Ma 450: Mathematics for Multimedia
Final Examination

Prof. Wickerhauser

Due 4:00 pm Friday, May 7th, 2010
in Cupples I, room 100

10 problems on 10 pages

You may use a calculator or computer. You may consult any existing written reference material, but you may not get help from any other person. Please write your complete answers in the space provided.

1. Find an integer x such that $2010x - 1$ is divisible by 507, or prove that none exists.

2. a. Define $Tf(x) = \int_0^{1-x} f(t)\,dt$ for $0 \le x \le 1$ and $f \in \mathbf{Lip} = \mathbf{Lip}([0,1])$. Prove that $T : \mathbf{Lip} \to \mathbf{Lip}$ and that T is a linear transformation.

   b. Determine with proof whether $\|T\|_{\text{op}} < 1$ with respect to the L^2 norm on $\mathbf{Lip} = \mathbf{Lip}([0,1])$ in Equation 2.23 on textbook page 34.

3. Compute the complex exponential Fourier series of the 1-periodic function

$$\operatorname{cas}(2\pi kt - d) = \cos(2\pi kt - d) + \sin(2\pi kt - d),$$

where $k > 0$ is an integer and d is a fixed real number.

4. Let T_n denote the n^{th} Chebyshev polynomial.

   a. Determine with proof all elements of the set $\{T_n(\frac{1}{2}) : n \in \mathbf{N}\}$.

   b. Determine with proof all elements of the set $\{n \in \mathbf{N} : T_n(0) = 0\}$.

5. Let $f(x) = x^7$, let $x_k = k/19$ for $k \in \mathbf{Z}$, and let $y_k = f(x_k)$. Evaluate the Lagrange polynomial through the points $\{(x_k, y_k) : k = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$, at the point $x = 1$, to six decimal places.

6. Draw the graphs of $w(2t+3)$ and $w(2(t+3))$ on one set of axes for the Haar function $w(t)$ defined in Equation 5.2 of the textbook.

7. For each $(a, b) \in \mathbf{Aff}$, define the linear operator $\hat{\sigma}(a, b) : L^2(\mathbf{R}) \to L^2(\mathbf{R})$ by

$$\hat{\sigma}(a, b)f(t) \stackrel{\text{def}}{=} \sqrt{a}\, \exp(2\pi i b t) f(at).$$

a. Show that $\hat{\sigma}$ is a representation of the group $\mathbf{Aff}$.

b. Show that $\hat{\sigma}$ is a faithful representation.

c. Show that $\hat{\sigma}$ is a unitary representation.

8. Compute the periodic discrete Haar wavelet transform on the 8-periodic signal $u = u(n)$ defined by $u(n) = \sin(n\pi/4)$ on the period interval $n = 0, 1, \ldots, 7$.

9. a. Construct a canonical Huffman code for the alphabet $A = \{1, 2, 3, 4, 5\}$ with occurrence probabilities $p = (.10, .15, .20, .25, .30)$, with the property that no letter has a codeword consisting of just 1-bits.

   b. Compute its bit rate.

   c. Compute the theoretical minimum bit rate for the information source in part (a).

10. a. Show that the mod-2 polynomial $t^3 + t^2 + 1$ is irreducible.
    b. Find a factorization of the mod-2 polynomial $t^3 + t^2 + t + 1$ into irreducible mod-2 polynomials.