MATH 217 – WORKSHEET 04

Q.1 The ODE

$$y^{(6)} + 4y^{(5)} + 3y^{(4)} - 10y^{(3)} - 26y'' - 24y' - 8y = 0$$

has characteristic equation

$$r^6 + 4*r^5 + 3*r^4 - 10*r^3 - 26*r^2 - 24*r - 8 = (r+1)^2(r^2+2r+2)(r^2-4) = 0.$$

Find the general solution.

Solution:

$$y(x) = A_1 e^{-x} + A_2 x e^{-x} + B_1 e^{-x}\cos x + B_2 e^{-x}\sin x + Ce^{2x} + De^{-2x}.$$

corresponding to the repeated real root -1 (A), complex conjugate roots $-1 \pm i$ (B), and distinct roots 2 (C) and -2 (D).

Q.2 Find the general solution of $y''' - 3y'' + 2y' = 10 + 18e^{3x}$

Solution: Characteristic equation is $r^3 - 3r^2 + 2r = r(r-1)(r-2) = 0$, which has three distinct real roots $0, 1, 2$, so the homogeneous part has general solution

$$y_0(x) = A + Be^x + Ce^{2x}.$$

Use undetermined coefficients to find a particular solution $y_p(x) = \beta x + \gamma e^{3x}$, which must satisfy

$$10 + 18e^{3x} = y_p''' - 3y_p'' + 2y_p' = 27\gamma e^{3x} - 3(9)\gamma e^{3x} + 2\beta + 2(3)\gamma e^{3x},$$

which implies $2\beta = 10$ and $6\gamma = 18$ or $y_p = 5x + 3e^{3x}$. Thus the general solution to the whole ODE is

$$y = y_p + y_0 = 5x + 3e^{3x} + y_0(x) + A + Be^x + Ce^{2x},$$

where A, B, C are arbitrary constants.

Q.3 Suppose that a straight tunnel is drilled through Mars connecting two points on the surface, which we will assume is a perfect sphere of radius $R = 3390$ km. We will also assume Mars has uniform density, and that the acceleration due to gravity is $g = 3.72$ meters per second squared at its surface.

(a) If a train coasts frictionlessly on tracks laid in the tunnel, show that the time required for a complete round trip does not depend on the locations of the surface entrances.

Solution: Let the x axis be the line connecting the two tunnel entrances with midpoint $x = 0$ being the deepest point, closest to the center of Mars. Let r be the distance from the center to the train. Let A_g be the acceleration due to gravity acting on the train in the direction $-x$ along the tracks. Let θ be the angle between the tunnel and the radius line (of length r) from the train to the center of Mars. Then

$$A_g = -a\frac{r}{R}\cos\theta = -a\frac{r}{R}\frac{x}{r} = -\frac{a}{R}x.$$

From Newton's law of motion,

$$x'' = A_g = -\frac{a}{R}x, \quad \Longrightarrow x'' + \frac{a}{R}x = 0$$

which yields the general solution

$$x(t) = C\cos\sqrt{\frac{a}{R}}t + D\sin\sqrt{\frac{a}{R}}t$$

which has a period $2\pi/\sqrt{a/R}$ depending only on the radius and surface gravity of Mars, not the tunnel entrances.

(b) Estimate the value of the round trip time in part (a).

Solution: For the given values, the period is $T = 2\pi/\sqrt{a/R} = 5998$ seconds, or roughly 100 minutes.

(c) If the tunnel has length $2L$, where $0 < L \leq R$, what is the maximum speed attained by the train? Check your answer as $L \to 0$.

Solution: First solve the IVP with $x(t) = L$ and $x'(0) = 0$ to get the solution

$$x(t) = L\cos\sqrt{\frac{a}{R}}t.$$

Then $x(t) = 0$ when $\sqrt{\frac{a}{R}}t = \pi/2$, at which time the speed of the train is

$$|x'(t)| = L\sqrt{\frac{a}{R}}\sin\sqrt{\frac{a}{R}}t = L\sqrt{\frac{a}{R}}.$$

This tends to 0 as expected as $L \to 0$. It is maximal at $L = R$ when the train passes through the center of Mars at $\sqrt{aR} = 3551$ meters per second.

Q.4 A cylindrical buoy d meters in diameter floats with its axis vertical in water with a density of ρ kg per cubic meter. When pushed down and released, it is observed to oscillate with a period of T seconds. This all happens on a watery planet where the acceleration due to gravity is g meters per second squared at the surface.

(a) What is the formula for the mass of the buoy in kg?

Solution: Let m be the mass of the buoy in kg. Let $z = z(t)$ be the depth, in meters, of a reference point on the buoy that, at rest, is at the water's surface.

If the buoy is submerged to depth $-z$, the restoring force due to its buoyancy is equal to g times the mass ρV of the displaced water volume V, namely

$$F = gV\rho = g\pi(d/2)^2 z\rho,$$

using the cylinder volume formula for V.

Then z satisfies the differential equation

$$mz'' = -F = -\pi g\rho \frac{d^2}{4} z, \quad \Longrightarrow z'' + \frac{\pi g\rho d^2}{4m} z = 0,$$

which has the general solution $z(t) = A\cos kt + B\sin kt$ with characteristic frequency k and period of oscillation $T = 2\pi/k$ satisfying

$$k^2 = \frac{\pi g\rho d^2}{4m}; \quad T^2 = \frac{16\pi m}{g\rho d^2}, \Longrightarrow m = \frac{g\rho T^2 d^2}{16\pi}.$$

(b) Use $g = 9.8$, $\rho = 1027$, $T = 1.9$, and $d = 0.6$ to compute the mass by your formula.

Solution: 260.22 kg using Octave.

Q.5 The *mean distance* of a planet from its star is the semimajor axis of its elliptical orbit.

(a) Mercury's "year" (namely its orbital period) is 88 days. What is its mean distance from the Sun?

Solution: Use Kepler's Third Law: $T^2 = k(2d)^3$, where T is the orbital period, d is the mean distance or semimajor axis, and k is a constant for all plants.

It is not necessary to determine k. Let $d_m = 1$ be Mercury's mean orbital distance in AU, so Earth's is 1, and compute

$$\frac{k(2d_m)^3}{k(2\cdot 1)^3} = \frac{(88)^2}{(365)^2}, \Longrightarrow d_m = \left(\frac{(88)^2}{(365)^2}\right)^{1/3} = 0.3874\text{AU} = 58,100,000\text{km}.$$

NOTE: These are approximations. Much more precise values are known with good confidence.

(b) Saturn's mean distance from the Sun is 9.54 times that of Earth. What is Saturn's orbital period?

Solution: Again it is not necessary to determine Kepler's constant of proportionality. Let T_s be Saturn's orbital period. Then

$$\frac{T_s^2}{(365)^2} = \frac{k(2\cdot 9.54)^3}{k(2\cdot 1)^3}, \Longrightarrow T_s = (365)(9.54)^{3/2} = 10,755 \text{ days},$$

or approximately $(9.54)^{3/2} = 29.46$ Earth years.