MATH 217 – WORKSHEET 05

Q.1 If $p \neq 0$ and p is not a positive integer, show that the power series

$$\sum_{j=1}^{\infty} \frac{p(p-1)(p-2)\cdots(p-j+1)}{j!}\, x^j$$

converges for $|x| < 1$ and diverges for $|x| > 1$.

Solution: Use the ratio test for $\sum a_j$ with

$$a_j = \frac{p(p-1)(p-2)\cdots(p-j+1)}{j!}\, x^j.$$

Then

$$\frac{a_{j+1}}{a_j} = \frac{p(p-1)(p-2)\cdots(p-(j+1)+1)}{p(p-1)(p-2)\cdots(p-j+1)}\,\frac{j!}{(j+1)!}\,\frac{x^{j+1}}{x^j} = \frac{p-j}{j+1}x \;\to\; -x,$$

as $j \to \infty$. By the ratio test, the series converges absolutely if $|x| < 1$ and diverges if $|x| > 1$. The case $|x| = 1$ is indeterminate.

Q.2 By the geometric sum formula, we have the power series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots, \qquad \text{and} \qquad \frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots,$$

and these converge absolutely if $|x| < 1$. Show the following:

(a) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

Solution: Integrate the power series for $1/(1+x)$ term by term.

(b) $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

Solution: Substitute $x \leftarrow x^2$ into the power series for $1/(1+x)$ and then integrate term by term.

(c) $\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$

Solution: Differentiate the power series for $1/(1-x)$ term by term.

Q.3 For each of the following ODEs, find a power series solution of the form $\sum_j a_j x^j$. Then solve the equation by some method from Chapter 2.

(a) $y' = xy$

Solution: Differentiate the power series term by term to get

$$\sum_{j=1}^{\infty} j a_j x^{j-1} = \sum_{j=0}^{\infty} a_j x^{j+1} \implies a_1 + \sum_{j=1}^{\infty}(j+1)a_{j+1}x^j = \sum_{j=1}^{\infty} a_{j-1}x^j.$$

Equate powers of x to get $a_1 = 0$ and $a_{j+1} = a_{j-1}/(j+1)$ for $j = 0, 1, 2, \ldots$. Solve the recursion to get

$$a_j = \begin{cases} 0, & j \text{ odd}, \\ \dfrac{a_0}{2^k k!}, & j = 2k \text{ even}, \end{cases}$$

where the first coefficient a_0 is arbitrary. Thus

$$y(x) = a_0 \sum_{k=0}^{\infty} \frac{x^{2k}}{2^k k!} = a_0 \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{x^2}{2}\right)^k = a_0 \exp \frac{x^2}{2},$$

recognizing the power series for e^x composed with $x^2/2$.

Solve by separation of variables:

$$y' = xy \implies \frac{dy}{y} = x\,dx \implies \ln|y| = \frac{x^2}{2} + C \implies y(x) = A \exp \frac{x^2}{2},$$

in agreement with the power series solution.

(b) $y' - y = x$

Solution: Differentiate the power series term by term to get

$$\sum_{j=1}^{\infty} j a_j x^{j-1} - \sum_{j=0}^{\infty} a_j x^j = x \implies \sum_{j=0}^{\infty} (j+1) a_{j+1} x^j - \sum_{j=0}^{\infty} a_j x^j = x.$$

Equate powers of x to get $a_1 = a_0$ and $a_j = (1 + a_0)/j!$ for $j = 2, 3, \ldots$. Then

$$y(x) = \sum_{j=0}^{\infty} \frac{1 + a_0}{j!} x^j - 1 - x = (1 + a_0)e^x - 1 - x,$$

recognizing the power series of e^x and noting that the x^0 and x^1 terms need adjustment since $a_0 = (1 + a_0)/0! - 1$ and

$$a_1 x = a_0 x = \frac{1 + a_0}{1!} x - x.$$

Alternatively, solve by integrating factor $\rho = \exp \int (-1)\,dx = \exp(-x)$ to get

$$y(x) = \frac{1}{\rho} \int x\rho\,dx = \frac{-xe^{-x} - e^{-x} + C}{e^{-x}} = -x - 1 + Ce^x,$$

in agreement with the power series solution.

(c) $xy' = y$

Solution: Differentiate the power series term by term to get

$$\sum_{j=1}^{\infty} j a_j x^j = \sum_{j=0}^{\infty} a_j x^j$$

Equate powers of x to get $a_0 = 0$, a_1 is arbitrary, and $a_j = 0$ for $j = 2, 3, \ldots$ because it must satisfy $a_j = j a_j$. Then $y(x) = a_1 x$ is the finite power series, or polynomial solution.

Solve by separation of variables:

$$xy' = y \implies \frac{dy}{y} = \frac{dx}{x} \implies \ln|y| = \ln|x| + C \implies y(x) = Ax,$$

for some constant A, in agreement with the power series solution.

(d) $y' - (1/x)y = x^2$

Solution: Rewrite as $xy' - y = x^3$, then differentiate the power series term by term to get

$$\sum_{j=0}^{\infty} j a_j x^j - \sum_{j=0}^{\infty} a_j x^j = x^3$$

Equate powers of x to get $a_0 = 0$, a_1 is arbitrary, $a_2 = 0$, $a_3 = 1/2$, and $a_j = 0$ for $j = 4, 5, 6 \ldots$ because it must satisfy $a_j = j a_j$. Then

$$y(x) = a_1 x + \frac{x^3}{2}$$

is the finite power series, or polynomial solution.

