Name:

ID:

- This is a timed examination. You are allowed 120 minutes to finish it.
- There are 12 questions worth 10 points each.
- This exam covers the entire course but with emphasis on Chapters 10 and 11 of the textbook.

- No calculators or other devices may be used.
- No books or notes other than a single letter-size sheet of notes and formulas are permitted, nor any collaboration.
- Read the statement of each problem carefully.
- Be sure to ask questions if anything is unclear.
- Show all your work for full credit.
- Your ability to make your solution clear will be part of your grade.

1. Write the solution $u(r,\theta)$ to the Dirichlet problem $\Delta u = 0$ on the unit disc $D = \{(r,\theta) : r \leq 1\}$, with boundary condition $u(1,\theta) = \sin^2\theta$, using the Poisson kernel, and then find the value $u(0,0)$.

Solution: The Poisson integral formula for the solution is

$$\boxed{u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(1-r^2)\sin^2\xi}{1-2r\cos(\theta-\xi)+r^2}\, d\xi}$$

When $r = 0$, regardless of θ, it reduces to

$$\boxed{u(0,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin^2\xi\, d\xi \;=\; \frac{1}{2},}$$

since $\sin^2\xi = \frac{1}{2} - \frac{1}{2}\cos 2\xi$ and the integral of $\cos 2\xi$ over a complete period $-\pi < \xi < \pi$ is zero.

2. Solve the boundary value problem

$$\begin{aligned} w_{xx} &= w_t, \qquad 0 < x < \pi,\ t \geq 0, \\ w(x,0) &= 2\sin(3x), \quad 0 < x < \pi, \\ w(0,t) &= 0, \quad t \geq 0, \\ w(\pi,t) &= 0, \quad t \geq 0. \end{aligned}$$

Solution: Following Section 11.3, use separation of variables to write the solution as

$$w(x,t) = \sum_{j=1}^{\infty} b_j e^{-j^2 t} \sin(jx)$$

which, for all sequences $\{b_j\}$, satisfies $w(0,t) = w(\pi,t) = 0$ for all $t \geq 0$. Then determine $\{b_j : j = 1, 2, \ldots\}$ from the Fourier sine series for f:

$$b_j = \frac{2}{\pi} \int_0^{\pi} 2\sin(3x)\sin(jx)\,dx = \begin{cases} 2, & j = 3 \\ 0, & j \neq 3. \end{cases}$$

Thus the solution is $\boxed{w(x,t) = 2e^{-9t}\sin(3x).}$

3. Consider the vibrating string problem for $y(x,t)$ on $t \geq 0$ and $0 \leq x \leq \pi$:

$$\frac{\partial^2 y}{\partial t^2} = k^2 \frac{\partial^2 y}{\partial x^2},$$

with boundary conditions $y(0,t) = 0$ and $y(\pi,t) = 0$ for all t, initially at rest with $\frac{\partial y}{\partial t}|_{t=0} = 0$. Find the solution if the initial position of the string is given by

$$y(x,0) = \sin x + \frac{1}{5}\sin 5x, \qquad 0 \leq x \leq \pi.$$

Solution: Separation of variables $y(x,t) = u(x)v(t)$ yields two eigenvalue problems:

$$\frac{u''(x)}{u(x)} = \frac{v''(t)}{k^2 v(t)} = \lambda.$$

The boundary conditions in $x \in [0,\pi]$ force the solution $u(x) = \sin nx$ for integer n, so $\lambda = -n^2$. The initially-at-rest condition is that $v'(0) = 0$, which forces the solution $v(t) = \cos nkt$. Using linearity, we get a general solution of the form

$$y = \sum_{n=1}^{\infty} b_n \sin nx \cos nkt,$$

with coefficients $\{b_n : n = 1, 2, \dots\}$ yet to be determined.

But the initial shape condition implies that

$$\sin x + \frac{1}{5}\sin 5x = y(x,0) = \sum_{n=1}^{\infty} b_n \sin nx,$$

from which we conclude, by comparing Fourier sine expansions, that $b_1 = 1$, $b_5 = 1/5$, and $b_n = 0$ for $n \notin \{1, 5\}$. Thus

$$\boxed{y(x,t) = \sin x \cos kt + \frac{1}{5}\sin 5x \cos 5kt.}$$

4. Consider the two-dimensional heat equation

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{\partial w}{\partial t}$$

for $w = w(x, y, t)$. Use the method of separation of variables to find the steady-state solution in the infinite half-strip of the xy plane consisting of $\{(x, y) : 0 \leq x \leq \pi, y \geq 0\}$ if the following boundary conditions are satisfied:

$$\begin{aligned} w(0, y, t) = w(\pi, y, t) &= 0, \quad t \geq 0, y \geq 0 \\ w(x, 0, 0) &= 2\sin 7x, \quad 0 \leq x \leq \pi \\ \lim_{y\to\infty} w(x, y, t) &= 0, \quad t \geq 0, 0 \leq x \leq \pi. \end{aligned}$$

