MATH 217 – WORKSHEET 10

Q.1 Let δ be the Dirac mass at 0, so $\delta(x-a)$ is the Dirac mass at a. Let h be the Heaviside function with a jump at $x=0$, so $h(x-a)$ is the Heaviside function with a jump at $x=a$.

(a) Find

$$L\left[\sum_{n=0}^{\infty}\delta(x-n)\right](p)$$

(b) Find

$$L\left[\sum_{n=0}^{\infty}h(x-n)\right](p)$$

(c) Find $L[s(x)](p)$, where $s:[0,\infty)\to\mathbf{R}$ is the square wave function

$$s(x)=\begin{cases}1, & 2k<x<2k+1,\\ -1, & 2k+1<x<2k+2,\end{cases} \qquad k=0,1,2,\ldots$$

(d) Find $L[|\sin x|](p)$, the Laplace transform of the "full wave rectified" sine wave $|\sin x|$. (Hint: use periodicity.)

Q.2 Find the eigenvalues λ_n and the eigenfunctions y_n for the equation $y''+\lambda y=0$ in each of the following instances:

(a) $y(0)=0$, $y(\pi/2)=0$.

(b) $y(0)=0$, $y(2\pi)=0$.

(c) $y(0)=0$, $y(1)=0$.

(d) $y(a)=0$, $y(b)=0$ for fixed $a<b$.

Q.3 Let F,G be arbitrary twice-differentiable functions of one variable. Fix $a>0$ and let $y(x,t)=F(x+at)+G(x-at)$.

(a) Show that y satisfies the partial differential equation

$$a^2\frac{\partial^2 y}{\partial x^2}=\frac{\partial^2 y}{\partial t^2}.$$

(b) Let $\alpha = x + at$ and $\beta = x - at$. Show that, with this change of variables for y, the equation from part (a) may be written as

$$\frac{\partial^2 y}{\partial\alpha\partial\beta} = 0.$$

(c) From the equation in part (b), integrate twice to derive the formula for y in terms of F and G. (Note: this is known as d'Alembert's solution of the wave equation.)

Q.4 Consider the vibrating string problem in the text (Eqs.11.9, 11.10, and 11.11 on p.322).

(a) Find the solution if the initial shape is $y(x, 0) = c \sin x$ on $x \in [0, \pi]$.

(b) Find the solution if the initial shape is $y(x, 0) = c \sin nx$ on $x \in [0, \pi]$, for some positive integer n.

Remark. The solution in part (b) factors into a function of x times a function of y called a "standing wave." Such a standing wave solution has $n + 1$ equispaced values $0 = x_0 < \cdots < x_n = \pi$ call "nodal points" where $y(x_k, t) = 0$ for all t.

Q.5 Solve the "struck string" problem, namely the wave equation $a^2 y_{xx} = y_{tt}$ with boundary conditions $y(0, t) = y(\pi, t) = 0$, all $t \geq 0$, and the initial conditions

$$y(x, 0) = 0; \qquad y_t(x, 0) = g(x),$$

for $x \in [0, \pi]$, with a given function $g = g(x)$ defined on $[0, \pi]$.