Math 410: Introduction to Fourier Series and Integrals
Final Examination

Name:________________________________

Due Monday, 18 December 2017 by 1:00pm at Cupples I, room 105a.

6 Problems on 1+6 Pages

You may use any pre-existing reference materials on the class website as well as the textbook, "Early Fourier Analysis" by Hugh L. Montgomery.

Recall that $e(x) = e^{2\pi i x}$ for any real number x, in Vinogradov's notation.

You may quote results from the textbook as justifications in your proofs.

Please cite theorems, corollaries, lemmas, equations and so on by number and page.

1. Suppose that f is a continuous function on $\mathbb{T}$.

   (a) Explain why $f \in L^1(\mathbb{T})$.

   (b) Show that $|\hat{f}(n)| \leq \|f\|_1$ for every $n \in \mathbf{Z}$.

   (c) Let $\sigma_N(x)$ be the Nth Cesàro sum of the Fourier series for f. Give a formula for $\sigma_N(x)$ in terms of the Fourier coefficients $\hat{f}(n)$.

   (d) State a theorem that asserts

$$f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e(nx) \qquad \text{(C)},$$

   for every $x \in \mathbb{T}$.

2. Suppose that $f \in L^2(\mathbb{T})$.

   (a) Prove that $f \in L^1(\mathbb{T})$ as well with the estimate $\|f\|_1 \leq \|f\|_2$.

   (b) Explain why $\hat{f}(n)$ exists with $|\hat{f}(n)| \leq \|f\|_2$ for every $n \in \mathbf{Z}$.

   (c) Further suppose that $\|f\|_2 = 3$. What theorem asserts that $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 = 9$?

3. Let $T \in \mathcal{T}_N$ be a trigonometric polynomial of degree N.

(a) Prove that T' is also a trigonometric polynomial of degree N.

(b) Suppose further that $|T(x)| \leq 1$ for all x. State a theorem asserting that

$$|T'(x)| \leq 2\pi N$$

for all x.

(b) Suppose further that $T(x)$ is purely real with $0 \leq T(x) \leq 1$ for all x. State a theorem that asserts the existence of numbers $a_0, a_1, \ldots, a_N$ such that

$$T(x) = \left| \sum_{n=0}^{N} a_n e(nx) \right|^2$$

for all x.

4. Let $f_N(x) = \sum_{n=1}^{N} (\frac{2}{3})^n \cos 2^n \pi x$ for positive integer N, and put

$$f(x) \stackrel{\text{def}}{=} \lim_{N \to \infty} f_N(x).$$

(a) Explain why $f(x)$ exists for every x, and why the resulting function f is continuous and thus belongs to $L^1(\mathbb{T})$.

(b) State a theorem asserting that f is everywhere continuous but nowhere differentiable.

5. Suppose $f \in L^1(\mathbf{R})$ and define the Fourier integral transform

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x) e(-tx)\, dx$$

for $t \in \mathbf{R}$.

(a) Prove that $|f(t)| \le \|f\|_1$ for all t.

(b) Suppose further that $\hat{f} \in L^1(\mathbf{R})$. Explain why f must be continuous and why

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(t) e(tx)\, dt$$

for every x.

(c) Suppose still further that $f \in L^2(\mathbf{R})$ as well. State a theorem asserting

$$\hat{f} \in L^2(\mathbf{R}) \quad \text{with} \quad \|\hat{f}\|_2 = \|f\|_2.$$

6. Suppose that $f(x) = e^{-\pi x^2}$ and let $\hat{f}$ denote the Fourier integral transform as in Problem 5.

   (a) Find $\hat{f}(t)$. (You may cite computations done in the textbook.)

   (b) Define $g(x) = f(2x - 3)$. Find $\hat{g}(t)$.

   (c) Define $h(x) = f''(x)$. Find $\hat{h}(t)$.