Math 450: Mathematics for Multimedia

Practice Midterm Examination

Friday, 14 March 2014

No materials other than this test and a pen or pencil are permitted. Please write your complete answers in the space provided.

1. (a) How many integers in the set $\{0, 1, \ldots, 54\}$ are relatively prime with 55?

   (b) Find an integer $x \in \{0, 1, \ldots, 54\}$ such that $7^{120} \equiv x \pmod{55}$.

2. Express the number $x = 6.666\ldots$ (base 16) as a decimal expansion in base 10.

3. Suppose that $\mathbf{x}, \mathbf{y}$ are vectors in an inner product space $\mathbf{X}$, with $\langle \mathbf{x}, \mathbf{y} \rangle = 6$ and $\|\mathbf{x}\| = 3$.

   (a) What is the minimum possible value of $\|\mathbf{y}\|$?

   (b) What is the minimum possible value of $\|\mathbf{x} + \mathbf{y}\|$?

   (c) What is the minimum possible value of $\|\mathbf{x} - \mathbf{y}\|$?

   (d) Given a fixed L, find vectors $\mathbf{x}, \mathbf{y} \in \mathbf{R}^2$ satisfying the given conditions plus $\|\mathbf{y}\| > L$.

4. Let X be an n-dimensional inner product space with basis $\mathbf{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$. Suppose that $\mathbf{B}' = \{\mathbf{b}'_1, \mathbf{b}'_2, \ldots, \mathbf{b}'_n\}$ is the biorthogonal dual basis for $\mathbf{B}$. Let $Y = \text{span}\,\{\mathbf{b}_n\}$. Find a basis for $Y^\perp$.

5. Suppose that $A \in \mathbf{Mat}(N \times N)$ is an upper triangular matrix with zeros on the main diagonal, namely, $A_{ij} = 0$ for all $1 \leq j \leq i \leq N$.

   (a) Show that A^2 has zeros on the main diagonal.

   (b) Show that A^N must be the zero matrix. (Hint: use induction, noticing that if $\mathbf{x} \in \mathbf{R}^N$ has zeros in its last n coordinates, then $A\mathbf{x}$ has zeros in its last $n + 1$ coordinates.)

6. Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation defined by $T(x, y) = (x + 2y, 2x - y)$.

   (a) Compute T^* with respect to the usual inner products.

   (b) Compute $\|T\|_{\text{op}}$ with respect to the usual Euclidean norms.

7. Find the complex exponential Fourier series of the 1-periodic function $\sin^2(\pi k t)$, where $k > 0$ is an integer.

8. Suppose that $\phi = \phi(x)$ has Fourier integral transform

$$\mathcal{F}\phi(\xi) = \begin{cases} 1, & \text{if } 0 < \xi < 1, \\ 0, & \text{otherwise.} \end{cases}$$

   Use the Fourier inversion theorem to compute $\phi(x)$ at all $x \in \mathbf{R}$.