Math 132, Spring 2003
Exam 2: Solutions

No calculators with a CAS are allowed. Be sure your calculator is set for "radians", not "degrees" if you do any calculus computations with trig functions.

Part I, Multiple Choice, 5 points/problem: blacken your answers on the answer card.

1. Find the value of $\int_1^\infty xe^{-5x^2}\,dx$

A) $\frac{1}{10}$ B) e^{-2} C) $\frac{1}{2}e$ **D)** $\boldsymbol{\frac{1}{10}e^{-5}}$ E) diverges to ∞

F) e^{-10} G) $\frac{1}{2}e^{-4}$ H) $2e$ I) 2e^{-5} J) diverges to $-\infty$

Let $u = -5x^2$, $du = -10x\,dx$, so $-\frac{1}{10}du = x\,dx$. Then $\int xe^{-5x^2}dx = -\frac{1}{10}\int e^u\,du = -\frac{1}{10}e^u + C = -\frac{1}{10}e^{-5x^2} + C.$

Then $\int_1^\infty xe^{-5x^2}dx = \lim_{t\to\infty}\int_1^t xe^{-5x^2}dx = \lim_{t\to\infty} -\frac{1}{10}e^{-5x^2}|_1^t = -\frac{1}{10}\lim_{t\to\infty}(e^{-t^2} - e^{-5}) = \frac{1}{10}e^{-5}.$

2. For a certain radioactive isotope, the amount remaining after 3 hours is 78% of the original amount. What is its half-life? (*Round your answer to 2 decimal places.*)

A) 6.21 hrs **B)** 8.37 hrs C) 8.85 hrs D) 9.12 hrs E) 9.24 hrs

F) 9.56 hrs G) 9.75 hrs H) 9.93 hrs I) 10.12 hrs J) 10.35 hrs

Let y = amount of isotope present at time t. Then $\frac{dy}{dt} = ky$ and $y = y_0e^{kt}$. Since $0.78y_0 = y_0e^{3k}$, we get that $0.78 = e^{3k}$, so $\frac{\ln(0.78)}{3} = k$.

If h = half-life, we have $\frac{1}{2}y_0 = y_0e^{kh}$. Then $\ln(\frac{1}{2}) = -\ln 2 = kh$, so $h = \frac{-\ln 2}{k} = \frac{-3\ln 2}{\ln(0.78)} \approx 8.37$ hrs.

3. Suppose we perform the partial fraction decomposition

$$\frac{4x^2-3x-4}{x(x-1)(x+2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2}$$

What is the value of B ?

A) $B=0$	**B)** $B=-1$	C) $B=\frac{1}{2}$	D) $B=1$
E) $B=\frac{2}{3}$	F) $B=\frac{3}{4}$	G) $B=2$	H) $B=-\frac{1}{2}$
I) $B=-2$	J) $B=-\frac{2}{3}$		

$$\frac{4x^2-3x-4}{x(x-1)(x+2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2} = \frac{A(x-1)(x+2)+Bx(x+2)+Cx(x-1)}{x(x-1)(x+2)}.$$

Equating numerators gives

$$4x^2 - 3x - 4 = A(x-1)(x+2) + Bx(x+2) + Cx(x-1)$$

Setting $x=1$ gives $-3=3B$, so $B=-1$.

4. During Lab 2, LoggerPro software collected the following data for t (time, in sec) and v (velocity, in m/sec). Use Simpson's Rule to estimate the change in position (displacement) of the moving person for $0 \le t \le 0.4$. (*Round your answer to 2 decimal places.*)

t	v
0	-0.51
0.1	-0.20
0.2	0.01
0.3	0.01
0.4	-0.02

A) -0.10 m	B) -0.04 m	C) -0.06 m	D) 0.01 m
E) 0.13 m	F) -0.25 m	G) 0.15 m	H) -0.22 m
I) 0.31 m	J) -0.22 m		

The displacement is given by $\int_0^{0.4} v(t)\,dt$.

The interval $[0,0.4]$ is divided into 4 subintervals, with $\Delta t = 0.1$. Simpson's approximation is

$$\int_0^{0.4} v(t)dt \approx S_4 = \frac{0.1}{3}(-0.51 + 4(-0.20) + 2(0.01) + 4(0.01) + (-0.02))$$
$$\approx -0.04\,\text{m}$$

5. Find the value of $\int_0^1 \frac{e^x}{e^x - 1}\, dx$ (if the integral converges).

A) 17.08 B) 19.03 C) $\frac{e}{2}$ D) $\frac{e}{e-1}$ **E)** diverges to ∞
F) $\frac{2e}{e^2-1}$ G) e H) $\frac{e^2}{e^2-1}$ I) $2e$ J) diverges to $-\infty$

The integral is improper (type II) because $\frac{e^x}{e^x-1}$ has a vertical asymptote at $x = 0$.

