Geometric Construction of Regulator Currents

with Applications to Algebraic Cycles

Matt Kerr

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Abstract

This is the first complete study of regulator maps on motivic cohomology from the standpoint of complex algebraic geometry. The classical Abel-Jacobi map is used to geometrically motivate the construction of maps from Milnor K-groups $K_n^M(\mathbb{C}(X))$ to Deligne cohomology. These maps are given in terms of some new, explicit $(n-1)$-currents. We study their behavior in families X_t and prove a rigidity result for the image of the Tame kernel, using techniques from the theory of variations of Hodge structure. This leads to an astonishing vanishing theorem for very general complete intersections.

The Milnor current formulas generalize to regulator maps (defined on the level of algebraic cycle complexes) on all the cubical higher Chow groups $CH^p(X,n)$, whose projections to real Deligne cohomology are (by involved computations) shown to be compatible with the Beilinson regulator on niveau-graded pieces. Connections with polylogarithms and higher Bloch groups are explored in several ways: for example, (1) by means of higher residue maps arising as differentials in relevant local-global spectral sequences, and (2) by way of a new approach to computing certain relative regulators.

We generalize the Milnor currents in another direction to produce explicit integrals detecting rational inequivalence to zero, for 0-cycles in the Albanese($=AJ$) kernel on a product of curves; concrete examples are provided. More generally, we combine and extend the work of Green-Griffiths and Lewis on higher Abel-Jacobi maps Ψ_i, and show that the above integrals compute (essentially) quotients of the invariants $\Psi_i(\mathcal{Z})$.

Preface

Complex algebraic geometry is the study of *varieties*, or the solution sets of algebraic equations (possibly many at once), whose coefficients belong to $\mathbb{C}$. A classical question, which has seen a resurgence of interest (e.g. [**Gr1**], [**GG5**], [**L2**], [**RS**]) in recent years, is that of *rational equivalence* of algebraic cycles on a smooth projective variety $X/\mathbb{C}$. For instance, suppose we take two collections $\mathcal{Z}_1$ and $\mathcal{Z}_2$ of N points (or 0-cycles) on X. When can we algebraically parametrize, in one variable, a "path" (essentially n simultaneous paths) between the two collections? That is, writing $\square := \mathbb{P}^1_{\mathbb{C}} \setminus \{1\}$ for affine space, when does there exist an algebraic 1-cycle $W \in Z_1(X \times \square)$ such that

$$\mathcal{Z}_1 - \mathcal{Z}_2 = \pi_*^X(W \cdot X \times \{0\}) - \pi_*^X(W \cdot X \times \{\infty\}) \ \ ?$$

For $\dim_{\mathbb{C}} X = 1$, this is just the question of when $\mathcal{Z}_1 - \mathcal{Z}_2 = (f)$ for $f \in \mathbb{C}(X)$, and it is "solved" by Abel's theorem (see §5.2.1 for a brief review). In fact, this holds in higher dimension provided $\mathrm{codim}_X \mathcal{Z}_i = 1$: the two invariants Ψ_0 :=cycle-class map and Ψ_1:=Abel-Jacobi map still completely detect $\underset{\text{rat}}{\not\equiv} 0$.

One of the "holy grails" of the subject of Algebraic Cycles $/\mathbb{C}$, then, is the explicit description of a series of Hodge-theoretically determined, higher AJ-type maps Ψ_i (defined on $\ker(\Psi_{i-1})$) which "completely capture" rational equivalence classes (modulo torsion) of codimension $p\,(\geq 2)$ cycles on a smooth projective variety X. That is, their successive kernels should give a descending filtration exhausting $CH^p(X(\mathbb{C}))_{\mathbb{Q}}$. As in the $p=1$ case we can define Ψ_0 and Ψ_1, which are then summarized (see [**Gr3**]) in the Deligne cycle-class map

$$c_{\mathcal{D}} : CH^p(X(k)) \otimes \mathbb{Q} \to H^{2p}_{\mathcal{D}}(X, \mathbb{Q}(p)), \quad k \subseteq \mathbb{C}.$$

This is injective (a) for $p = 1$ (by Abel), and also (b) for $k \subseteq \bar{\mathbb{Q}}$ (conjecturally, according to Bloch-Beilinson). Otherwise its kernel may be huge, and so one is naturally led into the arithmetic world in dealing with the field extension $k/\mathbb{Q}$. One can in fact exchange this for additional geometry by "spreading" $X \supseteq \mathcal{Z}^p/k$ over the generic point η_S of some projective $S/\mathbb{Q}$ with $\mathbb{Q}(S) \cong k$, obtaining $\mathcal{X} \supseteq \zeta^p/\mathbb{Q}$. We shall usually assume for simplicity that the coefficients of the defining equation of X belong to $\mathbb{Q}$, so that one has $\mathcal{X} = X \times \eta_S$.

Provided one is willing to assume a Bloch-Beilinson conjecture for *quasi*-projective varieties (of which $X \times \eta_S$ is a limit),

$$'c_{\mathcal{D}} : CH^p(X \times \eta_S(\mathbb{Q})) \otimes \mathbb{Q} \hookrightarrow H^{2p}_{\mathcal{D}}(X \times \eta_S, \mathbb{Q}(p)).$$

Following **[L2]** and **[GG5]**, in §5.1 we produce a series of invariants Ψ_i by placing a Leray filtration on [a suitable modification of] $H^{2p}_{\mathcal{D}}(X \times \eta_S, \mathbb{Q}(p))$, and composing the resulting "graded pieces" of $'c_{\mathcal{D}}$ with the act of spreading. So intuitively, the idea is to use the product structure of $X \times \eta_S$ to chop up the cycle- and AJ-classes of the spread ζ of $\mathcal{Z}$. The difficulty is in finding explicit formulae for the resulting maps

$$\Psi_i : Gr^i_{\Psi} CH^p(X(k))_{\mathbb{Q}} \longmapsto Gr^i_{\mathcal{L}} \underline{\underline{H}}^{2p}_{\mathcal{H}}(X \times \eta_S, \mathbb{Q}(p)).$$

Incidentally one should not take the assumption of even the Bloch-Beilinson conjecture lightly. There are very simple cases where it is not known. For example, say we present a $K3$ surface X as a double cover of $\mathbb{P}^2$ branched over a smooth sextic curve with affine equation $f(x, y) = 0$. Then for any $x, y \in \bar{\mathbb{Q}}$ not solving f,

$$\mathcal{Z}_{x,y} := (x, y, +\sqrt{f(x,y)}) - (x, y, -\sqrt{f(x,y)}) \in \ker(c_{\mathcal{D}}) \subseteq CH^2(X(\bar{\mathbb{Q}})),$$

since $h^{1,0}(X) = 0$; but we are not aware of any proof that (some multiple of) $\mathcal{Z}_{x,y} \underset{\text{rat}}{\equiv} 0$. If BBC fails (for quasi-projectives) then the series Ψ_i is still (well-)defined but their kernels do not exhaust $CH^p(X(k))_{\mathbb{Q}}$. So without the aid of conjectures, at present the best we can hope for is to detect $\mathcal{Z} \underset{\text{rat}}{\not\equiv} 0$ when $c_{\mathcal{D}}(\mathcal{Z}) = 0$, by showing e.g. $\Psi_2(\mathcal{Z})$ or $\Psi_3(\mathcal{Z}) \neq 0$. Some concrete examples are given at the end of §5.3.

