Ma 4111: Advanced Calculus
Solutions to Homework Assignment 3

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Due Tuesday, October 9th, 2012

Please return your solutions to the instructor by the end of class on the due date. You may collaborate on these problems but you must write up your own solutions. **Late homework will not be accepted.**

1. Prove that every closed subset of $\mathbf{R}$ is the intersection of a countable collection of open sets.

   Solution: Let $S \subset \mathbf{R}$ be closed. Then $\mathbf{R} - S$ is open, so by theorem 3.11 we can write $\mathbf{R} - S = \bigcup_{k=1}^{\infty} A_k$, where $\{A_k : k = 1, 2, \ldots\}$ is a countable collection of disjoint open intervals of $\mathbf{R}$. Thus $S = \mathbf{R} - \bigcup_{k=1}^{\infty} A_k = \bigcap_{k=1}^{\infty}(\mathbf{R} - A_k)$, where each $\mathbf{R} - A_k$ is the union of at most two disjoint closed intervals. If $A_k = (a_k, b_k)$, then we can call the two intervals of its complement $L_k = (-\infty, a_k]$ and $R_k = [b_k, \infty)$, though one or both of these might be empty.

   We now note that $(-\infty, a_k] = \bigcap_{j=1}^{\infty} L_{kj}$ where $L_{kj} \stackrel{\text{def}}{=} (-\infty, a_k + 1/j)$ is open, and $[b_k, \infty) = \bigcap_{j=1}^{\infty} R_{kj}$ where $R_{kj} \stackrel{\text{def}}{=} (b_k - 1/j, \infty)$ is also open. Thus $L_{kj} \cup R_{kj}$ is open and $S = \bigcap_{k=1}^{\infty} \bigcap_{j=1}^{\infty}(L_{kj} \cup R_{kj})$ is a countable intersection of open sets since the indices are taken from the countable set $\mathbf{Z}^+ \times \mathbf{Z}^+$. □ □

   For Problems 2–3, a set $S \subset \mathbf{R}^n$ is called *convex* if for every pair of points $\mathbf{x}, \mathbf{y} \in S$ and every real number θ satisfying $0 < \theta < 1$ we have $\theta\mathbf{x} + (1-\theta)\mathbf{y} \in S$.

2. Prove that (a) an n-ball is convex; and (b) an n-dimensional open interval is convex.

   Solution: For (b), we note that $\mathbf{x} \in (\mathbf{x}, \mathbf{y})$ and $\mathbf{y} \in (\mathbf{x}, \mathbf{y})$ implies $a_k \le x_k \le b_k$ and $a_k \le y_k \le b_k$ for all $k = 1, 2, \ldots, n$. But then since both θ and $1-\theta$ are positive, we have $\theta a_k + (1-\theta)a_k \le \theta x_k + (1-\theta)y_k \le \theta b_k + (1-\theta)b_k$ for all $k = 1, 2, \ldots, n$ which means that $\theta\mathbf{x} + (1-\theta)\mathbf{y} \in (\mathbf{x}, \mathbf{y})$.

   For (a), theorem 3.3 implies that if $\mathbf{x} \in B(\mathbf{x}, r)$ and $\mathbf{y} \in B(\mathbf{x}, r)$, then $\|\theta\mathbf{x} + (1-\theta)\mathbf{y} - \mathbf{x}\| = \|\theta[\mathbf{x} - \mathbf{x}] + (1-\theta)[\mathbf{y} - \mathbf{x}]\| \le \theta\|\mathbf{x} - \mathbf{x}\| + (1-\theta)\|\mathbf{y} - \mathbf{x}\| < r$.

   Remark. This proof also shows that the closed n-ball and the closed n-interval are convex. □

3. Prove that the intersection of any collection of convex sets is convex.

   Solution: Let $X = \bigcap_{K \in F} K$ be an arbitrary intersection of convex sets $K \in F$ and suppose x and y belong to X. Then x and y belong to K for each $K \in F$ and therefore $\theta x + (1-\theta)y \in K$ for every $K \in F$. Hence, $\theta x + (1-\theta)y \in X$. □

4. Prove that the collection of isolated points of a set $S \subset \mathbf{R}^n$ must be countable.

   Solution: Around each isolated point $\mathbf{x} \in S$ is an open n-ball $B(\mathbf{x})$ which contains no other points of S. By theorem 3.27, there is an n-ball $A_{\mathbf{x}}$ with rational radius and rational center coordinates such that $\mathbf{x} \in A_{\mathbf{x}} \subset B(\mathbf{x})$. The map $\mathbf{x} \mapsto A_{\mathbf{x}}$ is a one-to-one correspondence between the isolated points of S and a subset of the countable set of all open n-balls with rational center and radius. □

5. (a) Give an example of a closed subset $U \subset \mathbf{R}$ which is not bounded and an infinite open cover F of U which has no finite subcover. (b) Give an example of a bounded subset $B \subset \mathbf{R}$ which is not closed and an infinite open cover F of B which has no finite subcover.