Solve by integrating factor $\rho = \exp \int (-1/x)\, dx = 1/x$ to get

$$y(x) = \frac{1}{\rho} \int x\rho\, dx = \frac{x^2/2 + C}{1/x} = \frac{x^3}{2} + Cx,$$

in agreement with the power series solution.

Q.4 In each of the following problems, verify that 0 is an ordinary point. Then find the power series solution of the ODE.

(a) $y'' - xy' - y = 0$

Solution: Rewrite as $y'' + P(x)y' + Q(x)y = 0$ to identify coefficient functions $P(x) = -x$ and $Q(x) = -1$. These are both polynomials and have finite, hence convergent, power series about $x = 0$. Hence 0 is an ordinary point.

Now let $y = \sum_{j=0}^{\infty} a_j x^j$. Differentiate the power series term by term to get

$$\sum_{j=2}^{\infty} j(j-1) a_j x^{j-2} - \sum_{j=1}^{\infty} j a_j x^j - \sum_{j=0}^{\infty} a_j x^j = 0,$$

which implies, after reindexing, that

$$\sum_{j=0}^{\infty} \left[(j+2)(j+1) a_{j+2} - (j+1) a_j\right] x^j = 0.$$

Equating x^j terms for $j = 0, 1, 2, 3, \ldots$ gives the recursion $a_{j+2} = a_j/(j+2)$, solved by

$$a_j = \begin{cases} \frac{a_0}{2\times 4\times\cdots\times(2k)} = \frac{a_0}{2^k k!}, & j = 2k \geq 2 \text{ even}, \\ \frac{a_1}{3\times 5\times\cdots\times(2k+1)} = \frac{a_1 2^k k!}{(2k+1)!}, & j = 2k+1 \geq 3 \text{ odd}. \end{cases}$$

Thus

$$y(x) = a_0 \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{x^2}{2}\right)^k + a_1 x \sum_{k=0}^{\infty} \frac{k!}{(2k+1)!} \left(2x^2\right)^k$$

(b) $(1+x^2)y'' + xy' + y = 0$

Solution: Rewrite as $y'' + P(x)y' + Q(x)y = 0$ to identify coefficient functions $P(x) = x/(1+x^2)$ and $Q(x) = 1/(1+x^2)$. These are both rational functions with nonvanishing denominators and thus have convergent power series about $x = 0$. Hence 0 is an ordinary point.

Now let $y = \sum_{j=0}^{\infty} a_j x^j$. Differentiate the power series term by term to get

$$\sum_{j=2}^{\infty} j(j-1)a_j x^{j-2} + \sum_{j=2}^{\infty} j(j-1)a_j x^j + \sum_{j=1}^{\infty} j a_j x^j + \sum_{j=0}^{\infty} a_j x^j = 0,$$

which implies, after reindexing and reduction, that

$$\sum_{j=0}^{\infty} \left[(j+2)(j+1)a_{j+2} + (j^2+1)a_j\right] x^j = 0.$$

Equating x^j terms for $j = 0, 1, 2, 3, \ldots$ gives the recursion

$$a_{j+2} = \frac{j^2+1}{(j+1)(j+2)} a_j, \qquad j = 0, 1, 2, \ldots.$$

Compute

$$\lim_{j\to\infty} \frac{a_{j+2}}{a_j} = \lim_{j\to\infty} \frac{j^2+1}{(j+1)(j+2)} = 1$$

and apply the ratio test to conclude that the power series $y(x)$ converges for $|x| < 1$.

Q.5 *Chebyshev's equation* is $(1-x^2)y'' - xy' + p^2 y = 0$, where p is constant.

(a) Find two linearly independent solutions valid for $|x| < 1$.

Solution: Let $y = \sum_{j=0}^{\infty} a_j x^j$. Differentiate the power series term by term to get

$$\sum_{j=2}^{\infty} j(j-1)a_j x^{j-2} - \sum_{j=2}^{\infty} j(j-1)a_j x^j - \sum_{j=1}^{\infty} j a_j x^j + \sum_{j=0}^{\infty} p^2 a_j x^j = 0,$$

which implies, after reindexing and reduction, that

$$\sum_{j=0}^{\infty} \left[(j+2)(j+1)a_{j+2} + (j^2-p^2)a_j\right] x^j = 0.$$

Equating x^j terms for $j = 0, 1, 2, 3, \ldots$ gives the recursion

$$a_{j+2} = \frac{j^2-p^2}{(j+1)(j+2)} a_j, \qquad j = 0, 1, 2, \ldots.$$

The even-indexed coefficients a_{2k} are all multiples of a_0, while the odd-indexed a_{2k+1} are multiples of a_1. Thus the solution may be written as

$$y(x) = a_0 y_0(x) + a_1 y_1(x),$$

where y_0 is an even function (only even powers) and y_1 is an odd function (only odd powers). Both are nonzero, hence they must be linearly independent.

Compute

$$\lim_{j\to\infty} \frac{a_{j+2}}{a_j} = \lim_{j\to\infty} \frac{j^2 - p^2}{(j+1)(j+2)} = 1$$

and apply the ratio test to conclude that both $y_0(x)$ and $y_1(x)$ power series converge for $|x| < 1$.

(b) Show that if $p = n$ is a nonnegative integer, then there is a polynomial solution of degree n. (These solutions, when multiplied by suitable normalizing constants, are called *Chebyshev polynomials.*)

Solution: From the recursion in part (a), it follows that $a_{p+2} = 0$ if p is a positive integer, hence $a_j = 0$ for $j = p+2, p+4, \ldots$. Thus one of the solutions in part (a) (the y_0 solution if p is even, else the y_1 solution if p is odd) will be a polynomial of degree p.