More generally, what we find in Chapter 5 is that we can motivate a very explicit recipe for (quotients of) the graded pieces $[AJ\zeta]_i$, which is moreover computable for X a product of curves, by studying $AJ\zeta$ in the *degenerate* situation[1] $X = (\square^n, \partial\square^n)$ – where all but one Leray graded piece is zero. We can compute this piece exactly, essentially by "pushing it down" to integration of a current over integral cycles on the "base" η_S of the spread. These "regulator currents of Milnor type" define cohomology classes (or at worst differential characters $\Longleftrightarrow$ Deligne classes) on η_S; and the general philosophy of "pushing down" $[AJ\zeta]_i$ to i-currents on the base is what extends to X smooth projective.

Sticking with the degenerate situation now, which is to say relative cycles on $\zeta \subseteq \eta_S \times (\square^n, \partial\square^n)$, we backtrack a bit from Chapter 5 to Chapters 2-3 and change notation $\zeta \mapsto \mathcal{Z}$, and replace η_S by any quasi-projective (possibly projective) variety $Y/\mathbb{C}$, so that we are now dealing with $[\mathcal{Z}] \in CH^p(Y \times (\square^n, \partial\square^n))_{\mathbb{Q}}$ or (equivalently as far as cycle *classes* are concerned)

[1] affine n-space relative its faces. This is equivalent to projective n-space relative the coordinate hyperplanes, or even essentially to the *singular* variety given by the union of the coordinate hyperplanes in $\mathbb{P}^{n+1}$.

with higher Chow cycles $CH^p(Y,n)$. (In fact, in the text we write X in place of Y; we warn the reader that X has changed sides in the product.)

Again in this situation we can give formulas for AJ maps on $CH^p(Y,n)_{\mathbb{Q}}$ in terms of explicit currents "on the base" Y, $\mathcal{Z} \mapsto (T_{\mathcal{Z}}, \Omega_{\mathcal{Z}}, R_{\mathcal{Z}})$ (where in many cases one can omit all but the $R_{\mathcal{Z}}$). These are interesting in their own right (rather than just as preparation "in the degenerate case" for Chapter 5), since they may be interpreted as realization functors or "regulators" on motivic cohomology

$$[CH^p(Y,n)_{\mathbb{Q}} \cong]\ H^{2p-n}_{\mathcal{M}}(Y,\mathbb{Q}(p)) \to H^{2p-n}_{\mathcal{D}}(Y,\mathbb{Q}(p)),$$

and so are referred to alternately as AJ and $\mathcal{R}$. So we derived a formula for these maps (§2.4), whose geometric motivation is given in §1.3 – 2.3 (for $n = p$) and §5.1.1 (for $n \neq p$). According to §3.1 our formula gives a $\mathbb{C}/\mathbb{Q}(p)$-lift of the real regulator maps (see **[Be]** or **[Ra]**)

$$H^{2p-n}_{\mathcal{M}}(Y,\mathbb{Q}(p)) \to H^{2p-n}_{\mathcal{D}}(Y,\mathbb{R}(p))$$

for which Goncharov wrote explicit *real* currents[2] in **[Go1]**. (The Abel-Jacobi map given there was not correct.)

A very specific instance of these maps occupies our attention in Chapter 4 and most of 1 and 2 – namely, that given by setting $p = n$, $Y = \eta_X$ (or $Spec(\mathbb{C}(X))$), and usually $\dim_{\mathbb{C}} X = n-1$. The resulting invariant

$$K^M_n(\mathbb{C}(X)) \cong CH^n(\eta_X,n) \xrightarrow{\mathcal{R}} H^n_{\mathcal{D}}(\eta_X,\mathbb{Z}(n)) \cong H^{n-1}(\eta_X,\mathbb{C}/\mathbb{Z}(n))$$

is called the "Milnor regulator" (see §1.2 for a definition of the Milnor K-groups). Since the generators $\{f_1,\dots,f_n\}$ of $K^M_n(\mathbb{C}(X))$ involve explicit meromorphic functions $f_i \in \mathbb{C}(X)^*$, this is where we can *really* get our hands on the regulator currents "$R_{\mathbf{f}}$"; in fact, the original motivation in Chapter 2 for the construction of $R_{\mathcal{Z}}$ for $\mathcal{Z} \in Z^p(Y,n)$ with $p \neq n$, comes out of the desire to get our hands on the so-called "higher residues" $\mathrm{Res}^i R_{\mathbf{f}}$ for $i \geq 2$ and interpret them in terms of polylogarithms and Bloch groups.

Here is a concrete example of what a Milnor regulator current loooks like, for $n = 3$. If f, g, $h \in \mathbb{C}(S)$ are meromorphic functions on an algebraic surface, let $T_f = f^{-1}(\mathbb{R}^-)$ (where $\mathbb{R}^-$ is considered as the directed path $\overleftarrow{[0,\infty]}$ on $\mathbb{P}^1$) and $\log f$ = the branch with imaginary part $\in (-\pi,\pi]$ and jump along T_f, and so on for g and h. On the other hand $\mathrm{dlog} f$ will mean $\mathrm{d}f/f$; they are related by $\mathrm{d}[\log f] = \mathrm{dlog} f - 2\pi i \delta_{T_f}$. Then if $\mathcal{C}$ is a "topological" 2-chain ($\dim_{\mathbb{R}} \mathcal{C} = 2$) on S avoiding $|(f)| \cup |(g)| \cup |(h)|$, the period of $R_{\{f,g,h\}}$ on $\mathcal{C}$ is by definition

$$\int_{\mathcal{C}} \log f \,\mathrm{dlog} g \wedge \mathrm{dlog} h \;+\; 2\pi i \int_{\mathcal{C}\cap T_f} \log g \,\mathrm{dlog} h \;-\; 4\pi^2 \sum_{p\in \mathcal{C}\cap T_f \cap T_g} \log h(p).$$

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[2]More precisely, we show our regulator lifts Goncharov's exactly, while Goncharov's formula is known to agree with the Beilinson regulator on niveau *graded* pieces of $CH^p(X,n)_{\mathbb{Q}}$.

