MATH 217 – WORKSHEET 03

Q.1 Find the general solutions of the following second-order ODEs

(a) $y'' + 5y' + 6y = 0$

Solution: Characteristic equation $0 = r^2 + 5r + 6 = (r+2)(r+3)$ has two distinct real roots $-2, -3$ giving the general solution

$$y(x) = Ae^{-2x} + Be^{-3x}.$$

(b) $y'' + 2y' + y = 0$

Solution: Characteristic equation $0 = r^2 + 2r + 1 = (r+1)^2$ has one repeated real root -1 giving the general solution

$$y(x) = Ae^{-x} + Bxe^{-x}.$$

(c) $y'' + 4y = 0$

Solution: Characteristic equation $0 = r^2 + 4 = (r+2i)(r-2i)$ has two complex conjugate roots $2i, -2i$ giving the general solution (after applying Euler's formula)

$$y(x) = A\cos 2x + B\sin 2x.$$

(d) $y'' + 2y' + 5y = 0$

Solution: Characteristic equation $0 = r^2 + 2r + 5$ has two complex conjugate roots

$$\frac{-2 \pm \sqrt{(2)^2 - 4(1)(5)}}{2(1)} = -1 \pm 2i,$$

giving the general solution (after applying Euler's formula)

$$y(x) = e^{-x}(A\cos 2x + B\sin 2x).$$

Q.2 Use these initial conditions in the corresponding ODEs of the previous question to find particular solutions.

(a) $y(0) = 1$, $y'(0) = -1$

Solution: Substitute into the general solution to get the linear system $A + B = 1$, $-2A - 3B = -1$, whose unique solution is $A = 2$, $B = -1$. Thus $y(x) = 2e^{-2x} - e^{-3x}$.

(b) $y(0) = 1$, $y'(0) = 0$

Solution: Substitute into the general solution to get $A = 1$ and $-A + B = 0 \implies B = 1$. Thus $y(x) = e^{-x} + xe^{-x} = (1+x)e^{-x}$.

(c) $y(0) = 2$, $y'(0) = 2$

Solution: Substitute into the general solution to get $A = 2$, $2B = 2 \implies B = 1$. Thus $y(x) = 2\cos 2x + \sin 2x$.

(d) $y(0) = 3$, $y'(0) = 5$

Solution: Substitute into the general solution to get $A = 3$, $-A + 2B = 5 \implies B = 4$. Thus $y(x) = e^{-x}(3\cos 2x + 4\sin 2x)$.

Q.3 Find the general solutions of the following second-order ODEs

(a) $y'' + 5y' + 6y = 2e^{-x}$

Solution: Use the general solution to the homogeneous part from Q.1a:

$$y_0(x) = Ae^{-2x} + Be^{-3x}.$$

Let $y_p(x) = Ce^{-x}$, with undetermined coefficient C, be a particular solution to the full inhomogeneous equation. Then

$$2e^x = y_p'' + 5y_p' + 6y_p = C(1 - 5 + 6)e^x = 2Ce^x,$$

so $C = 1$, giving the general solution

$$y(x) = y_0(x) + y_p(x) = Ae^{-2x} + Be^{-3x} + e^{-x}.$$

(b) $y'' + 2y' + y = x$

Solution: Use the general solution to the homogeneous part from Q.1b:

$$y_0(x) = Ae^{-x} + Bxe^{-x}.$$

Let $y_p(x) = Cx + D$, with undetermined coefficients C, D, be a particular solution to the full inhomogeneous equation. Then

$$x = y_p'' + 2y_p' + y_p = 0 + 2C + Cx + D = Cx + (2C + D),$$

so $C = 1$ and $D = -1$, giving the general solution

$$y(x) = y_0(x) + y_p(x) = Ae^{-x} + Bxe^{-x} + x - 1.$$

(c) $y'' + 4y = \sin 2x$

Solution: Use the general solution to the homogeneous part from Q.1c:

$$y(x) = A\cos 2x + B\sin 2x.$$

Since $\sin 2x$ solves homogeneous equation from Q.1c, include an additional x factor and let $y_p(x) = Cx\cos x + Dx\sin x$, with undetermined coefficients C, D, be a particular solution. Then

$$\sin 2x = y_p'' + 4y_p = 4D\cos 2x - 4C\sin 2x,$$

so $C = -1/4$ and $D = 0$, giving the general solution

$$y(x) = y_0(x) + y_p(x) = A\cos 2x + B\sin 2x - \frac{1}{4}x\cos x.$$

(d) $y'' + 2y' + 5y = 16xe^x$

Solution: Use the general solution to the homogeneous part from Q.1c:

$$y(x) = e^{-x}(A\cos 2x + B\sin 2x).$$

Let $y_p(x) = (Cx + D)e^x$, with undetermined coefficieints C, D, be a particular solution. Then

$$16xe^x = y_p'' + 2y_p' + 5y_p = [8Cx + (4C + 8D)]e^x,$$

so $8C = 16$ and $4C + 8D = 0$, which implies $C = 2$ and $D = -1$, giving the general solution

$$y(x) = y_0(x) + y_p(x) = e^{-x}(A\cos 2x + B\sin 2x) + (2x - 1)e^x.$$

Q.4 Find a particular solution of each of the following differential equations.