Solution: The steady-state solution has $\partial w/\partial t = 0$, so there is no t-dependence, and the PDE reduces to

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{\partial w}{\partial t} = 0,$$

giving Dirichlet's equation. Now separate variables $w(x, y) = u(x)v(y)$ to get two eigenvalue problems from the PDE:

$$u'' = \lambda u, \quad u(0) = u(\pi) = 0; \qquad v'' = -\lambda v, \quad v(y) \to 0 \text{ as } y \to \infty.$$

The first implies that $\lambda = -n^2$ for some integer n and that $u(x) = \sin nx$. The second implies that $v(y) = e^{-ny}$ and that $n > 0$. Combining these gives

$$w(x, y) = \sum_{n=1}^{\infty} b_n e^{-ny} \sin nx,$$

where b_n is the Fourier sine coefficient of the boundary function $2\sin 7x$, found by setting $y = 0$ to get $w(x, 0) = \sum_{n=1}^{\infty} b_n \sin nx = 2\sin 7x$. But this has the finite sine series $b_7 = 2$, with $b_j = 0$ for all $j \neq 7$. Thus

$$\boxed{w(x, y) = 2e^{-7y} \sin 7x}$$

5. Write the second-order system

$$x'' = x'y + \frac{t^2}{x^2 + y^2} \qquad y'' = y'x + \frac{1 - t^2}{x^2 + y^2}$$

as an equivalent system of first-order equations.

Solution: Let $x_0 = x(t)$, $x_1 = x'$, $y_0 = y(t)$, and $y_1 = y'$. The equivalent system is then

$$\begin{aligned}
x_0' &= x_1 \\
x_1' &= x_1 y_0 + \frac{t^2}{x_0^2 + y_0^2} \\
y_0' &= y_1 \\
y_1' &= y_1 x_0 + \frac{1 - t^2}{x_0^2 + y_0^2}
\end{aligned}$$

6. Show that $(x_1, y_1) = (e^{-t}, e^{-t})$ and $(x_2, y_2) = (e^{3t}, -e^{3t})$ are solutions to the homogeneous first-order linear system

$$\begin{aligned} x' &= x - 2y \\ y' &= -2x + y \end{aligned}$$

and that they are linearly independent on every closed interval of t.

Solution: Check by differentiation and substitution:

$$\begin{aligned} x_1' = -e^{-t} &= e^{-t} - 2e^{-t} = x_1 - 2y_1; \\ y_1' = -e^{-t} &= -2e^{-t} + e^{-t} = -2x_1 + y_1, \end{aligned}$$

and likewise

$$\begin{aligned} x_2' = 3e^{3t} &= e^{3t} - 2(-e^{3t}) = x_2 + 2y_2; \\ y_2' = -3e^{3t} &= -2e^{3t} + (-e{-}^{3t}) = -2x_2 + y_2. \end{aligned}$$

For linear independence, check the Wronskian:

$$W(t) = \det \begin{pmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{pmatrix} = \det \begin{pmatrix} e^{-t} & e^{3t} \\ e^{-t} & -e^{3t} \end{pmatrix} = -2e^{2t} < 0, \quad \text{all } t.$$

Thus $W(t)$ is never zero for any t. Conclude that (x_1, y_1) and (x_2, y_2) are linearly independent on every closed interval of t.

7. Consider the Volterra predator-prey equation for $x(t)$ and $y(t)$ on $t \geq 0$:

$$x' = x - xy; \qquad y' = -y + xy,$$

Find the steady-state solution and compute x'' and y'' in terms of x and y.

Solution: To find the steady-state solution, put $x' = 0$ and $y' = 0$ and solve to get

$$\boxed{x(t) = 1, \qquad y(t) = 1,}$$

both independent of time.

Differentiate the Volterra system, substitute, and simplify:

$$x'' = x' - x'y - xy' = x - xy - (x - xy)y - x(-y + xy) = \boxed{x - xy + xy^2 - x^2y}$$

and

$$y'' = -y' + x'y + xy' = y - xy + (x - xy)y + x(-y + xy) = \boxed{y - xy - xy^2 + x^2y}$$

8. Solve the initial value problem $y' - y = e^{2x}$, $y(0) = 2$, using the Laplace transform.

Solution: Transforming both sides gives $L[y' - y](p) = L[e^{2x}](p)$, which evaluates to

$$(p-1)L[y](p) - y(0) = \frac{1}{p-2}.$$

But then

$$L[y](p) = \frac{2}{p-1} + \frac{1}{(p-1)(p-2)} = \frac{2}{p-1} + \left(\frac{1}{p-2} - \frac{1}{p-1}\right) = \frac{1}{p-1} + \frac{1}{p-2},$$

after adding $y(0) = 2$ to both sides, dividing by $p - 1$, and expanding into partial fractions and simplifying. This gives

$$\boxed{y(x) = L^{-1}\left[\frac{1}{p-1} + \frac{1}{p-2}\right](x) = e^x + e^{2x}}$$

using linearity and the precomputed Laplace transform $L[e^{ax}](p) = 1/(p-a)$.

