Ma 4111: Advanced Calculus
Solutions to Homework Assignment 6

Prof. Wickerhauser

Due Tuesday, November 20th, 2012

Please return your solutions to the instructor by the end of class on the due date. You may collaborate on these problems but you must write up your own solutions. **Late homework will not be accepted.**

1. Determine (with proof) whether the function $f(x) \stackrel{\text{def}}{=} x^{1/5}\cos(\pi/2x)$ if $x \neq 0$, with $f(0) \stackrel{\text{def}}{=} 0$, has bounded variation on the interval $[-1,1]$.

   Solution: We prove that this function f is not of bounded variation on any interval containing 0 in its interior. We let $x_n = \frac{1}{n}$ for $n = 1, 2, \ldots$, and write $P = \{x_n : n = 1, 2, \ldots\}$. Every subinterval of $[-1,1]$ containing 0 in its interior holds all x_n with sufficiently large n, so that f will have bounded variation on such a subinterval only if $\sum_n |\Delta f_n|$ converges. But $|\Delta f_{2n}| = |\Delta f_{2n+1}| = (1/2n)^{1/5}$, so $\sum_n |\Delta f_n|$ diverges as $n \to \infty$ by the comparison test. □

2. A function $\mu = \mu(x)$ defined on $\mathbf{R}^+$ is called a *Marcinkiewicz multiplier* if there is some $M < \infty$ such that $V_\mu(2^j, 2^{j+1}) < M$ for all integers j; that is, μ has uniformly bounded variation on intervals of the form $[2^j, 2^{j+1}]$.

   (a) Prove that $\mu(x) = \log x$ is a Marcinkiewicz multiplier. Thus such functions do not have to be bounded.

   (b) Prove that μ is a Marcinkiewicz multiplier if and only if there is some $\lambda > 1$ and some $N < \infty$ such that $V_\mu(a, \lambda a) < N$ for every $a > 0$.

   Solution: (a) Since $\log x$ is continuous and monotonic on $\mathbf{R}^+$, its total variation on $[2^j, 2^{j+1}]$ is $|\log 2^{j+1} - \log 2^j| = \log 2$. Thus we can choose $M = \log 2$ in the definition.

   (b) Suppose first that μ is a Marcinkiewicz multiplier for a given $\lambda > 1$ and $M > 0$. Fix $a > 0$; then $a \in [2^j, 2^{j+1}]$ and $\lambda a \in [2^k, 2^{k+1}]$ where $j = \lfloor \log_2 a \rfloor$ and $k = \lfloor \log_2 \lambda a \rfloor$ ($\lfloor z \rfloor$ should be read "the greatest integer less than or equal to z"). Since $\lfloor \log_2 \lambda a \rfloor \leq \lfloor \log_2 \lambda \rfloor + \lfloor \log_2 a \rfloor + 1$, we see that $k - j$ will be bounded by $\lfloor \log_2 \lambda \rfloor + 1$, which is independent of a. Since total variation is additive, $V_\mu(a, \lambda a) \leq V_\mu(2^j, 2^{k+1}) = V_\mu(2^j, 2^{j+1}) + \ldots + V_\mu(2^k, 2^{k+1}) \leq (\lfloor \log_2 \lambda \rfloor + 1) M$.

   Conversely, suppose $V_\mu(a, \lambda a) \leq N < \infty$ uniformly in $a > 0$. By Archimedes' principle applied to $\log_2 \lambda > 0$, we can choose $k \in \mathbf{Z}^+$ large enough so that $\lambda^k \geq 2$, with k clearly independent of the choice of a. Then for $a = 2^j$ we have $[2^j, 2^{j+1}] \subset [a, \lambda a]$, so by the additivity of total variation $V_\mu(2^j, 2^{j+1}) \leq V_\mu(a, \lambda^k a) = V_\mu(a, \lambda a) + \ldots + V_\mu(\lambda^{k-1} a, \lambda^k a) \leq kN < \infty$. Thus we can take $M = kN$ and show that μ is a multiplier. □

3. A real-valued function f defined on $[a,b] \subset \mathbf{R}$ is said to *absolutely continuous* on $[a,b]$ if for every $\epsilon > 0$ there is $\delta > 0$ such that for every finite collection of disjoint open subintervals $(a_i, b_i) \subset [a,b]$ with $\sum_i |b_i - a_i| < \delta$, we have $\sum_i |f(b_i) - f(a_i)| < \epsilon$.

Prove that a function which is absolutely continuous on $[a, b]$ is continuous and of bounded variation on $[a, b]$.

Solution: Suppose f is absolutely continuous. To show that f is continuous at any $x \in [a, b]$, let $\epsilon > 0$ be given and choose $\delta > 0$ to satisfy the definition. Then for any $y \in [a, b]$ with $|y - x| < \delta$ we will be sure that $|f(y) - f(x)| < \epsilon$.