If we let $u = e^x - 1,\ du = e^x dx,$ so $\int \frac{e^x}{e^x-1} dx = \int \frac{du}{u} = \ln|u| + C = \ln|e^x - 1| + C.$

Then $\int_0^1 \frac{e^x}{e^x-1}\, dx = \lim_{t\to 0^+} \int_t^1 \frac{e^x}{e^x-1}\, dx = \lim_{t\to 0^+} \ln|e^x - 1| \,|_t^1 =$
$\lim_{t\to 0^+} \ln(e-1) - \ln(e^t - 1) = \infty.$

6. Suppose a radioactive isotope is emitting radiation at a rate of $0.04e^{-2t}$ millirems/min, beginning at time $t = 0$. What is the total amount of radiation it will emit "over all time" – that is, if the process continues "forever" ?

A) 0.01 millirems B) 0.005 millirems **C)** 0.02 millirems D) 0.025 millirems
E) 0.04 millirems F) 0.05 millirems G) 2 millirems H) 10 millirems
I) 40 millirems J) an infinite amount

Total amount emitted $= \int_0^\infty 0.04e^{-2t}\, dt = \lim_{a\to\infty} \int_0^a 0.04e^{-2t}\, dt =$
$0.04 \lim_{a\to\infty} -\frac{1}{2}e^{-2t}|_0^a = 0.04 \lim_{a\to\infty} (-\frac{1}{2}e^{-2a} + \frac{1}{2}e^0) = 0.04(0 + \frac{1}{2}) = 0.02$ (millirems)

7. You use Simpson's rule with $n = 10$ to estimate the value of $\int_2^4 \frac{1}{x}\,dx$. Using the "error control formula," we get |ERROR| = $|\int_2^4 \frac{1}{x}\,dx - S_{10}| \leq$??? (*Find the best possible answer with this information; round your answer to 6 decimal places.*)

A) 0.000013 B) 0.000042 C) 0.000067 D) 0.000093 E) 0.000131
F) 0.000147 G) 0.000158 H) 0.000169 I) 0.000176 J) 0.000187

$|\int_2^4 \frac{1}{x}\,dx - S_{10}| \leq \frac{M(4-2)^5}{180(10)^4}$, where M is chosen so that $|f^{(4)}(x)| \leq M$ on $[2, 4]$.

We have $f(x) = x^{-1}$, $f'(x) = -x^{-2}$, ..., $f^{(4)}(x) = 24x^{-5} = \frac{24}{x^5}$. Since $\frac{24}{x^5}$ is positive and decreasing on $[2, 4]$, $f^{(4)}(x)$ has its max value at the point $x = 2$, so $|f^{(4)}(x)] \leq \frac{24}{32} = \frac{3}{4}$. This is the best choice of M we can make.

Therefore $|\int_2^4 \frac{1}{x}\,dx - S_{10}| \leq \frac{\frac{3}{4}(4-2)^5}{180(10)^4} = \frac{24}{180(10)^4} \approx 0.000013$

8. According to one simple physiological model, an adult male athlete needs 20 calories per day per pound of body weight to maintain his weight. If he consumes more or fewer calories than those required to maintain his weight, his weight changes at a rate proportional to the difference between the number of calories consumed and the number needed to maintain his current weight. If an athlete weighs 160 lbs. and consumes 3000 calories/day, which differential equation describes $W(t)$ = the athlete's weight at time t? (*In each equation, k is a constant.*)

A) $\frac{dW}{dt} = k(3000W - 20)$ B) $\frac{dW}{dt} = k(W - 20)$ C) $\frac{dW}{dt} = 2980\,kW$

D) $\frac{dW}{dt} = k(150 - 8W)$ **E)** $\frac{dW}{dt} = k(3000 - 20W)$ F) $\frac{dW}{dt} = k(\frac{3000}{W} - 20W)$

G) $\frac{dW}{dt} = k(3000 - \frac{W}{20})$ H) $\frac{dW}{dt} = (3000 - \frac{W}{160})$ I) $\frac{dW}{dt} = k(160 - 20W)$