   Solution: For (a) take $U = \mathbf{R}$ and consider the open cover $\{(n-1, n+1) : n \in \mathbf{Z}\}$. Any finite subcover must be bounded both above and below, and thus could not cover $\mathbf{R}$.

   For (b) take $B = (0,1)$ and consider the open cover $\{(\frac{1}{n}, \frac{2}{n}) : n \in \mathbf{Z}^+\}$. Any finite subcover will give a union that is bounded below by a positive number γ and will thus omit $\gamma/2 \in (0,1)$. $\square$

For Problems 6 and 7, let S be a subset of $\mathbf{R}^n$ and define a *condensation point of* S to be any point $\mathbf{x} \in \mathbf{R}^n$ such that every n-ball centered at $\mathbf{x}$ contains uncountably many points of S.

6. Prove that every uncountable subset $S \subset \mathbf{R}^n$ contains a condensation point of S. (Hint: use the fact that the countable union of countable sets is countable).

   Solution: Suppose that no point of S is a condensation point of S. We will show that S must be countable. But then for each $\mathbf{x} \in S$ there is some n-ball $B(\mathbf{x})$ centered at $\mathbf{x}$ such that $A_{\mathbf{x}} \stackrel{\text{def}}{=} B(\mathbf{x}) \cap S$ is countable. Then $\{B(\mathbf{x}) : \mathbf{x} \in S\}$ is an open cover of $S \subset \mathbf{R}^n$ and has a countable subcover $S \subset \bigcup_{k=1}^{\infty} B(\mathbf{x}_k)$, defined by a countable subset $\{\mathbf{x}_k : k = 1, 2, \ldots\} \subset S$. But then $S = S \cap \bigcup_{k=1}^{\infty} B(\mathbf{x}_k) = \bigcup_{k=1}^{\infty} [S \cap B(\mathbf{x}_k)] = \bigcup_{k=1}^{\infty} A_{\mathbf{x}_k}$ is a countable union of countable sets $A_{\mathbf{x}_k}$, and is countable by theorem 2.27. $\square$

7. Assume that S is an uncountable subset of $\mathbf{R}^n$. Let T be the collection of all condensation points of S. Prove the following: (a) $S - T$ is countable; (b) $S \cap T$ is uncountable; (c) T is closed; (d) T contains no isolated points.

   Solution: For (a), note that if $S - T$ is uncountable then by Solution 6 it must contain a condensation point of $S - T$ and thus of S. This contradicts the assumption that T contains all the condensation points of S.

   For (b), note that S is the disjoint union of $S \cap T$ and $S - T$. Since $S - T$ is countable by part (a) but the union S is uncountable, we conclude that $S \cap T$ must be uncountable.

   For (c), let $\mathbf{x}$ be any accumulation point of T. Then any ball $B(\mathbf{x}; r)$ contains a point of $T - \{\mathbf{x}\}$, say $\mathbf{y}$. Since $\|\mathbf{y} - \mathbf{x}\| < r$, we can find $q > 0$ such that $q < r - \|\mathbf{y} - \mathbf{x}\|$. We claim that $B(\mathbf{y}; q) \subset B(\mathbf{x}; r)$: if $\mathbf{x} \in B(\mathbf{y}; q)$, then $\|\mathbf{x} - \mathbf{x}\| \le \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y} - \mathbf{x}\| < q + \|\mathbf{y} - \mathbf{x}\| < r - \|\mathbf{y} - \mathbf{x}\| + \|\mathbf{y} - \mathbf{x}\| = r$. But $B(\mathbf{y}; q)$ contains uncountably many points of S since $\mathbf{y} \in T$, so $B(\mathbf{x}; r)$ must also contain uncountably many points of S. Since $r > 0$ was chosen arbitrarily we have shown that $\mathbf{x} \in T$ and we conclude that T is closed.

   For (d), note that if $\mathbf{x} \in T$ is an isolated point then there is a ball $B(\mathbf{x})$ which contains no points of T besides $\mathbf{x}$. Consider the set $S_{\mathbf{x}} \stackrel{\text{def}}{=} S \cap (B(\mathbf{x}) - \{\mathbf{x}\}) \subset S$. This is uncountable, so by Solution 6 $S_{\mathbf{x}}$ must contain a condensation point $\mathbf{y}$ of $S_{\mathbf{x}}$. But $\mathbf{y} \in B(\mathbf{x})$ will also be a condensation point of S since $S_{\mathbf{x}} \subset S$, so $\mathbf{y} \in T$. But $\mathbf{y} \neq \mathbf{x}$ since $\mathbf{x} \notin S_{\mathbf{x}}$, which contradicts our assumption that the only point of T in $B(\mathbf{x})$ is $\mathbf{x}$. $\square$

For Problems 8–10, let $\|\mathbf{x}\|$ be the usual norm of $\mathbf{x} \in \mathbf{R}^n$, and define

$$\|\mathbf{x}\|_1 \stackrel{\text{def}}{=} \sum_{i=1}^{n} |x_i|, \qquad \|\mathbf{x}\|_\infty \stackrel{\text{def}}{=} \max_{i=1,2,\ldots,n} |x_i|$$

for any $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n$.

