MATH 217 – WORKSHEET 02

Q.1 *Bernoulli's equation* has the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n,$$

where P and Q are given functions. It is first-order linear when $n = 0$ or $n = 1$. For other values of n it can be made linear in a new dependent variable $z = z(x)$ with the change of variable $z = y^{1-n}$ and the subsequent elimination of y.

Use this method to solve the differential equation $y' + xy = xy^4$ with initial condition $y = 1$ when $x = 1$.

Solution: Identify $n = 4$ and so put $z = y^{1-n} = y^{-3}$ and thus $y = z^{-1/3}$. Then $z' = -3y^{-4}y'$ or $y' = -\frac{1}{3}y^4 z'$. Rewrite the DE and eliminate y to get

$$-\frac{1}{3}y^4 z' + xy = xy^4, \quad \iff z' - 3xy^{-3} = -3x, \quad \iff z' - 3xz = -3x.$$

This is a first-order linear equation in z with integrating factor

$$\rho = \exp \int -3x\, dx = \exp(-\frac{3}{2}x^2),$$

so its solution is

$$z = \frac{1}{\rho}\int -3x\rho\, dx = \frac{1}{\rho}\int -3x\rho\, dx = \exp(\frac{3}{2}x^2)\int -3x \exp(-\frac{3}{2}x^2)\, dx$$

The integral may be evaluated by substituting $u = -\frac{3}{2}x^2$, so $du = -3x$:

$$\int -3x\exp(-\frac{3}{2}x^2)\, dx = \int \exp u\, du = \exp u + C = \exp(-\frac{3}{2}x^2) + C,$$

so

$$z = 1 + C\exp(\frac{3}{2}x^2), \quad \implies y = z^{-1/3} = \left(1 + C\exp(\frac{3}{2}x^2)\right)^{-1/3}.$$

The initial condition $y(1) = 1$ is satisfied if and only if $C = 0$, giving the unique solution $y(x) = 1$ for all x.

Q.2 Show that each of these equations is exact and find the solution.

(a) $(x + \frac{2}{y})\, dy + y\, dx = 0$

Solution: Check that

$$\frac{\partial}{\partial x}(x + \frac{2}{y}) = 1 = \frac{\partial}{\partial y}(y).$$

Combine the antiderivatives $\int (x + \frac{2}{y})\, dy = xy + 2\ln|y|$ and $\int y\, dx = xy$ to get the implicit solution

$$xy + 2\ln|y| = C$$

(b) $(y - x^3)\,dx + (x + y^3)\,dy = 0$

Solution: Check that

$$\frac{\partial}{\partial y}(y - x^3) = 1 = \frac{\partial}{\partial x}(x + y^3).$$

Combine the antiderivatives $\int (y - x^3)\,dx = xy - x^4/4$ and $\int (x + y^3)\,dy = xy + y^4/4$ to get the implicit solution

$$4xy - x^4 + y^4 = C$$

Q.3 See textbook section 2.4 for the definition of orthogonal trajectories.

(a) Find the orthogonal trajectories of the curves $y = Cx^4$.

Solution: Find the slope by differentiation, then eliminate $C = y/x^4$:

$$\frac{dy}{dx} = 4Cx^3 = \frac{4y}{x}.$$

Orthogonal trajectories have the negative reciprocals of these slopes:

$$\frac{dy}{dx} = -\frac{x}{4y}.$$

Solve this ODE by separation of variables to get the orthogonal trajectories:

$$4y\,dy = -x\,dx \implies 4y^2 + x^2 = C.$$

(b) Fix an integer $n \geq 1$. Find the orthogonal trajectories of the curves $y = Cx^n$.

Solution: Find the slope by differentiation, then eliminate $C = y/x^n$:

$$\frac{dy}{dx} = nCx^{n-1} = \frac{ny}{x}.$$

Orthogonal trajectories have the negative reciprocals of these slopes:

$$\frac{dy}{dx} = -\frac{x}{ny}.$$

Solve this ODE by separation of variables to get the orthogonal trajectories:

$$ny\,dy = -x\,dx \implies ny^2 + x^2 = C.$$

(c) Describe how the orthogonal trajectories in part (b) change as $n \to \infty$.

Solution: Writing them as

$$\frac{x^2}{n} + \frac{y^2}{1} = C,$$

to see that these curves are ellipses with a ratio (x semiaxis):(y semiaxis) equal to $\sqrt{n}$. As $n \to \infty$, these become flatter relative to their width.

Q.4 Verify that each of the following ODEs is homogeneous and then find its general solution.

(a) $(y + xe^{y/x})\,dx - x\,dy = 0$.

Solution: Factors before dx and dy are evidently both homogeneous of degree 1. Rewrite and substitute $z = y/x$ and $y' = z + xz'$ to get

$$\frac{dy}{dx} = \frac{y}{x} + e^{y/x}, \iff z + x\frac{dz}{dx} = z + e^z. \iff e^{-z}\,dz = x^{-1}\,dx.$$

Integrate both sides to obtain

$$-e^{-z} = \ln|x| + C, \iff Bx = e^{-e^{-z}}, \iff Bx = e^{-e^{-y/x}},$$

for some constant B

(b) $x^2y' - 3xy = 2y^2$

Solution: Rewrite as $x^2\,dy - (3xy + 2y^2)\,dx = 0$ and note that both x^2 and $-(3xy + 2y^2)$ are homogeneous of degree 2.