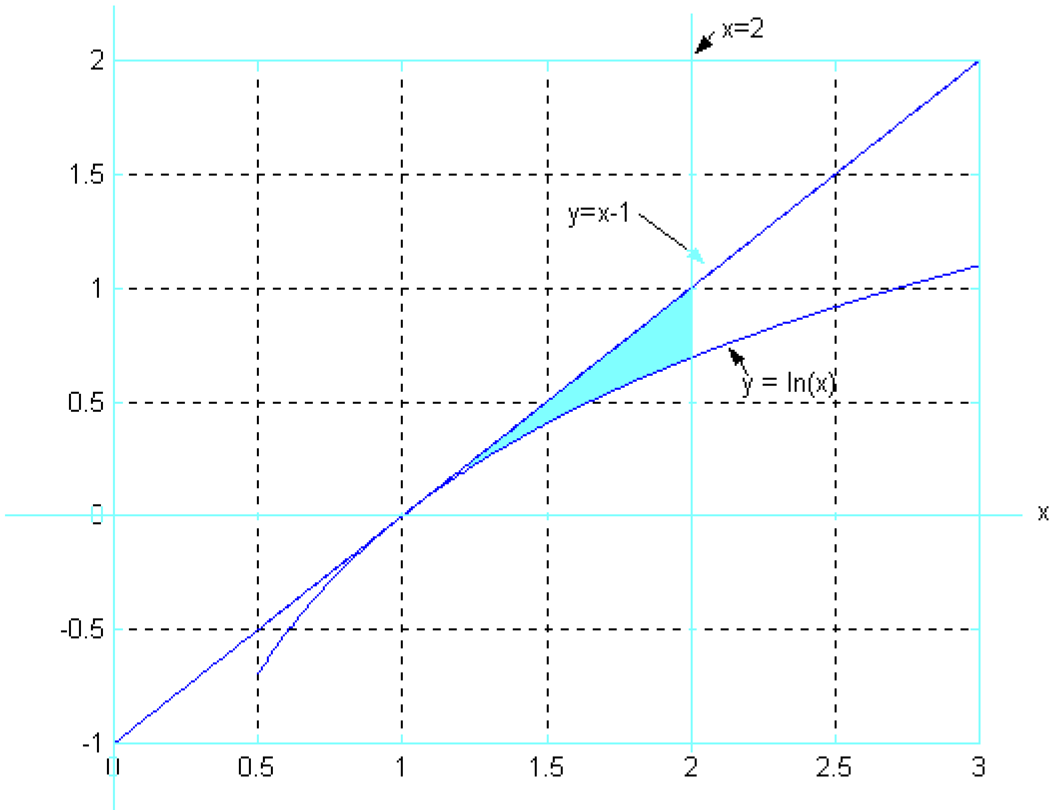
J) $\frac{dW}{dt} = k(3000 - 160W)$

$\frac{dW}{dt} = k$(number of calories consumed − number needed to maintain current weight W)
$= k(3000 - 20W)$

9. Find the area in the first quadrant bounded by the curves $y = \ln x$, $y = x - 1$ and $x = 2$.

A) $\frac{3}{4} - 4\ln 2$ B) $\ln 2$ C) 1 D) $\frac{5}{2} - \frac{(\ln 2)^2}{2}$ E) $\frac{1}{2}$
F) $\ln 2 - 2$ G) $\ln \frac{1}{2} - 3$ H) $2\ln 2$ I) $2\ln 2 - 1$ **J)** $\frac{3}{2} - 2\ln 2$

The figure shows the positions of the curves and the region (shaded):



The area of the region is $A = \int_1^2 (x-1) - \ln x\, dx$.

Using integration by parts, we get $\int \ln x\, dx = \int 1 \cdot \ln x\, dx =$
$= x\ln x - \int x \cdot \frac{1}{x}\, dx = x\ln x - x + C$.
Then $\int_1^2 (x-1) - \ln x\, dx = (\frac{x^2}{2} - x - (x\ln x - x))|_1^2 = (\frac{x^2}{2} - x\ln x)|_1^2 =$
$(2 - 2\ln 2) - (\frac{1}{2} - 0) = \frac{3}{2} - 2\ln 2$.