These currents (as well as the formulas of §2.4) generalize and bridge the gap between several maps in the literature, see [**GG1**], [**L4**], [**C1**], [**Es**], [**MS1**], etc.

Nontrivial periods of Milnor regulator currents are very hard to compute in practice, except when X is a relative variety like those considered in §3.2, or when the periods come from residues. In fact, in Chapter 4 we prove a vanishing theorem for the "residue-free" part of the map when X is smooth and sufficiently general in its moduli space, which is to say any interesting periods of the "holomorpic part" of the Milnor regulator arise arithmetically and not geometrically, except for $n = 2$ and X and elliptic curve. This in particular is a really beautiful result.

A Brisk Outline

We have been going backwards; let's now reverse course and give a section-by-section guide. This should be useful for the first three chapters in particular; a coherent but somewhat stream-of-consciousness style seems to have resulted from the desire to focus on not so much a series of results as a web of interrelationships. At the outset (in Chapters 1-2) we wish to extend the classical AJ maps to relative quasi-projective varieties $X \times (\square^n, \partial\square^n)$. Why? Because these maps have the aforementioned motivic interpretation, and motivate an understanding of AJ on spreads (and thus the Ψ_i).

In §1.1-2 we introduce the higher Chow groups $CH^p(X,n)_{[\mathbb{Q}]}$, which are just a geometric realization of $H^{2p-n}_{\mathcal{M}}(X, \mathbb{Q}(p))$ in terms of cycles on $X \times (\square^n, \partial\square^n)$, and specialize to the case $n = p$. In order to understand the situation algebraically and analytically in terms of meromorphic functions on X and its subvarieties, we proceed in two steps. We first break $CH^n(X, n)$ into coniveau-graded pieces linked by geometric "residue" maps. The pieces are *subquotients* of $\coprod_{x \in X^i} CH^i(\mathbb{C}(x), n-1)$ and one has successive Res^i mapping $\big(\ker(\mathrm{Res}^i) \subseteq\big)\, CH^n(\mathbb{C}(X), n)$ into them. Second, to "explain" the first two graded pieces we introduce the Milnor K-groups and the graph isomorphism

$$K_n^M(\mathbb{C}(X)) \underset{\gamma}{\overset{\cong}{\longrightarrow}} CH^n(\mathbb{C}(X), n),$$

and define the "holomorphic" or "residue free" subgroup

$$K_n^M(X) := \gamma^{-1}\left(\mathrm{im}\{CH^n(X,n) \to CH^n(\mathbb{C}(X),n)\}\right) = \gamma^{-1}\Big(\bigcap_{i=1}^{\dim(X)} \ker(\mathrm{Res}^i)\Big).$$

By partial degeneration of the local-global spectral sequence one shows that it is enough to take $\bigcap_{i=1}^{\mu}$ where $\mu = \min\{\dim X, \left[\frac{n}{2}\right]\}$. Finally, after interpreting Res^1 in terms of the "tame" map of [**BT**], in §1.2.4 we attempt to arrive at a similar understanding of the Res^i on $K_n^M(\mathbb{C}(X))$ directly (algebraically rather than geometrically); we see the first exciting (but conjectural) connections with polylogarithms and Bloch groups (of $\mathbb{C}(x),\ x \in X^i$).

In §1.3 we start developing our geometric approach to the realization functors (or regulators) $H^{2p-n}_{\mathcal{M}}(X, \mathbb{Q}(p)) \to H^{2p-n}_{\mathcal{M}}(X, \mathbb{Q}(p))$ that will culminate in §2.4. We establish precisely the sense in which relative cycle-class and AJ maps on relative cycles in $X \times (\square^n, \partial\square^n)$ are possible, by turning them into limits of topological cycles avoiding $X \times \partial\square^n$. Now computing $AJ(\mathcal{Z})$ always involves integrating forms over chains Γ with $\partial\Gamma = \mathcal{Z}$. The computational key here is that there is a standard way to write down Γ as a "geometric collapsing sum" $\theta(\mathcal{Z})$ (taking advantage of the fundamental domain of $\mathbb{C}^*$) plus a membrane term $T_{\mathbf{f}} \times (n\text{-torus}) \subseteq X \times (\mathbb{C}^*)^n$; the rest of the argument consists of Hodge-theoretic considerations. This procedure has its roots in Chapter 8 of Bloch's book **[B1]**, where he does $n = p = 2$. One can visualize θ as a homotopy contracting the fundamental domains of the various factors (=copies of $\mathbb{C}^*$) to $\{1\}$, one after the other.

In §1.4 we just restrict to $\eta_X \times (\square^n, \partial\square^n)$, and take AJ of the "multigraph"[3] $\gamma_{\mathbf{f}}$ in the same spirit as we did for $\mathcal{Z} \subseteq X \times (\square^n, \partial\square^n)$ in §1.3. the result can be interpreted as a map $K_n^M(\mathbb{C}(X)) \to H^n_{\mathcal{D}}(\eta_X, \mathbb{Z}(n))$ called the "Milnor regulator". Using the "standard homotopy" θ we can compute (eqn. 1.4.1) this very explicitly in terms of a current on X (in its first, very primitive form). So for the first time we have pushed an AJ map on a "product" down to integration against a current on the "base" η_X. We can tie all this to the local-global considerations of §1.2 by asking: how are the AJ maps on different coniveau "stitched together"? At least for codimension 1 (good enough for $n = 2$, 3) we can answer that

$$[AJ_X(\gamma_{\mathbf{f}})](\mathrm{d}\tilde{\alpha}) = 2\pi i [AJ_V(\gamma_{\mathrm{Tame}(\mathbf{f})})](\alpha)$$

using the Tame symbol. (Here $\tilde{\alpha}$ is an extension to X of a form α on divisor $V \subseteq X$.) We can't do codimension 2 residues for a while yet, because we won't have AJ formulas for $CH^{n-1}(F, n-1),\ n - i \neq n - 1$ until §2.4.