Rewrite again and substitute $z = y/x$ and $y' = z + xz'$ to get

$$\frac{dy}{dx} = \frac{-3xy - 2y^2}{x^2} = -\frac{3y}{x} - 2\frac{y^2}{x^2}, \quad \Longrightarrow z + xz' = -3z - 2z^2.$$

This gives an ODE for z solvable by separation of variables:

$$\frac{dz}{dx} = \frac{-4z - 2z^2}{x} \iff \frac{dz}{z^2 + 2z} = \frac{-2}{x}\,dx \iff \frac{1}{2}\left[\frac{1}{z} - \frac{1}{z+2}\right]dz = \frac{-2}{x}\,dx$$

(after expansion into partial fractions). Integrate both sides to get

$$\ln|z| - \ln|z+2| = -4\ln|x| + C, \Longrightarrow \frac{z}{z+2} = Bx^{-4}, \Longrightarrow \frac{y}{y+2x} = Bx^{-4},$$

for some constant B after replacing $z = y/x$.

Q.5 Solve these differential equations by finding an integrating factor.

(a) $(x + 3y^2)\,dx + 2xy\,dy = 0$

Solution: Compute the relevant partial derivatives for $M = x + 3y^2$ and $N = 2xy$:

$$\frac{\partial}{\partial y}(x + 3y^2) = 6y; \qquad \frac{\partial}{\partial x}(2xy) = 2y.$$

The ODE fails the test for exactness, and it is also not separable. However,

$$g = \frac{\partial M/\partial y - \partial N/\partial x}{N} = \frac{6y - 2y}{2xy} = \frac{2}{x}$$

is a function of x alone, so use the integrating factor

$$\mu(x) = \exp\int g(x)\,dx = \exp(2\ln|x|) = x^2$$

to get the exact equation $(x^3+3x^2y^2)\,dx+2x^3y\,dy=0$. Combine the antiderivatives

$$\int (x^3+3x^2y^2)\,dx = x^4/4 + x^3y^2, \qquad \int (2x^3y)\,dy = x^3y^2,$$

to get the implicit solution $x^4+4x^3y^2 = C$ for some constant C.

(b) $(y\ln y - 2xy)\,dx + (x+y)\,dy = 0$

Solution: Compute the relevant partial derivatives for $M = y\ln y - 2xy$ and $N = x+y$:

$$\frac{\partial M}{\partial y} = 1 + \ln y - 2x; \qquad \frac{\partial N}{\partial x} = 1.$$

The ODE fails the test for exactness, and it is also not separable. However,

$$h = -\frac{\partial M/\partial y - \partial N/\partial x}{M} = -\frac{\ln y - 2x}{y\ln y - 2xy} = -\frac{1}{y}$$

is a function of y alone, so use the integrating factor

$$\mu(y) = \exp\int h(y)\,dy = \exp(-\ln|y|) = \frac{1}{y}$$

to get the exact equation $(\ln y - 2x)\,dx + (1+x/y)\,dy = 0$. Combine the antiderivatives

$$\int (\ln y - 2x)\,dx = x\ln y - x^2, \qquad \int (1+x/y)\,dy = y + x\ln y,$$

to get the implicit solution $x\ln y + y - x^2 = C$ for some constant C.

Q.6 Solve these ODE initial value problems by reduction of order.

(a) $y'' = 3y'$, with $y(0)=0$ and $y'(0)=1$.

Solution: Let $z = y'$ to get $z' = y''$ and the reduced equation $z' = 3z$, which has the general solution $z(x) = Ae^{3x}$ for some constant A.

Use $1 = y'(0) = z(0) = A$ to get the particular solution $z(x) = e^{3x}$.

Integrate once more to find y from solve $y' = z = e^{3x}$, giving the general solution

$$y(x) = \frac{1}{3}e^{3x} + B,$$

for some constant B.

Use $0 = y(0) = \frac{1}{3} + B$ to compute $B = -1/3$, so

$$y(x) = \frac{1}{3}e^{3x} - \frac{1}{3}$$

solves the initial value problem.

(b) $xy'' + y' = 2x$, with $y'(1) = 2$ and $y(1) = 0$

Solution: Substitute $z = y'$ and thus $z' = y''$ to get the first order linear equation $xz' + z = 2x$, which in standard form is

$$z' + \frac{1}{x}z = 2.$$

This may be solved using the integrating factor

$$\rho = \exp \int \frac{1}{x}\, dx = \exp \ln x = x,$$

giving the solution

$$z(x) = \frac{1}{\rho} \int 2\rho\, dx = \frac{1}{x} \int 2x\, dx = x + \frac{A}{x},$$

for an arbitrary constant A. Use the initial condition $2 = y'(1) = z(1) = 1 + A$ to compute $A = 1$, so $z(x) = x + 1/x$.

Integrate once more to find

$$y(x) = \int z(x)\, dx = \frac{1}{2}x^2 + \ln|x| + B,$$

for another constant B. Use the initial condition $0 = y(1) = \frac{1}{2}(1)^2 + \ln 1 + B = \frac{1}{2} + B$ to compute $B = -1/2$, so

$$y(x) = \frac{1}{2}x^2 + \ln|x| - \frac{1}{2}.$$