10. Find S_4 (= the Simpson approximation with 4 subdivisions) for $\int_0^4 x^2 + x\,dx$.

A) $\frac{107}{6}$ B) $\frac{74}{6}$ C) $\frac{153}{6}$ D) $\frac{183}{6}$ E) $\frac{193}{6}$
F) $\frac{55}{3}$ **G)** $\frac{88}{3}$ H) $\frac{97}{3}$ I) $\frac{47}{3}$ J) $\frac{70}{3}$

Method I (longer): We have $\Delta x = \frac{4-0}{4} = 1$, so

$S_4 = \frac{1}{3}((0+0) + 4(1^2+1) + 2(2^2+2) + 4(3^2+3) + (4^2+4))$
$= \frac{1}{3}(0+8+12+48+20) = \frac{88}{3}$ (≈ 29.33)

Method II (shorter): Simpson's Rule approximates the integrand with pieces of parabolas. Since the integrand $x^2 + x$ is itself a parabola, Simpson's approximation S_n (with any choice of even n) will actually give the exact value of the integral: $S_4 = \int_0^4 x^2 + x\,dx = (\frac{x^3}{3} + \frac{x^2}{2})|_0^4 = \frac{64}{3} + \frac{16}{2} = \frac{128+48}{6} = \frac{88}{3}$.

11. Suppose the velocity $v = \frac{ds}{dt}$ (m/sec) of a point moving along a line satisfies

$$\begin{cases} \frac{ds}{dt} = \frac{3s}{2t} & (t > 0) \\ s(1) = 1 \end{cases}$$

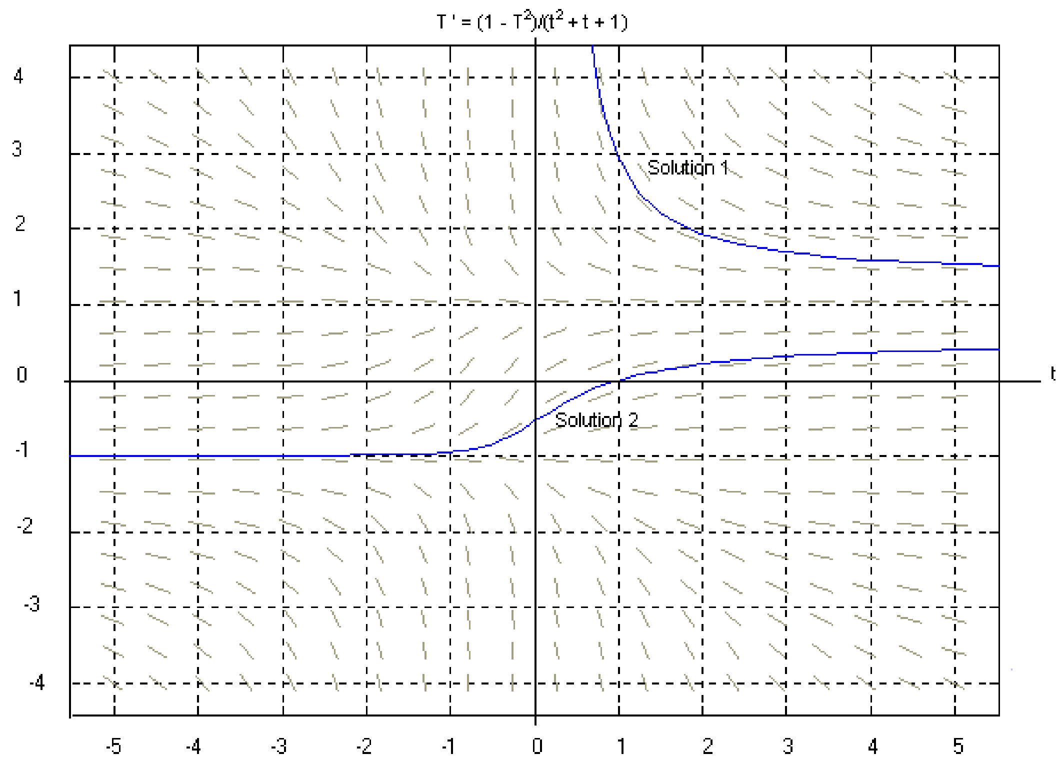
What is the position s of the particle when $t = 9$?

A) $-$ 22 m B) $-$ 10 m C) $-$ 5 m D) 0 m E) 35 m
F) 27 m G) 22 m H) 17 m I) 12 m J) 8 m

Separating variables in the d.e. and integrating both sides, we get $\int \frac{ds}{s} = \int \frac{3}{2}\frac{dt}{t}$, so $\ln|s| = \frac{3}{2}\ln|t| + D = \ln|t|^{3/2} + D$, so that $|s| = e^D|t|^{3/2}$. Then $s = \pm e^D|t|^{3/2} = C|t|^{3/2}$ (where $C = \pm e^D$).