In the first three sections of Chapter 2 we gradually abstract the milnor regulator from the AJ map. First of all, however, in Chapter 1 we only defined a map

$$\otimes^n \mathbb{Z}[\mathbb{P}^1_{\mathbb{C}(X)} \setminus \{0, \infty\}] \overset{\text{"AJ"}}{\longrightarrow} H^n_{\mathcal{D}}(\eta_X, \mathbb{Z}(n));$$

we must now show it kills the Steinberg relations (by which $K_n^M(\mathbb{C}(X))$ is the quotient). We do this first in §2.1 geometrically and then in §2.2 more analytically by exhibiting $R'_{\mathbf{f}}$, for $\mathbf{f}\in$Steinbergs, as coboundary-currents. In §2.3 we discuss how to *pair* the local-global spectral sequences for cohomology of X via (a) currents and (b) compactly supported C^∞-forms on codimension-i points, in order to understand residues; the formula for Res^1

[3] here $\mathbf{f} \in \otimes^n \mathbb{Z}[\mathbb{P}^1_{\mathbb{C}(X)} \setminus \{0, \infty\}]$ is a formal sum of "multifunctions".

takes the form $\mathrm{Res}^1 R'_{\mathbf{f}} = 2\pi\sqrt{-1}R'_{\mathrm{Tame}(f)}$. Finally we introduce the notion of the "holomorphic" part of the Milnor regulator

$$K_n^M(X) \longrightarrow \mathrm{im}\{H^{n-1}(X, \mathbb{C}/\mathbb{Z}(n)) \to H^{n-1}(\eta_X, \mathbb{C}/\mathbb{Z}(n))\}.$$

Now that we have pushed AJ on $CH^n(\eta_X, n)$ down to a current on X, in the last section §2.4 we write currents $R_{\mathcal{Z}}$ for all $[\mathcal{Z}] \in CH^p(X, n)$; that these do arise analogously from an AJ procedure similar to that in §1.3.4 is discussed at the beginning of §5.1. In order to define AJ on the level of complexes $z^n(X, \bullet)$, Deligne homology (**[Ja]**, **[L2]**)cannot be avoided. A formula

$$\frac{1}{(2\pi\sqrt{-1})^i}\mathrm{Res}^i R'_{\mathbf{f}} = R'_{\mathrm{Res}^i(\gamma_{\mathbf{f}})}$$

for higher Res^i of Milnor regulator currents makes sense now, and there are more conjectural connections with polylogarithms (as in §1.2.4). In §2.4.4 we propose a simple procedure that would use these results to "lift" the single-valued *real* cousins of the Li_n (the generalized Bloch-Wigner functions $\mathcal{L}_n$) to $\mathbb{C}/\mathbb{Z}(n)$-valued cousins on $\ker(\delta) \subseteq \mathcal{B}_n(\mathbb{C})$.

That this lift works for $n = 2$ is proved in §3.1 as an application of the main result there: that the composition

$$CH^p(X, n)_{\mathbb{Q}} \overset{AJ}{\longrightarrow} H_{\mathcal{D}}^{2p-n}(X, \mathbb{Q}(p)) \overset{\pi_{\mathbb{R}}^p}{\longrightarrow} H_{\mathcal{D}}^{2p-n}(X, \mathbb{R}(p))$$

identifies with Goncharov's real regulator map r, and so with r_{Be} on $Gr_N^i CH^p(X, n)$. The other application of this fact is an interpretation of the "vanishing theorem" of Chapter 4 in terms of real regulators. So lifting r_{Be} to a $\mathbb{C}/\mathbb{Z}(n)$-valued regulator, from this point of view, allows us to apply infinitesimal-invariant theory to prove rigidity results that end up having implications for the original real regulator.

§3.2 comprises some bizarre, but at points inspiring, attempts to compute some periods of "Milnor regulators" on simple relative varieties. In §3.2.1-2 we set up (in a rather *ad hoc* fashion) "relative Milnor K-groups" and "relative regulator maps"

$$K^M(\mathbb{C}(X, Y)) \to H^{n-1}(\eta_{(X,Y)}, \mathbb{C}/\mathbb{Q}(n)) \ , \quad K^M(X, Y) \to \underline{H}^{n-1}(\eta_{(X,Y)}, \mathbb{C}/\mathbb{Q}(n))$$

by appealing (for motivation) in the case $(X, Y) = (\bar{\square}^{n-1}, \partial\bar{\square}^{n-1})$ to AJ on $CH^n(\mathbb{C}, 2n-1)$. In §3.2.3 we modify the relative K^M-groups of the $(\bar{\square}^{n-1}, \partial\bar{\square}^{n-1})$ so that linear factorization of terms is possible (in $'K^M$); the terms of the regulator become (classical) polylogarithms and we compute an example. (The problem with $'K_n^M$ is that for $n > 2$ it involves getting rid of the $a \wedge (1-a) \wedge \ldots$ Steinberg relations, which is perhaps somewhat too *ad hoc*.) Now in analogy to §2.4.4/3.1.2, we show in §3.2.4 (for $n = 2$ and $X = (\mathbb{P}^1, \{0, \infty\})$ that if $\mathbf{f}$ satisfies certain conditions then its "abelian symbol" $\mathcal{N}_{\mathbf{f}} \in \mathcal{B}_2(\mathbb{C})$ actually lives in ker(st), and $\Im \int_0^\infty R_{\mathbf{f}} = \mathcal{L}_2(\mathcal{N}_{\mathbf{f}})$ where $\mathcal{L}_2$ is the Bloch-Wigner function. Finally in §3.2.5 we exhibit the Catalan constant (a famous transcendental number) as a relative regulator period.

In §3.3 we compare our regulator to yet another construction, by means of a nice sheaf-cohomology computation in a double complex. It is shown that over η_X, taking sections of the Milnor-sheaf regulator $\underline{K^M_{n,X}} \to \underline{\mathcal{H}^n_{\mathcal{D}}(n)}$ (defined using the product in the Deligne cohomology ring) gives a map equivalent to the Milnor regulator we have defined, making ours "compatible" in a certain sense with the product structure. The sheafified version has appeared for instance in **[Ra]** (for $n = 2$) and **[Es]**.

