MATH 217 – WORKSHEET 11

Q.1 In parts (a) and (b) below, constant $a > 0$ and function $f = f(x)$ are given, and $w = w(x,t)$ is to be found.

(a) Solve the boundary value problem

$$\begin{aligned} a^2 w_{xx} &= w_t, \qquad 0 < x < \pi,\ t \geq 0, \\ w(x,0) &= f(x), \quad 0 < x < \pi, \\ w(0,t) &= 0, \quad t \geq 0, \\ w(\pi,t) &= 0, \quad t \geq 0. \end{aligned}$$

(b) Solve the boundary value problem in part (a) with the modified end conditions

$$w(0,t) = w_1, \qquad w(\pi,t) = w_2, \quad t \geq 0,$$

where w_1 and w_2 are given real numbers.

Q.2 The two-dimensional heat equation is

$$a^2 \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) = \frac{\partial w}{\partial t}$$

for $w = w(x,y,t)$. Use the method of separation of variables to find the steady-state solution in the infinite half-strip of the xy plane consisting of $\{(x,y) : 0 \leq x \leq \pi, y \geq 0\}$ if the following boundary conditions are satisfied:

$$\begin{aligned} w(0,y,t) = w(\pi,y,t) &= 0, \quad t \geq 0, y \geq 0 \\ w(x,0,0) &= f(x), \quad 0 \leq x \leq \pi \\ \lim_{y\to\infty} w(x,y,t) &= 0, \quad t \geq 0, 0 \leq x \leq \pi. \end{aligned}$$

Q.3 Solve the Dirichlet problem for the unit disc, using the Poisson kernel, when the boundary function $f(\theta)$ is defined by

(a) $f(\theta) = \cos(\theta/2)$, $-\pi \leq \theta \leq \pi$.

(b) $f(\theta) = 0$ if $-\pi \leq \theta < 0$ and $f(\theta) = 1$ if $0 \leq \theta \leq \pi$.

(c) $f(\theta) = 0$ if $-\pi \leq \theta < 0$ and $f(\theta) = \sin\theta$ if $0 \leq \theta \leq \pi$.

Q.4 Given the ODE

$$P(x)y'' + Q(x)y' + R(x)y = 0,$$

there is an exact equation obtained by multiplying with an integrating factor $\mu(x)$:

$$0 = \mu(x)P(x)y'' + \mu(x)Q(x)y' + \mu(x)R(x)y = [\mu(x)P(x)y']' + [S(x)y]',$$

for some appropriate function $S = S(x)$.

(a) Show that $\mu(x)$ satisfies the *adjoint equation*

$$P(x)\mu'' + [2P'(x) - Q(x)]\mu' + [P''(x) - Q'(x) + R(x)]\mu = 0.$$

(b) Find the adjoint equation for Legendre's equation

$$(1 - x^2)y'' - 2xy' + p(p - 1)y = 0.$$

(c) Find the adjoint equation for Bessel's equation

$$x^2y'' + xy' + (x^2 - p^2)y = 0.$$

(d) Find the adjoint equation for Hermite's equation

$$y'' - 2xy' + 2py = 0.$$

(d) Find the adjoint equation for Laguerre's equation

$$xy'' + (1 - x)y' + py = 0.$$

Q.5 The equation $P(x)y'' + Q(x)y' + R(x)y = 0$ is called *self-adjoint* if its adjoint is just the same equation, possibly after a change of notation.

(a) Show that this equation is self-adjoint if and only if $P' = Q$.

(b) Show that such a self-adjoint equation may be written as

$$P(x)y'' + P'(x)y' + R(x)y = 0, \quad \Longleftrightarrow \quad [P(x)y']' + R(x)y = 0$$

(Hint: use part (a).)

(c) Determine which of the equations in Q.4 above are self-adjoint.