Since $s(1) = 1$, we have $1 = C \cdot 1$, that is, $C = 1$. Therefore $s = |t|^{3/2}$, so $s(9) = 9^{3/2} = 27$ (m)

12. The temperature T of a body (C°) at time t (min) is changing at a rate of $\frac{dT}{dt} = \frac{1-T^2}{t^2+t+1}$. The slope field (= "direction field") for the differential equation is shown below (*some "solution curves" were added here on the solutions sheet.*)



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Over the time interval $-5 < t < 5$, <u>which of the following are true</u> (*you may need to use either the slope field or the differential equation itself to answer).*

i) If the temperature is 3° when $t = 1$, then the body is cooler when $t = 2$.
ii) No matter what the temperature is when $t = 1$, the body is cooler after time $t = 1$.
iii) The temperature of the body at time t depends only on t, not on the initial temperature T_0 of the body.
iv) If the initial temperature of the body is $1°$, then the body never changes temperature.

A) only i) B) only ii) C) only iii) D) only iv)
E) only i), ii) F) only i),iii) **G)** only i), iv) H) only ii), iii)
I) only ii), iv) J) only iii), iv)

i) is true: look at the curve “Solution 1” in the figure

ii) is false: look at the curve “Solution 2” in the figure

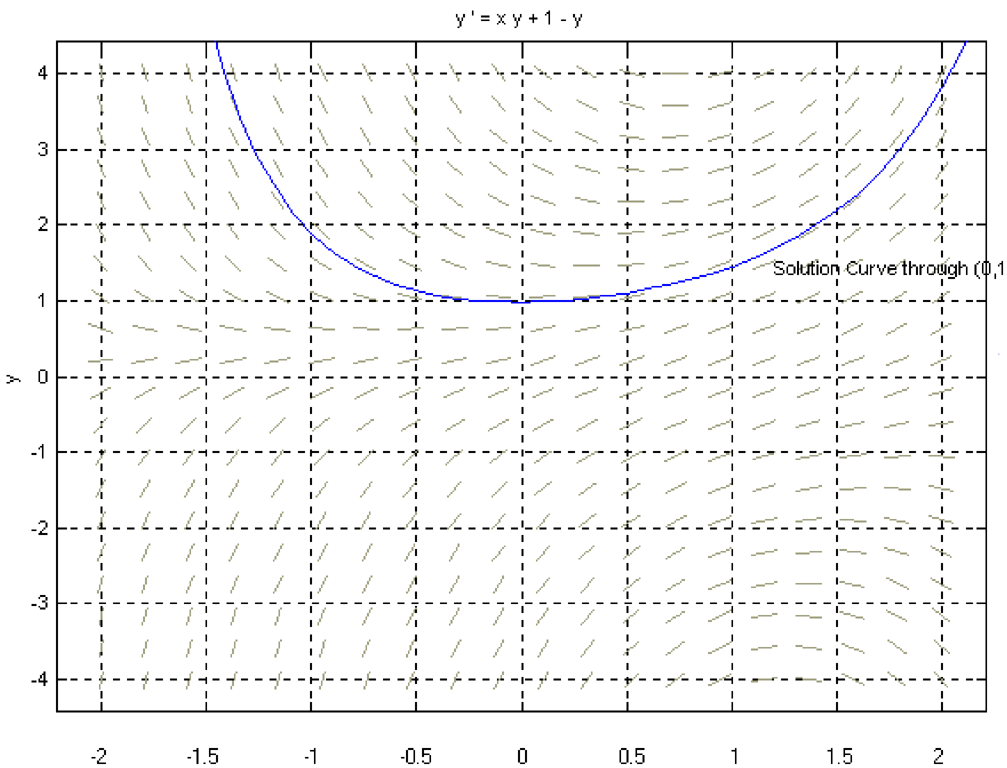
iii) is false: In the figure, “Solution 1” and “Solution 2” (just for example) show different temperatures for the same time ($t = 1$, say). They differ at $t = 1$ because they had different initial temperatures T_0.

iv) is true: you can see (approximately) the figure that the solution curve through $(0, 1)$ is horizontal. More rigorously: the constant curve $T = 1$ has $T_0 = T(0) = 1$ and it obviously satisfies the d.e., (plug it in!)