Now suppose that f has unbounded variation on $[a, b]$, and let $\epsilon > 0$ be given. If $\delta > 0$, we can find a positive integer N such that $0 < \delta' \stackrel{\text{def}}{=} \frac{b-a}{N} < \delta$ and we can write $[a, b] = \bigcup_{i=1}^{N} [a_i, b_i]$ where $a_i = a + (i-1)\delta'$, $b_i = a + i\delta'$. Then $V_f(a_i, b_i)$ must be $+\infty$ for some i, since otherwise by additivity the total variation of f on $[a, b]$ would be bounded. But then we could find a partition $P = \{a_i < x_1 < \ldots < x_n < b_i\}$ of $[a_i, b_i]$ such that $\sum_P |\Delta f_k| > \epsilon$, even though $|x_1 - a_i| + |x_2 - x_1| + \ldots + |b_i - x_n| = |b_i - a_i| < \delta$. Hence$f$ cannot be absolutely continuous on $[a, b]$. $\square$

4. Suppose that $\mathbf{x}$ is a rectifiable path of length L defined on $[a, b]$ and assume that $\mathbf{x}$ is not constant on any subinterval of $[a, b]$. Let $s(x) = \Lambda_{\mathbf{x}}(a, x)$ if $a < x \le b$ and put $s(a) = 0$. Prove that s^{-1} exists and is continuous on $[0, L]$.

   Solution: The function $s = s(x)$ is strictly increasing by theorem 6.19 of the text, hence s is one-to-one. Thus s has an inverse on its range $[0, L]$. By the same theorem, s is continuous on its domain $[a, b]$. Since this domain is compact, theorem 4.29 implies that the inverse function s^{-1} is continuous. $\square$

5. Give an example of a bounded function f and an increasing function α defined on $[a, b]$ such that $|f| \in R(\alpha)$ but $f \notin R(\alpha)$.

   Solution: Take $\alpha(x) = x$ and let

$$f(x) = \begin{cases} +1 & \text{if } x \text{ is a rational number,} \\ -1 & \text{if } x \text{ is an irrational number.} \end{cases}$$

Then $|f| \equiv 1$ is continuous and α has bounded variation on $[a, b]$, hence $f \in R(\alpha)$ on $[a, b]$ by Theorem 7.27. On the other hand, since every nonempty subinterval of $[a, b]$ contains both rational and irrational points, every partition $P \in \mathcal{P}[\dashv, \lfloor]$ will yield (by telescoping the upper and lower Stieltjes sums) that

$$U(P, f, \alpha) = +1[b - a]; \qquad L(P, f, \alpha) = -1[b - a].$$

Hence we cannot satisfy Riemann's condition with this f and any $\epsilon < 2[b - a]$, so $f \notin R(\alpha)$. $\square$

6. Assume that α has bounded variation on $[a, b]$ and $f \in R(\alpha)$ on $[a, b]$. Let $V(x)$ denote the total variation of α on $[a, x]$, where $a < x \le b$, and put $V(a) = 0$ as usual. Show that

$$\left| \int_a^b f \, d\alpha \right| \le \int_a^b |f| \, dV \le MV(b),$$

where $M = \sup\{|f(t)| : a \le t \le b\}$.

Solution: By Theorem 7.24, $f \in R(V)$ on $[a, b]$, so by Theorem 7.21 $|f| \in R(V)$ as well. We note that $|\Delta\alpha_k| \le \Delta V_k$, since the total variation of α on $[x_{k-1}, x_k]$ cannot be less than $|\alpha(x_k) - \alpha(x_{k-1})|$. Thus, for any partition $P \in \mathcal{P}[\dashv, \lfloor]$ we obtain the inequalities

$$|S(P, f, \alpha)| = \left| \sum_k f(t_k)\Delta\alpha_k \right| \le \sum_k |f(t_k)| \, |\Delta\alpha_k| \le \sum_k |f(t_k)| \Delta V_k = S(P, |f|, V).$$

Here we have used the triangle inequality. Since $|S(P, f, \alpha)| \le S(P, |f|, V)$ holds for arbitrary partitions P, it holds for the limits: $|\int_a^b f \, d\alpha| \le \int_a^b |f| \, dV$.