(a) $y'' + 5y' + 6y = e^{-2x}$

Solution: Use variation of parameters with the two solutions $y_1 = e^{-2x}$ and $y_2 = e^{-3x}$ of the homogeneous equation found in Q.1a. This gives the system

$$\begin{pmatrix} 0 \\ e^{-2x} \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} e^{-2x} & e^{-3x} \\ -2e^{-2x} & -3e^{-3x} \end{pmatrix} \begin{pmatrix} v_1' \\ v_2' \end{pmatrix},$$

which is solved by $v_1' = 1$ and $v_2' = -e^x$. These have antiderivatives $v_1 = x$ and $v_2 = -e^x$ giving the particular solution

$$y_p(x) = v_1y_1 + v_2y_2 = xe^{-2x} - e^xe^{-3x} = (x - 1)e^{-2x}.$$

(b) $y'' + 2y' + y = e^{-x}\ln x$

Solution: Use variation of parameters with the two solutions $y_1 = e^{-x}$ and $y_2 = xe^{-x}$ of the homogeneous equation found in Q.1b. This gives the system

$$\begin{pmatrix} 0 \\ e^{-x}\ln x \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & (1-x)e^{-x} \end{pmatrix} \begin{pmatrix} v_1' \\ v_2' \end{pmatrix},$$

which is solved by $v_1' = -x\ln x$ and $v_2' = \ln x$. These have antiderivatives $v_1 = (x^2 - 2x^2\ln x)/4$ and $v_2 = x\ln x - x$ (found using Macsyma), giving the particular solution

$$y_p(x) = v_1y_1 + v_2y_2 = e^{-x}(x^2 - 2x^2\ln x)/4 + xe^{-x}(x\ln x - x) = e^{-x}x^2[\frac{1}{2}\ln x - \frac{3}{4}],$$

after some simplification (also found using Macsyma).

(c) $y'' + 4y = \tan 2x$

Solution: Use variation of parameters with the two solutions $y_1 = \sin 2x$ and $y_2 = \cos 2x$ of the homogeneous equation found in Q.1c. This gives the system

$$\begin{pmatrix} 0 \\ \tan 2x \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} \sin 2x & \cos 2x \\ 2\cos 2x & -2\sin 2x \end{pmatrix} \begin{pmatrix} v_1' \\ v_2' \end{pmatrix},$$

which is solved by $v_1' = \frac{1}{2}\sin 2x$ and $v_2' = -\frac{1}{2}\tan 2x \sin 2x = \frac{1}{2}(\cos 2x - \sec 2x)$. These have antiderivatives $v_1 = -\frac{1}{4}\cos 2x$ and

$$v_2 = \frac{1}{4}\sin 2x - \frac{1}{4}\ln(\tan 2x + \sec 2x),$$

(found using Macsyma), giving the particular solution

$$y_p(x) = v_1 y_1 + v_2 y_2 = -\frac{1}{4}\cos 2x \ln(\tan 2x + \sec 2x),$$

after some simplification (also found using Macsyma).

(d) $y'' + 2y' + 5y = e^{-x}\sec 2x$

Solution: Use variation of parameters with the two solutions $e^{-x}\sin 2x$ and $e^{-x}\cos 2x$ of the homogeneous equation found in Q.1d. This gives the system

$$\begin{aligned}\begin{pmatrix} 0 \\ e^{-x}\sec 2x \end{pmatrix} &= \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} \\ &= \begin{pmatrix} e^{-x}\sin 2x & e^{-x}\cos 2x \\ 2e^{-x}\cos 2x - e^{-x}\sin 2x & -2e^{-x}\sin 2x - e^{-x}\cos 2x \end{pmatrix}\begin{pmatrix} v_1' \\ v_2' \end{pmatrix}.\end{aligned}$$

This may be row-reduced and divided by e^{-x} to give the equivalent system

$$\begin{pmatrix} 0 \\ \sec 2x \end{pmatrix} = \begin{pmatrix} \sin 2x & \cos 2x \\ 2\cos 2x & -2\sin 2x \end{pmatrix}\begin{pmatrix} v_1' \\ v_2' \end{pmatrix}.$$

Note the similarity with the system of part (c) above. It is solved by $v_1' = \frac{1}{2}\cos 2x \sec 2x = \frac{1}{2}$ and $v_2' = -\frac{1}{2}\sin 2x \sec 2x = -\frac{1}{2}\tan 2x$. These have antiderivatives $v_1 = \frac{1}{2}x$ and $v_2 = -\frac{1}{4}\ln(\sec 2x)$, (found using Macsyma), giving the particular solution

$$y_p(x) = v_1 y_1 + v_2 y_2 = -\frac{1}{2}xe^{-x}\sin 2x - \frac{1}{4}e^{-x}\ln(\sec 2x)$$

after some simplification (also found using Macsyma).

Q.5 The equation $xy'' + 3y' = 0$ has the trivial solution $y_1(x) = 1$. Using the method of Section 4.4, find a second linearly independent solution y_2 and then find the general solution.

Solution: Rewrite the equation into the form $y'' + py' + qy = 0$ with $p(x) = 3/x$ and $q(x) = 0$.

By the Section 4.4 method, put $y_2 = vy_1 = v$ in this case, and solve

$$v' = \frac{1}{y_1^2}\exp(-\int p(x)\,dx) = \exp(-3\ln x) = x^{-3}.$$

Integrate once more to find $v(x) = -\frac{1}{2}x^{-2} + C$, where C is an arbitrary constant.

Thus $y_2(x) = x^{-2}$ is another solution, and the general solution is any linear combination $y(x) = A + Bx^{-2}$.