The last two chapters are far more linear in organization. Chapter 4 is a study of the "residue free" part of the Milnor regulator essentially on families of varieties; by monodromy arguments the results have consequences for fixed, very general (arithmetically uninteresting) elements of the family. §4.3 is its heart, where the main theorems are proved for hypersurfaces (i.e. codimension 1 complete intersections) in $\mathbb{P}^n$: roughly speaking, that a family of symbols

$$\{\mathbf{f}_t\} \in \ker(\mathrm{Tame})[= H^0(X_t, \underline{K^M_{n,X_t}})]$$

has rigid (flat) regulator image, and that for t_0 very general in $\mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}(D))$,

$$\{\mathbf{f}_{t_0}\} \in K^M_n(X_{t_0}) \subseteq \ker(\mathrm{Tame}) \subseteq K^M_n(\mathbb{C}(X))$$

has trivial regulator image in

$$\mathrm{im}\{H^{n-1}(X_{t_0}, \mathbb{C}/\mathbb{Q}(n)) \to H^{n-1}(\eta_{X_{t_0}}, \mathbb{C}/\mathbb{Q}(n))\} \cong H^{n-1}_{var}(X_{t_0}, \mathbb{C}/\mathbb{Q}(n)).$$

§4.1 and §4.2 are background on the theory of variations of mixed Hodge structure. §4.5 extends the results to arbitrary codimension, in which form they are stated right on the very page of the chapter. Here we used recent work of Nagel and Dimca (see **[N]**), which extends Griffiths' "residue representation" (in **[G1]**) of the cohomology of a smooth projective hypersurface by quotients of polynomial rings (=Jacobi rings), to smooth complete intersections in any toric variety. §4.4 does codimension 2 in slightly greater depth using results of Green **[Gr2]** on pseudo-Jacobi rings.

Finally we come full circle to the higher AJ maps and Chapter 5. Starting again with relative affine space $X = (\square^n, \partial\square^n)$, we introduce the use of spreading to produce invariants – in this case for $CH^p(X(k))$, via $\mathcal{R}$ on $CH^p(X \times \eta_S(\bar{\mathbb{Q}}))$. These invariants are just the Milnor regulator for $p = n$ (the spread of a zero-cycle on this X just being a multigraph $\gamma_{\mathbf{f}}$) but e.g. for 1-cycles in $CH^3((\square^4, \partial\square^4)(k))$ already we have something interesting. Changing gears in §5.1.2, we take X smooth projective, and recall Mumford's theorem (**[Mu]**, generalized by **[Ro]**) which implies the "infinite-dimensionality" of the targets for the higher AJ maps Ψ_i. We summarize Lewis' construction (in **[L2]**) of the targets as limits of fimite-dimensional Hodge-theoretically defined objects, and relate his approach to that of Griffiths and Green (in **[GG5]**). In §5.1.3 we switch back to X =relative affine space, *formally* compute Lewis' target spaces and tie this to the maps in §5.1.1.

It doesn't take much imagination to extend the computations for $X = (\square^n, \partial\square^n)$ to $X' = (\mathbb{P}^1, \{0,\infty\})^n$ – the considerations are just "combinatorial". However this turns out to be an important step, since X' can be thought of as a product of degenerate elliptic curves. §5.2.1-2 is a reworked version of a talk where it is proved directly that $Gr^2 CH^2((\mathbb{P}^1, \{0,\infty\})^2(k)) \cong_{\otimes\mathbb{Q}} K_2^M(k)$, and argued by degeneration why this is relevant to $Gr^2 CH^2(X)$ for X smooth; §5.1.3 extends this to the case $n > 2$. In both cases the only real depth comes from *Suslin reciprocity*, a simple proof of which is included as an appendix. In §5.4 we make use of our *formal* computation of the targets for the graded pieces of AJ (of the spread), argue that these are just more Milnor regulators, and that they can be interpreted as differential characters arising from membrane integrals.

This latter property then becomes a *definition* when we pass back to X smooth projective in §5.3, for maps into (essentially) quotients of the Ψ_i-targets $Gr^i_{\mathcal{L}} \underline{H}^{2p}_{\mathcal{H}}(X \times \eta_S, \mathbb{Q}(p))$, which look like Deligne cohomology groups on η_S with coefficients in a lattice (given by periods of holomorphic forms on X). For $X = C_1 \times \ldots \times C_n$ a product of curves we find we can chop the C_i into (contractible) fundamental domains and produce a homotopy θ exactly as we did for $\mathbb{C}^* \times \ldots \times \mathbb{C}^*$ in §1.3. This standardizes the computation of the differential character, and Hodge-theoretic considerations let us push it down to a current on the base η_S. We apply this to $E \times E$ (and $E \times E \times E$) for an elliptic curve with complex multiplication (and defining equation $/\mathbb{Q}$), and exhibit 0-cycles in the Albanese kernel which we can now prove are $\underset{\text{rat}}{\not\equiv} 0$ by integrating a current (doing calculus) on the base of the spread.

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CHAPTER 1

Graphs as Algebraic Cycles

1.1. Introduction

Define over any infinite field k the *algebraic n-cube*

$$\square_k^n := \left(\mathbb{P}_k^1 \setminus \{1\}\right)^n =: (\mathbb{P}_k^1)^n \setminus \mathbb{I}^n,$$

with faces

$$\partial\square_k^n := \bigcup_{i,e} \rho_{i\,*}^e \square_k^{n-1},$$

and more generally subfaces $\rho_{i_1\ldots i_l\,*}^{e_1\ldots e_\ell}\square_k^{n-\ell}$, where for $e = 0, \infty$ the face inclusions $\rho_i^e : \square_k^{n-1} \to \square_k^n$ send $(z_1, \ldots, z_{n-1}) \mapsto \left(z_1, \ldots, \underset{i}{e}, \ldots, z_{n-1}\right)$, and so on. The n-cube is also equipped with projections $\pi_{i_1\ldots i_\ell} : \square_k^n \to \square_k^{n-\ell}$ where, e.g. for $\ell = 1$, π_i sends $(z_1, \ldots, z_n) \mapsto (z_1, \ldots, \hat{z}_i, \ldots, z_n)$. When the field is clear we will omit the subscript k.

Let $X/\mathbb{C}$ be a smooth quasiprojective variety and $\mathbf{f} = \{f_{i\alpha}\} \in \mathbb{C}(X)^*$ a collection of meromorphic functions; they make sense as maps into $\mathbb{P}^1$ over the generic point $E = Spec\mathbb{C}(X) = \varprojlim\{U \subset X \,\text{Zariski open}\}$, with the caveat that the limit only makes sense under some functor like CH. Let $|\mathbf{f}| = \cup|(f_{i,\alpha})|$ and consider the graph cycle in $Z^n(\{X \setminus |\mathbf{f}|\} \times \square^n)$

$$\gamma_{\mathbf{f}} := \sum n_\alpha(f_{1\alpha}, \ldots, f_{n\alpha}) :=$$

$$\left[\sum n_\alpha(id_X; f_{1\alpha}, \ldots, f_{n\alpha})_*\,(X \setminus |\mathbf{f}|)\right] \cap (\{X \setminus |\mathbf{f}|\} \times \square^n)\,.$$

In the next section we develop some infrasructure for understanding and attacking the following basic questions:

(a) Under what conditions can the closure on $X \times \square^n$ (of a suitable modification) of $\gamma_{\mathbf{f}}$ be completed, by addition of more *algebraic* cycles, to a "relative" cycle Γ? Roughly speaking, this means that there should be a continuous family of closed cycles Γ_ϵ, compactly supported on $X \times (\square^n \setminus N_\epsilon(\partial\square^n))$, with $\lim_{\epsilon\to 0}\Gamma_\epsilon = \Gamma$ in such a way that $\int_{\Gamma_\epsilon}\omega \to \int_\Gamma \omega$ for certain forms ω.

