MATH 217 – WORKSHEET 06

Q.1 For each of the following ODEs, verify that the origin is a regular singular point and find two linearly independent Frobenius series solutions.

(a) $2xy'' + (3 - x)y' - y = 0$

Solution: Standard form $y'' + py' + qy = 0$:

$$y'' + \frac{3-x}{2x}y' - \frac{1}{2x}y = 0, \implies p(x) = \frac{1}{x}(\frac{3}{2} - \frac{1}{2}x), q(x) = \frac{1}{x^2}(0 - \frac{1}{2}x).$$

Both $xp(x)$ and $x^2q(x)$ are polynomials, hence real analytic. Thus 0 is a regular singular point, with $p_0 = \frac{3}{2}$ and $q_0 = 0$, so the indicial equation is

$$0 = f(m) = m(m-1) + p_0 m + q_0 = m^2 + \frac{1}{2}m = m(m + \frac{1}{2}),$$

with roots $m = 0$ and $m = -1/2$.

For $m = 0$ suppose $y(x) = \sum_{j=0}^{\infty} a_j x^j$. Then the ODE implies

$$\sum_{j=0}^{\infty}[2j(j-1)a_j x^{j-1} + 3ja_j x^{j-1} - ja_j x^j - a_j x^j] = 0$$

which, after re-indexing and collection of terms in x^j yields

$$\sum_{j=0}^{\infty}[(2(j+1)j + 3(j+1))a_{j+1} - (j+1)a_j]x^j = 0,$$

from which we conclude that $a_{j+1} = a_j/(2j+3)$, so for $j = 1, 2, \ldots,$

$$a_j = \frac{a_0}{(3)(5)\cdots(2j+1)} = \frac{a_0}{(2j+1)!!} = \frac{2^j j!}{(2j+1)!}a_0.$$

This gives a Frobenius solution

$$\boxed{y_1(x) = \sum_{j=0}^{\infty} \frac{x^j}{(2j+1)!!}}$$

which is real analytic with infinite radius of convergence.

For $m = -1/2$ suppose $y(x) = \sum_{j=0}^{\infty} a_j x^{j-1/2}$. Then the ODE implies

$$x^{-1/2}\sum_{j=0}^{\infty}[2(j - \frac{1}{2})(j - \frac{3}{2})a_j x^{j-1} + 3(j - \frac{1}{2})a_j x^{j-1} - (j - \frac{1}{2})a_j x^j - a_j x^j] = 0$$

which, after canceling $x^{-1/2}$, re-indexing, and collection of terms in x^j yields

$$\sum_{j=0}^{\infty}[(2(j + \frac{1}{2})(j - \frac{1}{2}) + 3(j + \frac{1}{2}))a_{j+1} - (j + \frac{1}{2})a_j]x^j = 0,$$

from which we conclude that $a_{j+1} = a_j/(2j+2)$, so for $j = 1, 2, \ldots,$

$$a_j = \frac{a_0}{(2)(4)\cdots(2j)} = \frac{a_0}{2^j j!}.$$

Recognize the resulting power series for $e^{x/2}$ to obtain another linearly independent Frobenius solution

$$\boxed{y_2(x) = x^{-1/2}\sum_{j=0}^{\infty}\frac{x^j}{2^j j!} = x^{-1/2}e^{x/2}}$$

Remark. Using the technique from Section 4.4 of our textbook, it is possible to find a functional expression for y_1:

$$y_1(x) = y_2(x)\mathrm{erf}\left(\sqrt{\frac{x}{2}}\right) = x^{-1/2}e^{x/2}\mathrm{erf}(\sqrt{x/2}),$$

where $\mathrm{erf}(u) = \frac{2}{\sqrt{\pi}}\int_0^u e^{-t^2}\,dt$ is a well-known special function.

