MATH 217 – WORKSHEET 01

Q.1 Use the method of separation of variables to solve each of the following ordinary differential equations:

(a) $x^3y' - y^3 = 0$

Solution: Rewrite as $y^{-3}\,dy = x^{-3}\,dx$, integrate to get $y^{-2} = x^{-2} + C$.

(b) $y' \cos y = x$

Solution: Rewrite as $\cos y\,dy = x\,dx$, integrate to get $\sin y = \frac{1}{2}x^2 + C$.

Q.2 Find the particular solution to each of the following differential equations that satisfies the additional property:

(a) $y' = x^2y^2$, and $y = 3$ when $x = 1$.

Solution: Rewrite as $y^{-2}\,dy = x^2\,dx$, integrate to get $-y^{-1} = \frac{1}{3}x^3 + C$, solve $-\frac{1}{3} = \frac{1}{3}(1)^3 + C$ to get $C = -2/3$. Conclude that $y = 3/(2 - x^3)$.

(b) $y'y^2 = x + 1$, and $y = 3$ when $x = 0$.

Solution: Rewrite as $y^2\,dy = (x+1)\,dx$, integrate to get $\frac{1}{3}y^3 = \frac{1}{2}x^2 + x + C$, solve $\frac{1}{3}(3)^3 = 0 + C$ to get $C = 9$. Conclude that $y = [\frac{3}{2}x^2 + 3x + 27]^{1/3}$.

Q.3 Consider the differential equation $y''y' = e^x$.

(a) Make the substitution $y' = p$, $y'' = p'$ to obtain a first-order differential equation and solve it for $p = p(x)$.

Solution: Rewrite as $p'p = e^x$, then $p\,dp = e^x\,dx$. Integrate to get $\frac{1}{2}p^2 = e^x + C$.

(b) Solve the equation in part (a) for y satisfying the additional conditions $y'(0) = \sqrt{2}$ and $y(0) = 1$.

Solution: Use $p(0) = y'(0) = \sqrt{2}$ in the solution to part (a) to solve for $C = 0$, so $\frac{1}{2}(y')^2 = \frac{1}{2}p^2 = e^x$, so $y' = \sqrt{2}e^{x/2}$.

Integrate again to get $y = 2\sqrt{2}e^{x/2} + C$. Use $y(0) = 1$ to find $C = 1 - 2\sqrt{2}$. Conclude that $y(x) = 2\sqrt{2}e^{x/2} + 1 - 2\sqrt{2}$.

Q.4 Find the general solution to the following first order linear differential equations. Then find the particular solution with the given additional condition:

(a) $xy' - 3y = x^4$, additional condition $y = 2$ when $x = 1$.

Solution: Rewrite as $y' + (-3/x)y = x^3$ to identify the integrating factor

$$\rho = \exp \int (-3/x)\,dx = \exp(-3\ln x) = x^{-3}.$$

Mutiply the rewritten equation by ρ to get $x^{-3}y' - 3x^{-4}y = 1$. Recognize that the left-hand side is $(x^{-3}y)'$, so integrate to get $x^{-3}y = x + C$ as the general solution.

Using the additional condition, solve for $C = x^{-3}y - x = (1)^{-3}(2) - (1) = 1$ to get the particular solution $x^{-3}y = x + 1$, or $y = x^4 + x^3$.

(b) $y' + 4y = e^{-x}$, additional condition $y = 1$ when $x = 0$.

Solution: Identify the integrating factor

$$\rho = \exp \int (4)\, dx = \exp(4x).$$

Mutiply the rewritten equation by ρ to get $e^{4x}y' + 4e^{4x}y = e^{3x}$. Recognize that the left-hand side is $(e^{4x}y)'$, so integrate to get $e^{4x}y = \frac{1}{3}e^{3x} + C$ as the general solution.

Using the additional condition, solve for $C = e^{4x}y - \frac{1}{3}e^{3x} = 1 - \frac{1}{3} = 2/3$ to get the particular solution $e^{4x}y = \frac{1}{3}e^{3x} + \frac{2}{3}$, or $y = \frac{1}{3}e^{-x} + \frac{2}{3}e^{-4x}$.

Q.5 One solution of the differential equation $y' \sin 2x = 2y + 2\cos x$ remains bounded as $x \to \pi/2$. Find this solution.

Solution: Rewrite into standard first-order linear form:

$$y' - \frac{2}{\sin 2x} y = \frac{2\cos x}{\sin 2x} = \frac{2\cos x}{2\sin x \cos x} = \frac{1}{\sin x} = \csc x.$$

Find the integrating factor by substitution ($u \leftarrow 2x$) and a table of integrals:

$$\rho = \exp \int \frac{-2}{\sin 2x}\, dx = \exp \int -\csc u\, du = \csc 2x + \cot 2x = \frac{1 + \cos 2x}{\sin 2x} = \frac{\cos x}{\sin x}.$$

Solve for y using the integrating factor:

$$y = \frac{1}{\rho} \int \rho \csc x\, dx = \frac{1}{\rho} \int \frac{\cos x}{\sin^2 x}\, dx.$$

The indefinite integral may be solved using the substitution $u \leftarrow \sin x$:

$$\int \frac{\cos x}{\sin^2 x}\, dx = \int u^{-2}\, du = -u^{-1} + C = \frac{-1}{\sin x} + C,$$

where C is a constant. Hence

$$y = \frac{-1}{\rho \sin x} + \frac{C}{\rho} = \frac{C \sin x - 1}{\cos x}.$$

The denominator vanishes at $x = \pi/2$, so the solution is bounded as $x \to \pi/2$ if and only if the numerator also vanishes at $x = \pi/2$, which happens if and only if $C = 1$. Hence the unique solution bounded near $x = \pi/2$ is $y = (\sin x - 1)/\cos x$.