13. Using Euler's method, with stepsize $h\ (= dx) = 0.25$, estimate $y(0.5)$ if

$$\begin{cases} \frac{dy}{dx} = xy + 1 - y \\ y(0) = 1 \end{cases}$$

(*Round your answer to 2 decimal places. The slope field pictured below is not needed for solving the problem – but if you understand it, it will help you narrow down your choices.*)



A) 1.0202 B) 1.0465 **C)** 1.0625 D) 1.0842 E) 0.4322
F) 0.3222 G) 0.2212 H) 2.0212 I) 2.0432 J) 2.4024

The slope field (with solution curve through $(0,1)$ now included on the solution version) indicates that $y(0.5)$ should be just a little larger than 1. (*This rules out answers E-J*) .

Format 1 for solution: for Euler's method we have $h = dx = 0.25 = \frac{1}{4}$ and (in text's notation) the d.e. $\frac{dy}{dx} = F(x,y) = xy + 1 - y$. We begin at $(x_0, y_0) = (0,1)$. The next two steps in Euler's method would give us $(x_1, y_1) = (\frac{1}{4}, y_1)$ and $(x_2, y_2) = (\frac{1}{2}, y_2)$, and we need to know y_2. Euler's method gives $y_1 = y_0 + hF(x_0, y_0) = 1 + \frac{1}{4}F(0,1) = 1 + \frac{1}{4}(0) = 1$, and then
$y_2 = y_1 + hF(x_1, y_1) = 1 + \frac{1}{4}(F(\frac{1}{4}, 1)) = 1 + \frac{1}{4}(\frac{1}{4}) = 1 + \frac{1}{16} = 1.0625$

Format 2 for solution:

x	y	dx	$dy = (xy + 1 - y)\,dx$	$x + dx$	$y + dy$
0	1	$\frac{1}{4}$	0	$\frac{1}{4}$	1
$\frac{1}{4}$	1	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{2} = 0.5$	$\frac{17}{16} = 1.0625$

14. Suppose that $\int_1^e \frac{1}{x} \cdot f'(x)\,dx = 1$ and that $f'(e) = 0$. What is the value of $\int_1^e f''(x) \ln x\,dx$? (*Hint: integration by parts.*)

A) -2 **B)** -1 C) 0 D) $\frac{1}{2}$ E) 1

F) $\frac{3}{2}$ G) 2 H) $\frac{5}{2}$ I) 3 J) $\frac{7}{2}$

Let $u = \ln x, \;\; dv = f''(x)\,dx$
$du = \frac{1}{x}dx, \; v = f'(x).$

Using the formula for integration by parts gives

$\int_1^e f''(x) \ln x\,dx = (\ln x)(f'(x))|_1^e - \int_1^e f'(x)\frac{1}{x}dx.$ Substituting gives

$$\int_1^e f''(x) \ln x\,dx = (\ln e) \cdot f'(e) - (\ln 1) \cdot f'(1) - 1 =$$
$$= 1 \cdot 0 - 0 \cdot f'(1) - 1 = -1.$$

Part II, True or False, 1 point each. Blacken your answers on the answer card.

15. In Lab 2 you used a motion detector to gather position and velocity data. When you backed away from the sensor, LoggerPro recorded negative velocities.

A) True **B) False**

False: LoggerPro measured distance from the sensor. When you backed away, this distance was increasing, so your velocity was positive.

16. We can conclude that $\int_1^\infty \frac{1}{\sqrt{x+\sqrt{x}}}\,dx$ diverges by comparing it to the integral $\int_1^\infty \frac{1}{\sqrt{2x}}\,dx$.

A) True B) False

True: since $x \geq 1$ in these integrals, we know that $\sqrt{x} \leq x$, so $\frac{1}{\sqrt{x+\sqrt{x}}} \geq \frac{1}{\sqrt{x+x}} = \frac{1}{\sqrt{2x}}$. We know $\int_1^\infty \frac{1}{\sqrt{2x}}\,dx = \frac{1}{\sqrt{2}}\int_1^\infty \frac{1}{x^{1/2}}\,dx$ diverges (why?). Therefore because $\int_1^\infty \frac{1}{\sqrt{x+\sqrt{x}}}\,dx \geq \int_1^\infty \frac{1}{\sqrt{2x}}\,dx$, we conclude that the first integral diverges by the Comparison Test.