(b) $2x^2y'' + xy' - (x+1)y = 0$

Solution: Standard form $y'' + py' + qy = 0$:

$$y'' + \frac{1}{2x}y' - \frac{x+1}{2x^2}y = 0, \implies p(x) = \frac{1}{x}(\frac{1}{2}), q(x) = \frac{1}{x^2}(-\frac{1}{2} - \frac{1}{2}x).$$

Both $xp(x)$ and $x^2q(x)$ are polynomials, hence real analytic. Thus 0 is a regular singular point, with $p_0 = \frac{1}{2}$ and $q_0 = -\frac{1}{2}$, so the indicial equation is

$$0 = f(m) = m(m-1) + p_0m + q_0 = m^2 - \frac{1}{2}m - \frac{1}{2} = (m + \frac{1}{2})(m-1),$$

with roots $m = 1$ and $m = -1/2$.

For $m = -1/2$ suppose $y(x) = \sum_{j=0}^{\infty} a_j x^{j-1/2}$. Then the ODE implies

$$x^{-1/2}\sum_{j=0}^{\infty}[2(j-\frac{1}{2})(j-\frac{3}{2})a_jx^j + (j-\frac{1}{2})a_jx^j - a_jx^{j+1} - a_jx^j] = 0$$

which, after canceling $x^{-1/2}$, re-indexing, and collection of terms in x^j yields

$$\sum_{j=0}^{\infty}[(2(j-\frac{1}{2})(j-\frac{3}{2}) + (j-\frac{1}{2}) - 1)a_j - a_{j-1}]x^j = 0,$$

from which we conclude that $a_j = a_{j-1}/(2j^2 - 3j)$, so $a_1 = -a_0$, and for $j = 2, 3, \ldots,$

$$a_j = \frac{a_0}{j!(-1)(1)(3)\cdots(2j-3)} = \frac{-a_0}{j!(1)(3)\cdots(2j-3)}.$$

This gives a Frobenius solution

$$\boxed{y_1(x) = x^{-1/2}\left[1 - x - \sum_{j=2}^{\infty} \frac{x^j}{j!(2j-3)!!}\right]}$$

where $(2j-3)!! = (1)(3)(5)\cdots(2j-3)$ is the product of the odd integers up to $2j-3$.

For $m = 1$ suppose $y(x) = \sum_{j=0}^{\infty} a_j x^{j+1}$. Then the ODE implies

$$x^1 \sum_{j=0}^{\infty} [2j(j+1)a_j x^j + (j+1)a_j x^j - a_j x^{j+1} - a_j x^j] = 0$$

which, after re-indexing, cancellation of the leading x^1 factor, and collection of terms in x^j yields

$$\sum_{j=0}^{\infty} [(2(j+1)j + (j+1) - 1)a_j - a_{j-1}]x^j = 0,$$

from which we conclude that $a_j = a_{j-1}/(2j^2 + 3j)$, so for $j = 1, 2, \ldots,$

$$a_j = \frac{a_0}{j!(5)\cdots(2j+3)} = \frac{3a_0}{j!(2j+3)!!}.$$

This gives a Frobenius solution

$$\boxed{y_2(x) = x \sum_{j=0}^{\infty} \frac{3x^j}{j!(2j+3)!!}}$$

which is real analytic with infinite radius of convergence.

Remark. The factor 3 in each summand's numerator is not necessary.

Q.2 Find the Fourier series for the functions below:

(a) $f(x) = 0$ if $-\pi \le x < 0$, while $f(x) = \sin x$ if $0 \le x < \pi$.

Solution: We have

$$a_j = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos jx\, dx = \frac{1}{\pi}\int_0^{\pi} \sin x \cos jx\, dx,$$

so $a_0 = 2/\pi$. For $j > 0$, use the identity $2\sin A\cos B = \sin(A+B) + \sin(A-B)$ to compute

$$a_j = \frac{1}{\pi}\int_0^{\pi} \frac{1}{2}[\sin(1+j)x + \sin(1-j)x]\, dx = \begin{cases} 0, & j > 0 \text{ odd}, \\ \frac{-2}{\pi(j^2-1)}, & j > 0 \text{ even}. \end{cases}$$

Similarly,

$$b_j = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin jx\, dx = \frac{1}{\pi}\int_0^{\pi} \sin x \sin jx\, dx,$$

so $b_1 = 1/2$. For $j > 1$, use the identity $2\sin A \sin B = \cos(A-B) - \cos(A+B)$ to compute

$$b_j = \frac{1}{\pi}\int_0^\pi \frac{1}{2}[\cos(1-j)x - \cos(1+j)x]\,dx = \frac{1}{2\pi}\left[\frac{\sin(1-j)x}{1-j} - \frac{\sin(1+j)x}{1+j}\right]_0^\pi = 0,$$

since $\sin 0 = 0$ and $\sin n\pi = 0$ at all integers n.