9. Solve the initial value problem $y' - y = e^{2x}$, $y(0) = 0$, using an integrating factor.

Solution: This is a first-order linear equation in the form $y' + py = q$ with $p(x) = -1$ and $q(x) = e^{2x}$. The integrating factor is

$$\mu(x) = \exp \int p(x)\, dx = e^{-x}$$

and the general solution is

$$y(x) = \frac{1}{\mu} \int \mu(x) q(x)\, dx = e^x \int e^x\, dx = e^x \left[e^x + C\right] = e^{2x} + Ce^x.$$

Solve for $C = -1$ from the initial condition $y(0) = 0 = 1 + C$ to get

$$\boxed{y(t) = e^{2x} - e^x.}$$

10. Find the Fourier series for the function g defined by

$$g(x) = \begin{cases} -1, & -\pi \leq x < 0, \\ 1, & 0 \leq x < \pi. \end{cases}$$

Solution: First note that g is an odd function. Hence the Fourier series for g has no cosine components, so $a_j = 0$ for all $j \geq 1$, and its sine components are

$$b_j = \frac{2}{\pi} \int_0^\pi \sin jx \, dx = \frac{2}{\pi} \left[\frac{-\cos jx}{j} \right] \Big|_0^\pi = \frac{2}{\pi} \left[\frac{-\cos j\pi + 1}{j} \right] = \begin{cases} \frac{4}{j\pi}, & \text{if } j \text{ is odd}, \\ 0, & \text{if } j \text{ is even}, \end{cases}$$

for $j = 1, 2, 3, \ldots$. Thus,

$$\boxed{g(x) = \sum_{\text{odd } j \geq 1} \frac{4}{j\pi} \sin jx = \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi} \sin(2k+1)x.}$$

Here we write odd $j \geq 1$ as $j = 2k + 1$ for $k \geq 0$.

11. Explain why $x = 0$ is an ordinary point for the differential equation $y' - y = e^{2x}$, and then find the recursion for $\{a_j : j = 0, 1, 2, \dots\}$ in the power series solution

$$y(x) = \sum_{j=0}^{\infty} a_j x^j.$$

(It is not necessary to solve the recursion.)

Solution: Write the equation as $y' + p(x)y = q(x)$ and observe that the coefficient functions $p(x) = -1$ and $q(x) = e^{2x}$ are analytic in an open interval around 0. Hence $x = 0$ is an ordinary point by definition.

The power series for $y' - y$ is obtained by differentiating term by term:

$$y'(x) - y(x) = \sum_{j=0}^{\infty} j a_j x^{j-1} - \sum_{j=0}^{\infty} a_j x^j = \sum_{j=0}^{\infty} [(j+1)a_{j+1} - a_j] x^j,$$

after reindexing and combining terms in x^j. This must equal the power series

$$e^{2x} = 1 + \frac{2x}{1!} + \frac{(2x)^2}{2!} + \cdots = \sum_{j=0}^{\infty} \frac{2^j}{j!} x^j,$$

so the coefficients of x^j must be equal in both series, so

$$\boxed{(j+1)a_{j+1} - a_j = \frac{2^j}{j!}, \qquad \Longrightarrow\ a_{j+1} = \frac{a_j}{j+1} + \frac{2^j}{(j+1)!}}$$

for $j = 0, 1, 2, \dots$.

12. The ODE

$$y^{(6)} + 4y^{(5)} + 3y^{(4)} - 10y^{(3)} - 26y'' - 24y' - 8y = 0$$

has characteristic equation

$$r^6 + 4r^5 + 3r^4 - 10r^3 - 26r^2 - 24r - 8 = (r+1)^2(r^2+2r+2)(r^2-4) = 0.$$

Find the general solution.

Solution:

$$y(x) = A_1e^{-x} + A_2xe^{-x} + B_1e^{-x}\cos x + B_2e^{-x}\sin x + Ce^{2x} + De^{-2x}.$$

corresponding to the repeated real root -1 ($A_1e^{-x} + A_2xe^{-x}$), complex conjugate roots $-1 \pm i$ ($B_1e^{-x}\cos x + B_2e^{-x}\sin x$), and distinct real roots 2 (Ce^{2x}) and -2 (De^{-2x}).