17. Let $w = f(u)$ be a solution of the differential equation $\frac{dw}{du} = \frac{w^2+1}{u^2+2}$. Then $f(2) > f(3)$.

A) True **B) False**

False: From the d.e. $\frac{dw}{du} = \frac{w^2+1}{u^2+2}$, we see $\frac{dw}{du}$ is always positive. Therefore any solution $w = f(u)$ is an increasing function – so $f(2) < f(3)$.

18. $\int_1^\infty \frac{3}{(2t)^{2k+1}}\,dt$ converges if and only if $k > 0$.

A) True B) False

True: $\int_1^\infty \frac{3}{(2t)^{2k+1}}\,dt = \int_1^\infty \frac{3}{2^{2k+1}t^{2k+1}}\,dt = \frac{3}{2^{2k+1}}\int_1^\infty \frac{1}{t^{2k+1}}\,dt$. But we know that $\int_1^\infty \frac{1}{t^p}\,dt$ converges if an only if $p = 2k+1 > 1$, that is, if and only if $k > 0$.

19. Let y be the amount of a radioactive isotope present at time t. If the isotope has half-life of 0.39 years, then (*rounded to 2 decimal places*) $\frac{dy}{dt} = -1.78\,y$.

A) True B) False

True: We know that $\frac{dy}{dt} = ky$, which has solution $y = y_0 e^{kt}$. If the half-life is 0.39, then $\frac{1}{2}y_0 = y_0 e^{k(0.39)}$, which gives $k = \frac{-\ln 2}{0.39} = -1.78$ (rounded to two places), so $\frac{dy}{dt} = ky = -1.78y$.

Part III : These are two "free response" problems #20, #21) worth a total of 25 points. Write your answers on the test pages. <u>Show your work</u> neatly and cross out irrelevant scratchwork, false starts, etc.

Please put your <u>name</u> on each of the following pages, since they may be separated during grading. Also, please add your <u>Discussion Section Letter</u> (available on your exam front cover sheet) on each page so that we can return papers through discussion sections.

Name________________________________ Discussion Section Letter _________

20. (*Note: there may be several ways to do some of these integrals--some harder, some easier.*)

a) Find $\int \sin^9 x \cos^3 x \, dx$

Let $u = \sin x,$ so $du = \cos x \, dx$. Then $\int \sin^9 x \cos^3 x \, dx = \int \sin^9 x \cos^2 x \cos x \, dx =$

$\int \sin^9 x \, (1 - \sin^2 x) \cos x \, dx = \int u^9 (1 - u^2) \, du = \int u^9 - u^{11} \, du = \frac{u^{10}}{10} - \frac{u^{12}}{12} + C =$

$\frac{1}{10} \sin^{10} x - \frac{1}{12} \sin^{12} x + C.$

b) Evaluate $\int_0^2 \sqrt{4 - x^2} \, dx$

<u>Method I</u> (easy): the integral represents the area under $y = \sqrt{4 - x^2}$ and above $[0, 2]$. This region is a quarter circle with radius 2, so it has area $\frac{1}{4}\pi(2)^2 = \pi$.

<u>Method II</u> (trig substitution): let $x = 2\sin\theta$, $dx = 2\cos\theta \, d\theta$. Then $\int \sqrt{4 - x^2} \, dx =$
$\int \sqrt{4 - 4\sin^2\theta} \; 2\cos\theta \, d\theta = \int 2\sqrt{\cos^2\theta} \; 2\cos\theta \, d\theta = \int 4\cos^2\theta \, d\theta =$
$\int 4 \cdot \frac{1 + \cos 2\theta}{2} d\theta = 2\int 1 + \cos 2\theta \, d\theta = 2(\theta + \frac{1}{2}\sin 2\theta) + C = 2\theta + \sin 2\theta + C.$

Since $\theta = \arcsin(\frac{x}{2})$, $2\theta + \sin 2\theta = 2\arcsin(\frac{x}{2}) + \sin(2\arcsin(\frac{x}{2}))$.