(b) $f(x) = 0$ if $-\pi \le x < 0$, while $f(x) = \cos x$ if $0 \le x < \pi$.

Solution: We have

$$a_j = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos jx\,dx = \frac{1}{\pi}\int_0^\pi \cos x \cos jx\,dx,$$

so $a_0 = 0$ and $a_1 = 1/2$. For $j > 1$, use $2\cos A\cos B = \cos(A+B) + \cos(A-B)$ to compute

$$a_j = \frac{1}{\pi}\int_0^\pi \frac{1}{2}[\cos(1+j)x + \cos(1-j)x]\,dx = \frac{1}{2\pi}\left[\frac{\sin(1+j)x}{1+j} + \frac{\sin(1-j)x}{1-j}\right]_0^\pi = 0,$$

since $\sin 0 = 0$ and $\sin n\pi = 0$ at all integers n. Similarly,

$$b_j = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin jx\,dx = \frac{1}{\pi}\int_0^\pi \cos x \sin jx\,dx.$$

Use the identity $2\sin A\cos B = \sin(A+B) + \sin(A-B)$ as in part (a) above to compute

$$b_j = \frac{1}{\pi}\int_0^\pi \frac{1}{2}[\sin(j+1)x + \sin(j-1)x]\,dx = \begin{cases} 0, & j > 0 \text{ odd},\\ \frac{2j}{\pi(j^2-1)}, & j > 0 \text{ even}.\end{cases}$$

Q.3 Find the Fourier series for the functions below:

(a) $f(x) = -1$ if $-\pi \le x < 0$, while $f(x) = +1$ if $0 \le x < \pi$.

Solution: Clearly $a_0 = 0$ by antisymmetry. For $j > 0$, compute

$$a_j = \frac{1}{\pi}\int_{-\pi}^0 (-1)\cos jx\,dx + \frac{1}{\pi}\int_0^\pi (+1)\cos jx\,dx = 0,$$

while

$$b_j = \frac{1}{\pi}\int_{-\pi}^0 (-1)\sin jx\,dx + \frac{1}{\pi}\int_0^\pi (+1)\sin jx\,dx = \frac{4}{\pi j}.$$

(b) $g(x) = x - \frac{\pi}{2}$ if $-\pi \le x < 0$, while $g(x) = \frac{\pi}{2} - x$ if $0 \le x < \pi$.

Solution: First notice that $a_0 = 0$ by cancellation.

The remaining coefficients may be found directly using integration by parts as in Example 7.7.7, p.167 of the textbook, but it is faster to notice that $g'(x) = -f(t)$, where f is the function from part (a) above. The Fourier series for g may be differentiated term by term to give the series for $-f$, and corresponding coefficients may be identified: $b_j = 0$ for all $j > 0$ since these are proportional to the cosine coefficients of f, while

$$a_j = \frac{4}{\pi j^2}, \qquad j = 1, 2, 3, \dots$$

since, when multiplied by j, these are the sine coefficients of f.

Q.4 Let f be the 2π periodic function defined on $[-\pi, \pi)$ by

$$f(x) = \begin{cases} 0, & -\pi \le x < 0, \\ x^2, & 0 \le x < \pi \end{cases}$$

(a) Find the Fourier series for f.

Solution: Integrate twice by parts, which may be done by computer algebra software such as Macsyma, to get

$$b_j = \frac{1}{\pi}\int_0^\pi x^2 \sin jx\, dx = \begin{cases} -\pi/j, & j \text{ even}, \\ \\ \dfrac{\pi^2 j^2 - 4}{\pi j^3}, & j \text{ odd}, \end{cases}$$

while $a_0 = \pi^2/3$ and for $j > 0$,

$$a_j = \frac{1}{\pi}\int_0^\pi x^2 \cos jx\, dx = \frac{2(-1)^j}{j^2}.$$

(b) Use Dirichlet's theorem (Th.7.2.7, p.177 in the textbook) with the results from part (a) at $x = 0$ to prove that

$$\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$$

Solution: By Dirichlet's theorem, the Fourier series for f converges to $f(x)$ at each point x where f is continuous. Thus we may evaluate $f(0) = 0$ using the Fourier seriews from part (a).