The second inequality follows because V increases: if we replace $|M_k(f)|$ with the larger number M in any upper Stieltjes sum $U = U(P, |f|, V)$, we obtain a larger value than U. The resulting series telescopes and gives us $U \leq M[V(b) - V(a)] = MV(b)$. Since this is true for each approximation U, it is true for the infimum $\int_a^b |f|\, dV$. $\square$

7. Let f be a positive continuous function in $[a, b]$. Let M denote the maximum value of f on $[a, b]$. Show that

$$\lim_{n\to\infty} \left(\int_a^b f(x)^n \, dx \right)^{1/n} = M.$$

Solution: Since f is continuous on $[a, b]$ it attains its maximum value M at some point in that interval. Call that point ξ, and let $\epsilon > 0$ be given. For any such ϵ there is a $\delta > 0$ such that $|x - \xi| < \delta \Rightarrow f(x) > M - \epsilon/2$. But then comparing f with $M - \epsilon/2$ on $[\xi - \delta, \xi + \delta]$ and with 0 off that subinterval, Theorem 7.20 gives us the following inequality:

$$\left(\int_a^b f(x)^m \, dx \right)^{\frac{1}{m}} \geq \left(\int_{\xi-\delta}^{\xi+\delta} [M - \epsilon/2]^m \, dx \right)^{\frac{1}{m}} = (2\delta)^{\frac{1}{m}} [M - \epsilon/2].$$

Since $(2\delta)^{\frac{1}{m}} \to 1$ as $m \to \infty$, for sufficiently large m the right hand side will be greater than $M - \epsilon$. Since ϵ was arbitrary, we conclude that $\left(\int_a^b f(x)^m \, dx \right)^{\frac{1}{m}} \geq M$.

For the opposite inequality, we use $0 \leq f \leq M$ in the comparison theorem to conclude the following:

$$0 \leq \left(\int_a^b f(x)^m \, dx \right)^{\frac{1}{m}} \leq \left(\int_a^b M^m \, dx \right)^{\frac{1}{m}} = (b - a)^{\frac{1}{m}} M.$$

Again, since $(b - a)^{\frac{1}{m}} \to 1$ as $m \to \infty$, we see that for sufficiently large m the right-hand side will be less than $M + \epsilon$. Since ϵ is arbitrary, $\left(\int_a^b f(x)^m \, dx \right)^{\frac{1}{m}} \leq M$. $\square$

8. Assume that f has a decreasing derivative which satisfied $f'(x) \geq m > 0$ for all $x \in [a, b]$. Prove that

$$\left| \int_a^b \cos f(x) \, dx \right| \leq \frac{2}{m}.$$

(Hint: Multiply and divide the integrand by $f'(x)$ and use Theorem 7.37(ii)).

Solution: Following the hint, we divide by $f'(x) > 0$ to get, for some $x_0 \in [a, b]$,

$$\left| \int_a^b \frac{\cos f(x)}{f'(x)} f'(x) \, dx \right| = \frac{1}{m} \left| \int_{x_0}^b \cos f(x) f'(x) \, dx \right| = \frac{1}{m} \left| \int_{x_0}^b d \sin f(x) \right| = \frac{1}{m} |\sin f(b) - \sin f(x_0)| \leq \frac{2}{m}.$$

The leftmost equality follows from Theorem 37(ii) and the assumption that $0 \leq \frac{1}{f'(x)} \leq \frac{1}{m}$. The middle equality follows from two applications of the first fundamental theorem of calculus, Theorem 7.33. The rightmost inequality follows from the second fundamental theorem of calculus, Theorem 7.34. The inequality at the far right follows from the triangle inequality, since $|\pm \sin \theta| \leq 1$ for any θ. $\square$

9. Prove that the following function is Riemann integrable on $[0, 1]$:

$$f(x) = \begin{cases} 1, & \text{if } x = 0; \\ 0, & \text{if } x \in (0, 1) \text{ is irrational;} \\ 1/n, & \text{if } x \in (0, 1] \text{ is rational, with } x = m/n \text{ in lowest terms.} \end{cases}$$

(Hint: compute the oscillation $\omega_f(x)$ of f at each $x \in [0,1]$.)

Solution: Following the hint, we suppose that $x \in [0,1]$ and compute

$$\omega_f(x) = \lim_{h \to 0+} \sup\{|f(s) - f(t)| : s, t \in B(x;h)\} = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{if } x \text{ is irrational,} \\ 1/n, & \text{if } x = m/n > 0 \text{ in lowest terms.} \end{cases}$$

Thus f is continuous except at the rational points in $[0,1]$, which are a set of measure zero. We conclude by Lebesgue's criterion (Theorem 7.48, p.171) that f belongs to R. □

10. Define

$$g(x) = \begin{cases} 0, & \text{if } x = 0; \\ 1, & \text{if } 0 < x \le 1. \end{cases}$$

(a) Prove that g is Riemann integrable on $[0,1]$.

(b) Let $f = f(x)$ be as in Problem 9. Prove that $g \circ f$ is not Riemann integrable on $[0,1]$, despite both $f \in R$ and $g \in R$ on $[0,1]$.

Solution: (a) Since g is continuous except at the single point 0, g is Riemann integrable by Lebesgue's criterion.

(b) For $x \in [0,1]$, compute

$$g \circ f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational;} \\ 1, & \text{if } x \text{ is rational.} \end{cases}$$

But this function is discontinuous at every $x \in [0,1]$, since the rationals are dense in that interval. Hence by the necessity of Lebesgue's criterion, $g \circ f$ cannot be Riemann integrable. □