## MATH 217 – WORKSHEET 12

*Q.1* Replace each of the following differential equations by an equivalent system of of first-order equations.

(a) $y'' - xy' - xy = 0$

(b) $y''' = y'' - x^2(y')^2$

(c) $y^{(4)} - xy''' + x^2y'' - x^3y = 1$

*Q.2* Consider the homogeneous first-order linear system

$$\begin{aligned} x' &= x + 3y \\ y' &= 3x + y \end{aligned}$$

(a) Show that $(x, y) = (e^{4t}, e^{4t})$ and $(x, y) = (e^{-2t}, -e^{-2t})$ are solutions.

(b) Show in two ways that the given solutions are linearly independent on every closed interval. Then write the general solution of this system.

(c) Find the particular solution for which $x(0) = 5$ and $y(0) = 1$.

*Q.3* Consider the first-order linear system

$$\begin{aligned} x' &= x + 2y + t - 1 \\ y' &= 3x + 2y - 5t - 2 \end{aligned}$$

(a) Show that $(x, y) = (2e^{4t}, 3e^{4t})$ and $(x, y) = (e^{-t}, -e^{-t})$ are solutions to the homogeneous part of the equation.

(b) Show that the homogeneous solutions from part (a) are linearly independent on every closed interval.

(c) Show that $x(t) = 3t - 2$ and $y(t) = -2t + 3$ give a particular solution to the full equation. Then write the general solution to the full equation.

(d) Find the solution to the full equation satisfying $x(0) = 2$ and $y(0) = 4$.

*Q.4* Find the general solution to each of the following homogeneous linear first order systems with constant coefficients, using the method of eigenvalues:

(a)

$$\begin{aligned} x' &= -3x + 4y \\ y' &= -2x + 3y \end{aligned}$$

(b)

$$\begin{aligned} x' &= 4x - 3y \\ y' &= 8x - 6y \end{aligned}$$

(c)

$$\begin{aligned} x' &= 4x - 2y \\ y' &= 5x + 2y \end{aligned}$$

(d)

$$\begin{aligned} x' &= 5x + 4y \\ y' &= -x + y \end{aligned}$$

*Q.5* Consider the Volterra predator-prey equation for $x(t)$ and $y(t)$ on $t \geq 0$:

$$\begin{aligned} x' &= ax - bxy \\ y' &= -cy + gxy, \end{aligned}$$

where $a, b, c, g$ are all positive.

(a) Find the equilibrium solution in terms of $a, b, c, d$.

(b) Show that if $x'(t) > 0$, then $y'' > 0$.

(c) Give an interpretation of the result from part (b) in the model for prey $x$ and predators $y$.

*Q.6 (Optional)* Let

$$A = \begin{pmatrix} -3 & 4 \\ -2 & 3 \end{pmatrix}, \qquad u(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \qquad u_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix},$$

Solve the linear system $u' = Au$ with initial condition $u(0) = u_0$ by finding the matrix exponential $e^{tA}$.

(Hint: use the eigenvalues and eigenvectors from Q.4(a). )