Since all terms $b_j \sin jx = 0$ at $x = 0$, only the cosine terms remain. But $\cos jx = 1$ for all j if $x = 0$, giving

$$0 = \frac{1}{2}\frac{\pi^2}{3} + \sum_{j=1}^\infty \frac{2(-1)^j}{j^2}, \qquad \Longrightarrow \frac{\pi^2}{12} = \sum_{j=1}^\infty \frac{(-1)^{j+1}}{j^2}$$

as claimed.

(c) Use Dirichlet's theorem with the part (a) series at $x = \pi$ to prove that

$$\sum_{n=1}^\infty \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$

Solution: For this result, observe that the Fourier series for $f(\pi)$ converges to the midpoint $[f(\pi-)+f(\pi-)]/2 = \pi^2/2$ for the periodic function f, which is discontinuous at $x=\pi$ with limit $f(\pi+)=0$. But then, since the $\sin jx$ terms all vanish at $x=\pi$, we have

$$\frac{\pi^2}{2} = \frac{1}{2}\frac{\pi^2}{3} + \sum_{j=1}^{\infty} \frac{2(-1)^{j+1}}{j^2}\cos j\pi = \frac{\pi^2}{6} + 2\sum_{j=1}^{\infty}\frac{1}{j^2},$$

from which the result follows.

(d) Derive part (c) from part (b). (Hint: add $2\sum_n (1/[2n]^2)$ to both sides.)

Solution: First, write $B = \sum_{n=1}^{\infty} 1/n^2$ and note that

$$2\sum_{n=1}^{\infty}\frac{1}{[2n]^2} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{2}B.$$

Then, following the hint, add this quantity to the alternating series in part (b) to get

$$B = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} + 2\sum_{n=1}^{\infty}\frac{1}{[2n]^2} = \frac{\pi^2}{12} + \frac{1}{2}B,$$

from which the result follows.

Q.5 Determine whether the following functions are odd, even, or neither.

$$x^5\sin x,\quad x^2\sin 2x,\quad e^x,\quad (\sin x)^3,\quad \sin x^2,\quad \cos(x+x^3),\quad \frac{\sin x}{x},\quad x+x^2+x^3,\quad \ln\frac{1+x}{1-x}$$

Solution: Odd:

$$x^2\sin 2x,\quad (\sin x)^3,\quad \ln\frac{1+x}{1-x}$$

Even:

$$x^5\sin x,\quad \sin x^2,\quad \cos(x+x^3),\quad \frac{\sin x}{x},$$

Neither:

$$e^x,\quad x+x^2+x^3,$$

Q.6 Let $f(x)=\pi/4$ be the constant function.

(a) Show that the sine series for f is

$$\frac{\pi}{4} = \sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots, \qquad 0 < x < \pi.$$

Solution: Compute the sine series coefficients

$$b_n = \frac{2}{\pi}\int_0^{\pi}\frac{\pi}{4}\sin nx\,dx = \begin{cases} 0, & n \text{ even},\\ 1/n, & n \text{ odd}.\end{cases}$$

Thus, since only the odd terms are nonzero, on the right half interval,

$$\frac{\pi}{4} = f(x) = \sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots, \qquad 0 < x < \pi,$$

as claimed.

(b) Let $x = \pi/2$ in part (a) and deduce an infinite sum formula.

Solution: By Dirichlet's theorem, the Fourier sine series for $f(x)$ converges to $f(\pi/2) = \pi/4$ at $x = \pi/2$. But $\sin[(2k+1)\pi/2] = (-1)^k$ in that series, so we have

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

(c) Find the cosine series for f.

Solution: The even extension of f to $[-\pi, \pi]$ is the constant periodic function $f(x) = \pi/4$, all x. Therefore its Fourier cosine series contains only one nonzero coefficient, the constant term $a_0 = \pi/2$, with $a_n = 0$ for all $n > 0$.