(b) In that case, can we define an algebraic invariant similar to the Deligne class for the completed cycle, and if so, what geometric information does it carry? (For instance, graphs arise as spreads of zero-cycles on the relative variety $(\square^n, \partial\square^n)$ and one may care about rational equivalence classes of these.) This question is treted in §1.3-1.4, and motivates much of the subsequent work (Chapter 2).

(c) Can the conditions in (a) be expressed "algebraically", in terms of relations on the functions or their "residues" – that is, can we avoid the geometry of the graph?

Regarding (a), the following example (although trivial) is instructive and gives a concrete feel for what is going on. The philosophy here is that, provided components of a cycle $\Gamma \in Z^n(X \times \square^n)$ intersect subfaces $X \times \rho^{\mathbf{e}}_{\mathbf{i}\,*}\square^{n-k}$ (of all codimensions) properly, "cancellation" of their intersections with (codimension 1) faces $X \times \partial\square^n$ should be sufficient for Γ to be a relative cycle.

EXAMPLE 1.1.1. Let X be a projective surface with $f,\, g,\, h \in \mathbb{C}(X)^*$, and consider a Zariski open neighborhood U of a normal crossing of components V_1 and V_2 of $|(g)_0|$ and $|(h)_0|$, respectively. That is, all other components of $|(f)|$, $|(g)|$, and $|(h)|$ are tossed out from X, to get U. Henceforth we also write V_i for $V_i \cap U$, and write $p = V_1 \cap V_2$ for the normal crossing. Consider, over $U \setminus V_1 \cup V_2$, the cycle

$$\gamma^0 = (f, g, h) - (h, f, g)$$

(the superscripts indicate codimension of support). Its closure on U to $\overline{\gamma^0} \in Z^3(U \times \square^3)$ intersects all subfaces properly: over $V_1 \setminus \{p\}$,

$$\overline{\gamma^0} \cap \left[(V_1 \setminus \{p\}) \times \partial\square^3\right] = (f, g, 0) - (0, f, g)$$

and this may be cancelled there by the addition of

$$\gamma^1_{V_1} = (z, \frac{z-f}{z-1}, g) - (f, z, \frac{z-g}{z-1}) \in Z^2((V_1 \setminus \{p\}) \times \square^3)$$

where for example the second term means

$$\left[\left(id_{X*}; f, z, \frac{z-g}{z-1}\right)_* \left((V_1 \setminus \{p\}) \times \mathbb{P}^1\right)\right] \cap \left((V_1 \setminus \{p\}) \times \square^3\right)$$

counted with multiplicity -1. Similarly, over $V_2 \setminus \{p\}$ one has

$$\overline{\gamma^0} \cap \left[(V_2 \setminus \{p\}) \times \partial\square^3\right] = (f, 0, h) - (h, f, 0) = -\gamma^1_{V_2} \cap \left[(V_2 \setminus \{p\}) \times \partial\square^3\right]$$

where we define

$$\gamma^1_{V_2} = \left(z, \frac{fh}{z}, \frac{(z-f)(z-h)}{(z-fh)(z-1)}\right) - (f, z, \frac{z-h}{z-1}).$$

(There are other choices.) Now $\overline{\gamma^1_{V_2}}$ and $\overline{\gamma^1_{V_1}}$ intersect subfaces properly, but there is more to cancel at $\{p\}$: writing $a = f(p)$,

$$\left(\overline{\gamma^1_{V_1}} + \overline{\gamma^1_{V_2}}\right) \cap \left(p \times \partial\square^3\right) = (z, \frac{z-a}{z-1}, 0) + (z, 0, \frac{z-a}{z-1}) + (0, z, \frac{z-a}{z-1})$$

$$= \left(\gamma_p^2 := \{p \times (z_1, z_2, z_3) \mid (1-z_1)(1-z_2)(1-z_3) = 1-a\}\right) \cap \left(p \times \partial\square^3\right).$$

So put $\Gamma := \overline{\gamma^0 + \gamma_{V_1}^1 + \gamma_{V_2}^1 + \gamma_p^2}$ on U; as a cycle, Γ has zero intersection with all faces of the algebraic cube (components of $U \times \partial\square^3$), and the supports of all of its components intersect all subfaces properly. This is what we are after. We were lucky here, because closures of cycles intersecting subfaces properly do *not* in general retain this property: one needs to "move" them by adding a "trivial" cycle before closing them in this case. In order to pin this statement down we introduce two presentations of (cubical) higher Chow groups.

1.2. Extending Graphs to Higher Chow Cycles

1.2.1. Bloch's higher Chow groups. For any smooth quasi-projective X/k define

$c^p(X,n)$:=the subgroup of $Z^p(X \times \square^n)$ generated by subvarieties intersecting all subfaces $X \times \rho_{\mathbf{i}\,*}^{\mathbf{e}}\square^{n-\ell}$ properly,

i.e. in the right codimension. (Note that anything can happen at $\mathbb{I}^n$ if one looks at the closure of such a cycle on $X \times (\mathbb{P}^1)^n$.) We will sometimes refer to these cycles as "admissible". Let

$d^p(X,n)$:=the subgroup of $c^p(X,n)$ generated by subvarieties pulled back from $X \times \square^{n-1}$ by some $\pi_{\mathbf{i}}$.

We neglect these cycles and write

$$Z^p(X,n) := c^p(X,n)/d^p(X,n),$$

which forms a complex with differential

$$\partial_{\mathcal{B}} := \sum_{i=1}^{n} (-1)^i \left(\rho_i^{\infty *} - \rho_i^{0*}\right) : Z^p(X,n) \to Z^p(X,n-1);$$

in particular note that $\partial_{\mathcal{B}} \circ \partial_{\mathcal{B}} = 0$ so we have a complex (the Bloch complex). Define (after [**B3**]) the higher Chow groups as its homology:

$$CH^p(X,n) := H_n\{Z^p(X,\bullet)\}.$$

Our cycle groups and Chow groups are always $\otimes\mathbb{Q}$ (i.e. modulo torsion); we write $Z^p(k,n)$ for $Z^p(Spec(k),n)$, and note in particular that $Z^p(\mathbb{C}(X),n) = Z^p(\eta_X,n)$ is where the graph cycles γ^0 live. Since they are trivially $\partial_{\mathcal{B}}$-closed they define elements $[\gamma^0] \in CH^n(\mathbb{C}(X),n)$. In general a "$\partial_{\mathcal{B}}$-cycle" or "higher Chow cycle" on X will mean a $\partial_{\mathcal{B}}$-closed element of $Z^p(X,n)$.