8. Show that $\|\cdot\|_1$ satisfies the norm axioms:

(i) $\|\mathbf{x}\|_1 \geq 0$, with equality if and only if $\mathbf{x} = 0$.

(ii) $\|a\mathbf{x}\|_1 = |a|\,\|\mathbf{x}\|_1$, for any scalar a and vector $\mathbf{x}$.

(iii) $\|\mathbf{x}+\mathbf{y}\|_1 \leq \|\mathbf{x}\|_1 + \|\mathbf{y}\|_1$ for any vectors $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$.

Solution: For (i), note that the sum of nonnegative numbers $|x_1| + \cdots + |x_n|$ is nonnegative and is zero if and only if all of them are zero. For (ii), compute

$$\|a\mathbf{x}\|_1 = \sum_{i=1}^{n} |ax_i| = \sum_{i=1}^{n} |a|\,|x_i| = |a| \sum_{i=1}^{n} |x_i| = |a|\,\|\mathbf{x}\|_1$$

For (iii), use the triangle inequality in $\mathbf{R}$: $|x_i + y_i| \leq |x_i| + |y_i|$ for each $i = 1, \ldots, n$, so

$$\sum_{i=1}^{n} |x_i + y_i| \leq \sum_{i=1}^{n} (|x_i| + |y_i|) = \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n} |y_i|.$$

$\square$

9. Show that $\|\cdot\|_\infty$ satisfies the three norm axioms of Problem 8.

Solution: For (i), note that the largest element of a set of nonnegative numbers $|x_1|, \ldots, |x_n|$ is nonnegative and is zero if and only if all of them are zero. For (ii), compute

$$\|a\mathbf{x}\|_\infty = \max_i |ax_i| = \max_i |a|\,|x_i| = |a| \max_i |x_i| = |a|\,\|\mathbf{x}\|_\infty$$

For (iii), use the triangle inequality in $\mathbf{R}$: $|x_i + y_i| \leq |x_i| + |y_i|$ for each $i = 1, \ldots, n$, so

$$\max_i |x_i + y_i| \leq \max_i (|x_i| + |y_i|) \leq \max_i |x_i| + \max_i |y_i|.$$

$\square$

10. Find four positive constants A, B, C, D such that for every $\mathbf{x}$ in $\mathbf{R}^n$ we have

$$A\|\mathbf{x}\| \leq \|\mathbf{x}\|_1 \leq B\|\mathbf{x}\| \qquad \text{and} \qquad C\|\mathbf{x}\| \leq \|\mathbf{x}\|_\infty \leq D\|\mathbf{x}\|.$$

For full credit, find the largest A and C and the smallest B and D for which these inequalities hold, and prove that no "better" numbers exist. (Hint: find examples in $\mathbf{R}^1$ or $\mathbf{R}^2$ which require the "best" constants A, B, C, D.)

Solution: We claim that $A = 1$, $B = \sqrt{n}$, $C = \frac{1}{\sqrt{n}}$ and $D = 1$ are the best constants:

$$\|\mathbf{x}\| \leq^1 \|\mathbf{x}\|_1 \leq^2 \sqrt{n}\|\mathbf{x}\| \qquad \text{and} \qquad \frac{1}{\sqrt{n}}\|\mathbf{x}\| \leq^3 \|\mathbf{x}\|_\infty \leq^4 \|\mathbf{x}\|.$$

But first we show that they work:

(1) $\|\mathbf{x}\| \leq \|\mathbf{x}\|_1$ holds because the squares of the two nonnegative sides satisfy

$$|x_1|^2 + \ldots + |x_n|^2 \leq \Big(|x_1| + \ldots + |x_n|\Big)^2.$$

All the additional cross terms on the right hand side are nonnegative.

(2) $\|\mathbf{x}\|_1 \leq \sqrt{n}\|\mathbf{x}\|$ holds by the Cauchy–Schwarz inequality (theorem 1.23), using $a_i = x_i$ and $b_i = 1$ for all $i = 1, \ldots, n$.

(3) $\frac{1}{\sqrt{n}}\|\mathbf{x}\| \leq \|\mathbf{x}\|_\infty$ holds because $|x_1|^2 + \ldots + |x_n|^2 \leq n \max\{|x_i|\}^2$.

(4) $\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|$ holds because $\max\{|x_i|^2\} \leq |x_1|^2 + \ldots + |x_n|^2$.

To show that no better constants exist, take $\mathbf{x} = (1, 0, \ldots, 0)$ for (1) and (4), and take $\mathbf{x} = (1, 1, \ldots, 1)$ for (2) and (3). $\square$