Name:

ID:

- This is a timed examination. You are allowed 120 minutes to finish it.
- There are 12 questions worth 10 points each.
- This exam covers the entire course but with emphasis on Chapters 10 and 11 of the textbook.

- No calculators or other devices may be used.
- No books or notes other than a single letter-size sheet of notes and formulas are permitted, nor any collaboration.
- Read the statement of each problem carefully.
- Be sure to ask questions if anything is unclear.
- Show all your work for full credit.
- Your ability to make your solution clear will be part of your grade.

1. Write the solution $u(r, \theta)$ to the Dirichlet problem $\Delta u = 0$ on the unit disc $D = \{(r, \theta) : r \leq 1\}$, with boundary condition $u(1, \theta) = \sin^2 \theta$, using the Poisson kernel, and then find the value $u(0, 0)$.

2. Solve the boundary value problem

$$
\begin{aligned}
w_{xx} &= w_t, \qquad 0 < x < \pi,\ t \geq 0, \\
w(x,0) &= 2\sin(3x), \quad 0 < x < \pi, \\
w(0,t) &= 0, \quad t \geq 0, \\
w(\pi,t) &= 0, \quad t \geq 0.
\end{aligned}
$$

3. Consider the vibrating string problem for $y(x,t)$ on $t \geq 0$ and $0 \leq x \leq \pi$:

$$\frac{\partial^2 y}{\partial t^2} = k^2 \frac{\partial^2 y}{\partial x^2},$$

with boundary conditions $y(0,t) = 0$ and $y(\pi,t) = 0$ for all t, initially at rest with $\frac{\partial y}{\partial t}|_{t=0} = 0$. Find the solution if the initial position of the string is given by

$$y(x,0) = \sin x + \frac{1}{5}\sin 5x, \qquad 0 \leq x \leq \pi.$$

4. Consider the two-dimensional heat equation

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{\partial w}{\partial t}$$

for $w = w(x, y, t)$. Use the method of separation of variables to find the steady-state solution in the infinite half-strip of the xy plane consisting of $\{(x, y) : 0 \leq x \leq \pi, y \geq 0\}$ if the following boundary conditions are satisfied:

$$\begin{aligned} w(0, y, t) = w(\pi, y, t) &= 0, \quad t \geq 0, y \geq 0 \\ w(x, 0, 0) &= 2 \sin 7x, \quad 0 \leq x \leq \pi \\ \lim_{y \to \infty} w(x, y, t) &= 0, \quad t \geq 0, 0 \leq x \leq \pi. \end{aligned}$$

5. Write the second-order system

$$x'' = x'y + \frac{t^2}{x^2 + y^2} \qquad y'' = y'x + \frac{1 - t^2}{x^2 + y^2}$$

as an equivalent system of first-order equations.

6. Show that $(x_1, y_1) = (e^{-t}, e^{-t})$ and $(x_2, y_2) = (e^{3t}, -e^{3t})$ are solutions to the homogeneous first-order linear system

$$\begin{aligned} x' &= x - 2y \\ y' &= -2x + y \end{aligned}$$

and that they are linearly independent on every closed interval of t.

7. Consider the Volterra predator-prey equation for $x(t)$ and $y(t)$ on $t \geq 0$:

$$x' = x - xy; \qquad y' = -y + xy,$$

Find the steady-state solution and compute x'' and y'' in terms of x and y.

8. Solve the initial value problem $y' - y = e^{2x}$, $y(0) = 2$, using the Laplace transform.

9. Solve the initial value problem $y' - y = e^{2x}$, $y(0) = 0$, using an integrating factor.

10. Find the Fourier series for the function g defined by

$$g(x) = \begin{cases} -1, & -\pi \leq x < 0, \\ 1, & 0 \leq x < \pi. \end{cases}$$

11. Explain why $x = 0$ is an ordinary point for the differential equation $y' - y = e^{2x}$, and then find the recursion for $\{a_j : j = 0, 1, 2, \dots\}$ in the power series solution

$$y(x) = \sum_{j=0}^{\infty} a_j x^j.$$

(It is not necessary to solve the recursion.)

12. The ODE

$$y^{(6)} + 4y^{(5)} + 3y^{(4)} - 10y^{(3)} - 26y'' - 24y' - 8y = 0$$

has characteristic equation

$$r^6 + 4r^5 + 3r^4 - 10r^3 - 26r^2 - 24r - 8 = (r+1)^2(r^2+2r+2)(r^2-4) = 0.$$

Find the general solution.