There is a subcomplex of "antisymmetrized" cycles with the same homology groups ([**MS2**]). Let $\sigma \in \mathcal{S}_n \ltimes (\mathcal{S}_2)^{\oplus n}$ act on $\square^n$ by permuting, then inverting coordinates (this is consistent with composition in the semidirect product), and let $\mathrm{sgn} : \mathcal{S}_n \ltimes (\mathcal{S}_2)^{\oplus n} \to \{\pm 1\} = \mathbb{Z}_2$ be given by the product of characters $\mathrm{sgn} \cdot \prod \mathrm{sgn}_i$. There is an idempotent operator $Alt_n : Z^p(X, n) \to Z^p(X, n)$ given by the formula

$$Alt_n \mathcal{Z} = \frac{1}{2^n n!} \sum_{\sigma \in \mathcal{S}_n \ltimes (\mathcal{S}_2)^{\oplus n}} \mathrm{sgn}(\sigma)\, \sigma^* \mathcal{Z}.$$

Let $C^p(X, \bullet) \subset Z^p(X, \bullet)$ be the subcomplex generated by the image of $Alt_\bullet$, with the differential induced by inclusion. $Alt_\bullet$ commutes with the differential:

$$Alt_{n-1} \partial_{\mathcal{B}} \mathcal{Z} = \partial_{\mathcal{B}} Alt_n \mathcal{Z},$$

and so not only the inclusion above but also the projection $Z^p(X, \bullet) \xrightarrow{Alt_\bullet} C^p(X, \bullet)$ give maps of complexes.

PROPOSITION 1.2.1. *(Levine) The inclusion* $C^p(X, \bullet) \to Z^p(X, \bullet)$ *is a quasi-isomorphism of complexes.*

The content of this statement is that closed cycles are equivalent (modulo a boundary) to an antisymmetrized cycle: $\partial_{\mathcal{B}} \mathcal{Z} = 0 \implies \mathcal{Z} = \partial_{\mathcal{B}} \mathcal{Z}' + \mathcal{C}$. In fact, applying Alt to both sides one has $Alt\mathcal{Z} = \partial_{\mathcal{B}} Alt \mathcal{Z}' + \mathcal{C} = \partial_{\mathcal{B}} Alt \mathcal{Z}' + (\mathcal{Z} - \partial \mathcal{Z}') = \mathcal{Z} + \partial_{\mathcal{B}} \mathcal{Z}''$, where $\mathcal{Z}'' = (Alt\, \mathcal{Z}') - \mathcal{Z}'$.

COROLLARY 1.2.2. *(a)* $\partial_{\mathcal{B}} \mathcal{Z} = 0 \implies [\mathcal{Z}] = [Alt\mathcal{Z}]$*, from which it follows that*
(b) The projection $Z^p(X, \bullet) \xrightarrow{Alt_\bullet} C^p(X, \bullet)$ *is quasi-isomorphism.*

Writing ρ_i^{e*} for intersection with the corresponding face, notice that

$$2n \cdot \rho_i^{e*} Alt_n \mathcal{Z} = Alt_{n-1} \partial_{\mathcal{B}} \mathcal{Z},$$

and so if $Alt_n \mathcal{Z} = \mathcal{C} \in C^p(X, n)$ is $\partial_{\mathcal{B}}$-closed, then

$$0 = \frac{1}{2n} \partial_{\mathcal{B}} Alt_n \mathcal{Z} = \frac{1}{2n} Alt_{n-1} \partial_{\mathcal{B}} \mathcal{Z} = \rho_i^{e*} Alt_n \mathcal{Z} = \rho_i^{e*} \mathcal{C}$$

and so $\mathcal{C} \cap \partial \square^n = 0$ "on the nose" (as a cycle), which is of course stronger than $\partial_{\mathcal{B}} \mathcal{C} = 0$. This observation will be useful later in obtaining "relative cycles" in $X \times (\square^n, \partial \square^n)$.

Now we introduce the coniveau filtration. From now on X denotes a smooth projective variety (unless otherwise specified). Writing $|\mathcal{Z}|$ for the support of a cycle (its underlying subvariety), let

$$N^i Z^p(X, n) := \left\{ \mathcal{Z} \in Z^p(X, n) \;\middle|\; \mathrm{codim} |\pi_*^X \mathcal{Z}| \geq i \;\right\}.$$

Note that $N^1 Z^p(\eta_X, n) = 0$. We will also use the notation "$N^i X$" and "$X \setminus N^i X$" to denote $\varinjlim_{\{x_\alpha\} \subset X^i} (\cup \overline{x_\alpha})$ and $\varprojlim_{\{x_\alpha\} \subset X^i} (X \setminus \cup \overline{x_\alpha})$, with the same caveat as in the case of $\eta_X (= X \setminus N^1 X)$. (This will be useful when we want to ignore the behavior of a higher Chow cycle over subvarieties of codimension $\geq i$ in X.) It seems attractive to think of our example in **1.1** as a cycle in a double complex: writing $\gamma^1 = \gamma^1_{V_1} + \gamma^1_{V_2}$ and $Gr^i_N := N^i / N^{i+1}$,

$$
\begin{tikzcd}
0 & & \\
Gr_N^0 Z^3(X,3) \arrow[u] \arrow[r] & Gr_N^1 Z^3(X,2) & \\
& Gr_N^1 Z^3(X,3) \arrow[u] \arrow[r] & Gr_N^2 Z^3(X,2) & \\
& & Gr_N^2 Z^3(X,3) \arrow[u] \arrow[r] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
0 & & \\
\overline{\gamma^0} \arrow[u] \arrow[r] & \partial_{\mathcal{B}}\overline{\gamma^0} = -\partial_{\mathcal{B}}\overline{\gamma^1} & \\
& \overline{\gamma^1} \arrow[u] \arrow[r] & \partial_{\mathcal{B}}\overline{\gamma^1} = -\partial_{\mathcal{B}}\overline{\gamma^2} & \\
& & \overline{\gamma^2} \arrow[u] \arrow[r] & 0
\end{tikzcd}
$$

