MATH 217 – WORKSHEET 07

Q.1 This exercise illustrates how to compute Fourier series using trigonometric identities.

(a) Use the double angle formulas $\cos 2x = 1 - 2\sin^2 x = \cos^2 x - 1$ to compute the Fourier series of $f(x) = \sin^2 x$ and $g(x) = \cos^2 x$ without any integration.

(b) Use Euler's formula $e^{ix} = \cos x + i \sin x$ to prove that

$$\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x, \qquad \cos^3 x = \frac{3}{4}\cos x + \frac{1}{4}\cos 3x,$$

and use these to find the Fourier series for $\sin^3 x$ and $\cos^3 x$ without integration.

Q.2 Find the Fourier series for the functions on the given intervals:
(a) $f(x) = x$, $[-1, 1]$.

(b) $g(x) = \cos 2x$, $[-\pi/3, \pi/3]$.

(c) $h(x) = \sin(2x - \pi/3)$, $[-1, 1]$.

Q.3 Fix $L > 0$. Show that

$$\frac{L}{2} - x = \frac{L}{\pi}\sum_{j=1}^{\infty}\frac{1}{j}\sin\frac{2j\pi x}{L}, \qquad 0 < x < L.$$

Q.4 Verify that each pair of functions below is orthogonal on the given interval $[a, b]$ using the inner product $\langle f, g\rangle = \int_a^b f(x)g(x)\,dx$.
(a) $f(x) = \sin 2x$, $g(x) = \cos 3x$, $[-\pi, pi]$.

(b) $f(x) = \sin 2x$, $g(x) = \sin 4x$, $[0, pi]$.

(c) $f(x) = x^2$, $g(x) = x^3$, $[-1, 1]$.

(d) $f(x) = x$, $g(x) = \cos x$, $[-2, 2]$.

Q.5 Prove the following identities for functions $f, g \in L^2$:
(a) Parallelogram law: $2\|f\|^2 + 2\|g\| = \|f + g\|^2 + \|f - g\|^2$.

(b) Pythagoras' theorem and its converse: f, g are orthogonal if and only if $\|f - g\|^2 = \|f\|^2 + \|g\|^2$.

(c) Cauchy-Schwarz-Bunyakovsky inequality: $|\langle f, g\rangle| \leq \|f\| \cdot \|g\|$.
(Hint: let $\phi(t) = \|f + tg\|^2$ and find the minimum of ϕ, noting that it must be greater than or equal to zero.)