Therefore $\int_0^2 \sqrt{4 - x^2} \, dx = 2\arcsin(\frac{x}{2}) + \sin(2\arcsin(\frac{x}{2})) \,|_0^2 =$

$$2 \cdot \frac{\pi}{2} + \sin(2 \cdot \frac{\pi}{2}) - 0 - 0 = \pi.$$

(Note that $\sin(2\arcsin(\frac{x}{2}))$ can be simplified to $\frac{x}{2}\sqrt{4 - x^2}$, but that's not needed to complete the problem.)

c) Find $\int_{-1}^{1} \frac{x}{(x+2)(x-2)} dx$

Method I (easiest): The function $f(x) = \frac{x}{(x+2)(x-2)}$ is an odd function (i.e., its graph is symmetric with respect to the origin), so $\int_{-1}^{1} \frac{x}{(x+1)(x-1)} dx = 0.$

Method II (not much harder): $\int_{-1}^{1} \frac{x}{(x+2)(x-2)} dx = \int_{-1}^{1} \frac{x}{(x^2-4)} dx.$
Let $u = x^2 - 4,\ du = 2x\,dx$, so that $x\,dx = \frac{1}{2} du.$

Then (adjusting the limits) $\int_{-1}^{1} \frac{x}{(x^2-1)} dx = \int_{-3}^{-3} \frac{1}{2} \frac{1}{u} du \ = \ 0.$

Method III: (use partial fractions) $\frac{x}{(x+2)(x-2)} = \frac{A}{(x+2)} + \frac{B}{(x-2)} = \frac{A(x-2)+B(x+2)}{(x+2)(x-2)}$

Equating numerators gives $x = A(x - 2) + B(x + 2)$:

Let $x = 2$	to get $2 = 4B$	so $B = \frac{1}{2}$
Let $x = -2$	to get $-2 = -4A$	so $A = \frac{1}{2}$

Then $\int_{-1}^{1} \frac{x}{(x+2)(x-2)} dx = \int_{-1}^{1} \frac{1/2}{x+2} + \frac{1/2}{x-2} dx = \frac{1}{2} \left(\ln|x+2| + \ln|x-2| \right) \Big|_{-1}^{1} =$

$\frac{1}{2}(\ln 3 + \ln 1 - \ln 1 - \ln 3) = 0.$

21. A tank initially contains 200 L of pure water. Brine containing 5g of salt/L enters the tank at a rate of 2L/min. There is instant mixing and solution leaves the tank at 2L/min.

Let y denote the amount of salt in the tank at time t.

a) Write a differential equation that describes this situation: $\frac{dy}{dt} = ...$

$$\frac{dy}{dt} = (\text{rate salt enters}) - (\text{rate salt exits}) = (5\text{ g/L}) \cdot (2\text{ L/min}) - \left(\frac{y}{200}\text{ g/L}\right) \cdot (2\text{ L/min}) =$$

$$= 10\text{ g/min} - \frac{y}{100}\text{g/min}$$

so

$$\frac{dy}{dt} = 10 - \frac{y}{100} \quad (\text{g/min})$$

b) Solve the differential equation and find the amount of salt in the tank after 100 minutes.

$$\frac{dy}{dt} = 10 - \frac{y}{100} = \frac{1}{100}(1000 - y)$$

$$\frac{dy}{1000 - y} = \frac{1}{100}dt$$

so $\int \frac{dy}{1000 - y} = \int \frac{1}{100}dt$

so $-\ln|1000 - y| = \frac{1}{100}t + D$, or $\ln|1000 - y| = -\frac{1}{100}t - D$.

Then $|1000 - y| = e^{-\frac{1}{100}t - D} = e^{-D}e^{-\frac{1}{100}t}$,

so $1000 - y = \pm e^{-D}e^{-\frac{1}{100}t} = Ce^{-\frac{1}{100}t}$ (where $C = \pm e^{-D} =$ constant).

Therefore $y = 1000 - Ce^{-\frac{1}{100}t}$.

Since $y(0) = 0$, we get $C = 1000$, so $y = 1000 - 1000e^{-\frac{1}{100}t}$.

Therefore $y(100) = 1000 - 1000e^{-\frac{1}{100}100} = 1000 - 1000e^{-1} \approx 632.12$ g.

c) What is the "steady state amount" – that is, the limit of the amount of salt y as $t \to \infty$? (*If you can't do parts a) and b), you should nevertheless be able to give a reasonable explanation that arrives at the correct answer.*)

$$\lim_{t\to\infty} y = \lim_{t\to\infty} 1000 - 1000e^{-\frac{1}{100}t} = 1000\,(\text{g}).$$

This is completely plausible. As time passes, the brine in the tank "looks more and more like" the brine entering the tank, so, in the long run, the amount of salt in the tank should approach $(200\text{ L}) \cdot (5\text{ g/L}) = 1000$ g of salt.