MATH 217 – WORKSHEET 08

Q.1 Use the table of Laplace transforms on p.208 to evaluate the following without integration:

(a) $L[\sinh ax](p) = \dfrac{a}{p^2 - a^2}$.

Solution: Since $\sinh z = \frac{1}{2}[e^z - e^{-z}]$, use linearity with the formula $L[e^{ax}](p) = \frac{1}{p-a}$ to get

$$L[\sinh ax](p) = \frac{1}{2}[\frac{1}{p-a} - \frac{1}{p+a}]$$

and then simplify to get the claimed result.

(b) $L[\cosh ax](p) = \dfrac{p}{p^2 - a^2}$.

Solution: Since $\cosh z = \frac{1}{2}[e^z + e^{-z}]$, use linearity with the formula $L[e^{ax}](p) = \frac{1}{p-a}$ to get

$$L[\cosh ax](p) = \frac{1}{2}[\frac{1}{p-a} + \frac{1}{p+a}]$$

and then simplify to get the claimed result.

(c) Use the double angle formula $\cos 2z = \cos^2 z - \sin^2 z$ to find $L[\cos^2 ax](p)$ and $L[\sin^2 ax](p)$ without integration.

Solution: Write $\cos 2z = 1 - 2\sin^2 z = 2\cos^2 x - 1$ and use linearity with the formula $L[\cos ax](p) = \frac{p}{p^2 a^2}$ and $L[1](p) = \frac{1}{p}$ to get

$$L[\cos^2 ax](p) = \frac{1}{2}[\frac{1}{p} + \frac{p}{p^2 + 4a^2}]$$

and

$$L[\sin^2 ax](p) = \frac{1}{2}[\frac{1}{p} - \frac{p}{p^2 + 4a^2}]$$

As a check, compute

$$\frac{1}{p} = L[1](p) = L[\sin^2 ax + \cos^2 ax](p) = L[\sin^2 ax](p) + L[\cos^2 ax](p)$$

using the two formulas.

Q.2 Find each function $f(x)$ whose Laplace transform $L[f](p)$ is given.

(a) $30/p^4$

Solution: If $f(x) = 5x^3$, then $L[f](p) = 5(3!)/p^{3+1} = 30/p^4$.

(b) $1/(p^2 + p)$

Solution: Write

$$\frac{1}{p^2 + p} = \frac{1}{p(p+1)} = \frac{1}{p} - \frac{1}{p+1}$$

from which it follows that $f(x) = 1 - e^{-x}$.

(c) $1/(p^4+p^2)$

Solution: Write

$$\frac{1}{p^4+p^2} = \frac{1}{p^2(p^2+1^2)} = \frac{1}{p^2} - \frac{1}{p^2+1^2}$$

from which it follows that $f(x) = x - \sin x$.

(d) $2/(p+3)$

Solution: Write

$$\frac{2}{p+3} = 2\frac{1}{p-(-3)}$$

from which it follows that $f(x) = 2e^{-3x}$.

Q.3 Find the Laplace transform $L[f](p)$ for each of the functions $f(x)$ given.

(a) 17

Solution: $17/p$

(b) $x^2 + \cos 5x$

Solution: $2/p^3 + p/(p^2+25)$

(c) $3e^{2x} - 4\sin x \cos x$

Solution: Use $2\sin x \cos x = \sin 2x$ to compute

$$L[3e^{2x} - 4\sin x\cos x](p) = 3L[e^{2x}](p) - 2L[\sin 2x](p) = \frac{3}{p-2} - \frac{2(2)}{p^2+2^2}$$

(d) $x^5\cos^2 5x + x^5 \sin^2 5x$

Solution: Use $\cos^2\theta = \sin^2\theta = 1$ for all θ to simplify:

$$L[x^5\cos^2 5x + x^5\sin^2 5x](p) = L[x^5](p) = \frac{5!}{p^{5+1}} = \frac{120}{p^6}$$

Q.4 Find the Laplace transforms of the following functions:

(a) x^5e^{5x}

Solution: By Eq.8.4, p.212, and the table on p.208, this is $\dfrac{5!}{(x-5)^6}$.

(b) $(1-x^2)e^{-x}$

Solution: By Eq.8.4, p.212, and the table on p.208, this is

$$\frac{1}{p+1} - \frac{2!}{(p+1)^3} = \frac{p^2+2p-3}{(p+1)^3} = \frac{(p+3)(p-1)}{(p+1)^3}$$

(c) $x\sin 3x$

Solution: This may be done directly from the definition. Let

$$Y = L[x\sin 3x](p) = \int_0^\infty e^{-px} x \sin 3x \, dx$$

and use integration by parts twice to get an equation for Y. Namely, putting $u = x\sin 3x$ and $dv = e^{-px}\,dx$, get $v = -\frac{1}{p}e^{-px}$ and $dv = (\sin 3x + x\cos 3x)$, so

$$\begin{aligned} Y &= uv|_{x=0}^{\infty} - \int_{x=0}^{\infty} v\,du \\ &= 0 + \frac{1}{p}\int_{x=0}^{\infty} e^{-px}(\sin 3x + 3x\cos 3x)\,dx, \\ &= \frac{1}{p}\int_{x=0}^{\infty} e^{-px}\sin 3x\,dx + \frac{3}{p}\int_{x=0}^{\infty} e^{-px}x\cos 3x\,dx, \end{aligned}$$

