x \in \mathcal{M},$$

which is finite and differentialble since each term is differentiable and, at each x, there are only finitely many α with $b_{G_\alpha}(x) > 0$ in the sum.

In addition, $g(x) > 0$ since there is at least one strictly positive summand at each $x \in \mathcal{M}$.