since $uv = 0$ at $x = 0$ and $uv \to 0$ as $x \to \infty$. Now

$$\int_{x=0}^{\infty} e^{-px}\sin 3x\,dx = L[\sin 3x](p) = \frac{3}{p^2+3^2}$$

from the table on p.208, while a second integration by parts gives

$$\begin{aligned} \int_{x=0}^{\infty} e^{-px}x\cos 3x\,dx &= 0 + \frac{1}{p}\int_{x=0}^{\infty} e^{-px}(\cos 3x - 3x\sin 3x)\,dx \\ &= \frac{1}{p}\int_{x=0}^{\infty} e^{-px}\cos 3x\,dx - \frac{3}{p}\int_{x=0}^{\infty} e^{-px}x\sin 3x\,dx, \\ &= \frac{1}{p}\left[\frac{p}{p^2+3^3} - 3Y\right], \end{aligned}$$

using the fact that $e^{-px}x\cos 3x$ vanishes at $x = 0$ and as $x \to \infty$, with $L[\cos 3x](p)$ from the table on p.208. Combining this with the earlier computation gives

$$Y = \frac{1}{p}\left[\frac{3}{p^2+3^2}\right] + \frac{3}{p^2}\left[\frac{p}{p^2+3^2} - 3Y\right] = \frac{6}{p(p^2+3^2)} - \frac{3^2}{p^2}Y,$$

which is easily solved for Y to give

$$Y = L[x\sin 3x](p) = \frac{6p}{(p^2+3^2)^2}.$$

Alternatively, we may use Euler's formula to write

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}, \implies x\sin 3x = \frac{1}{2i}\left[xe^{i3x} - xe^{-i3x}\right],$$

and then use Eq.8.4 on p.212 of the text book to compute

$$\begin{aligned} L[x\sin 3x](p) &= \frac{1}{2i}\Big(L[e^{i3x}x](p) - L[e^{-i3x}x](p)\Big) \\ &= \frac{1}{2i}\left[\frac{1}{(p-3i)^2} - \frac{1}{(p+3i)^2}\right] = \frac{1}{2i}\left[\frac{(p+3i)^2 - (p-3i)^2}{(p-3i)^2(p+3i)^2}\right] \\ &= \frac{1}{2i}\left[\frac{(p^2+6ip+(3i)^2) - (p^2-6ip+(3i)^2)}{(p^2-(3i)^2)^2}\right] \\ &= \frac{1}{2i}\left[\frac{12ip}{(p^2+3^2)^2}\right] = \frac{6p}{(p^2+3^2)^2}, \end{aligned}$$

after some algebra, in agreement with the earlier method.

Yet another way to compute this Laplace transform is to write $g(a,x) = -\cos ax$ and note that

$$f(x) = x\sin ax = \frac{\partial}{\partial a} g(a,x).$$

Since $L[-\cos ax](p) = -p/(p^2+a^2)$, we may find $L[x\sin 3x](p)$ by differentiating $-p/(p^2+a^2)$ with respect to a and then evaluating at $a=3$:

$$L[x\sin 3x](p) = \frac{\partial}{\partial a}\left[\frac{-p}{p^2+a^2}\right]\Big|_{a=3} = \frac{2ap}{(p^2+a^2)^2}\Big|_{a=3} = \frac{6p}{(p^2+3^2)^2},$$

just as by the other two methods.

Remark. Macsyma has a command `laplace(f(x),x,p)` that finds the Laplace transform for $f(x)$ and returns a function of p.

Q.5 Find the function $f(x)$ given the Laplace transform $L[f](p)$ below:

(a) $\dfrac{6}{(p+2)^2+9}$

Solution: $f(x) = 2e^{-2x}\sin 3x$.

(b) $\dfrac{p}{4p^2+1}$

Solution: $f(x) = \frac{1}{4}\cos(x/2)$.

(c) $\dfrac{p+3}{p^2+2p+5}$

Solution: Observe that

$$\frac{p+3}{p^2+2p+5} = \frac{p}{(p+1)^2+4} + \frac{3}{(p+1)^2+4}$$

Recognizing these transforms from the table on p.208, and using Eq.8.4 on p.212, gives $f(x) = e^{-x}\cos 2x + \frac{3}{2}e^{-x}\sin 2x$.

(d) $\dfrac{1}{p^4+3p^2+2}$

Solution: Factor the denominator polynomial and then use partial fractions to decompose

$$\frac{1}{p^4+3p^2+2} = \frac{1}{(p^2+1)(p^2+2)} = \frac{1}{p^2+1} - \frac{1}{p^2+2}$$

Recognize the Laplace transforms of $\sin x$ and $\sin(\sqrt{2}x)$ to get

$$f(x) = L^{-1}\left[\frac{1}{p^2+1} - \frac{1}{p^2+2}\right](x) = \sin x - \frac{1}{\sqrt{2}}\sin(\sqrt{2}x)$$

Remark. Macsyma has a command `ilt(g(p),p,x)` that finds the inverse Laplace transform of $g(p)$ and returns a function of x.