REMARK 1.2.3. We should be cautious here: one can't get by with a double complex picture in general (all we have is a filtered complex). The horizontal differentials are not always defined, though they are defined on the kernel of the vertical differentials. More generally, by a snake-lemma type argument on the sequence

$$N^{i+k}Z(X,\bullet) \to N^i Z(X,\bullet) \to \frac{N^i}{N^{i+k}} Z^p(X,\bullet)$$

(which is exact on each term of the complex, *not* in the derived category), one obtains maps:

$$H_n(\frac{N^i}{N^{i+k}} Z^p(X,\bullet)) \to H_n(N^{i+k} Z^p(X,\bullet))$$

from what appear to be "diagonal chains" in the double complex (which have zeroes in the first i columns and look like cycles up to the $(i+k)$-th column), to the $(i+k+1)$-st column. In the simple normal-crossing situation of the example above, the map $H_3(\frac{N^0}{N^2} Z^3(X,\bullet)) \to H_3(N^2 Z^3(X,\bullet))$, which is *a priori* only defined on $\overline{\gamma^0+\gamma^1}$, reduces to a horizontal differential on $\overline{\gamma^1}$. However, one can certainly imagine a situation where $\overline{\gamma^0} = \overline{(f,g,h)-(h,f,g)}$ has essential singularities and is still admissible – e.g., if components of $|(f)_0|$, $|(f)_\infty|$, and $|(g)_0|$ all interesect in a point. Consequently $\partial_{\mathcal{B}}\overline{\gamma^0}$ can have a nontrivial $N^2 Z^3(X,2)$ component. (On the other hand $\overline{\gamma^0}$ will not be admissible when $|(f)_0|$, $|(g)_0|$, and $|(h)_0|$ all intersect in a point.)

EXAMPLE 1.2.4. It turns out that $\overline{\gamma^0+\gamma^1+\gamma^2}$ is also a boundary: in terms of the "double complex" picture,

$$\begin{array}{ccccccc}
Gr_N^0 Z^3(X,3) & & & \gamma^0 & & & \\
\uparrow & & & \uparrow & & & \\
Gr_N^0 Z^3(X,4) & \longrightarrow & Gr_N^1 Z^3(X,3) & \eta^0 & \longrightarrow & \gamma_a^1 + (\gamma_b^1 + \gamma_c^1) = \gamma^1 & \\
& & \uparrow & & & \uparrow & \\
& & Gr_N^1 Z^3(X,4) \longrightarrow Gr_N^2 Z^3(X,3) & & & \eta^1 & \longrightarrow \gamma^2.
\end{array}$$

Here we put

$$\eta^0 = \left(f, z, \frac{hg}{z}, \frac{(z-h)(z-g)}{(z-hg)(z-1)}\right) - \left(z, \frac{hf}{z}, \frac{(z-h)(z-f)}{(z-hf)(z-1)}, g\right),$$

$$\eta^1 = \eta_a^1 + \eta_b^1 = \left[\left(f, z, w, \frac{(1-z)(1-w)-(1-h)}{(z-h)(w-h)}\right)_{V_1} + \left(f, z, w, \frac{(1-z)(1-w)-(1-g)}{(z-g)(w-g)}\right)_{V_2}\right]$$

$$+ \left[\left\{(z_1, z_2, z_3, g) \mid \prod (1-z_i) = (1-f)\right\}_{V_2}\right],$$

$$\partial_{\mathcal{B}} \eta^0 = [(f,g,h) - (h,f,g)] + \left[\left(z, \frac{hf}{z}, \frac{(z-f)(z-h)}{(z-hf)(z-1)}\right)_{V_1}\right] = \overline{\gamma^0} + \overline{\gamma_a^1},$$

$$\partial_{\mathcal{B}} \eta_a^1 = \left[-\left(f, z, \frac{z-h}{z-1}\right)_{V_1} - \left(f, z, \frac{z-g}{z-1}\right)_{V_2}\right] + \left[\begin{array}{cc} \pi_3^*(f,h)_{V_1} + & \pi_2^*(f,h)_{V_1} \\ + \ \pi_3^*(f,g)_{V_2} + & \pi_2^*(f,g)_{V_2} \end{array}\right] = \overline{\gamma_b^1} + d^3(X,3),$$

and $\partial_{\mathcal{B}} \eta_b^1 = -\left\{(z_1, z_2, z_3) \mid \prod (1-z_i) = (1-f)\right\}_p + \left(z, \dfrac{z-f}{z-1}, g\right)_{V_2} = \overline{\gamma^2 + \gamma_c^1}$

and we are working modulo $d^*(X, \bullet)$. So $\overline{\gamma^0 + \gamma^1 + \gamma^2} = \partial_{\mathcal{B}}(\eta^0 + \eta^1)$ and this example is "trivial"; there is a great deal of cancellation of cycles in the computations. Their messy nature certainly motivates the work below, which will enable us to see by merely glancing at γ^0 that this example was trivial to start with.

1.2.2. Moving lemma, the local-global picture and "residue" maps. In the event that a graph $\gamma^0 \in Z^n(\eta_X, n)$ has inadmissible closure $\overline{\gamma^0}$, we want to be able to modify γ^0 by addition of a "trivial" cycle in $Z^n(\eta_X, n)$, in order that its closure meet subfaces properly (and so yield an element of $Z^n(X,n)$). $Im(\partial_{\mathcal{B}})$ gives us a concrete subgroup of trivial cycles to work with, and there is the following standard result, for any quasi-projective k-scheme $\mathcal{X}$.

PROPOSITION 1.2.5. [Moving Lemma (Bloch)]. *Let $U \subset \mathcal{X}$ be Zariski open, and let $r_U : Z^p(\mathcal{X}, \bullet) \to Z^p(U, \bullet)$ be the restriction. Then the complex $Z^p(U,\bullet)/im(r_U)$ is acyclic.*

COROLLARY 1.2.6. *The map $Z^p(\mathcal{X},\bullet)/Z^p(\mathcal{X}-U,\bullet) \to Z^p(U,\bullet)$ induced by r_U is a quasi-isomorphism.*

PROOF. ($\twoheadrightarrow$ in the d.c.): If $\mathcal{Z}_0 \in Z^p(U,n)$ is a $\partial_{\mathcal{B}}$-cycle then the lemma $\implies$ $\mathcal{Z}_0 = r_U\mathcal{Z} + \partial_{\mathcal{B}}\beta$. Apply $\partial_{\mathcal{B}}$ to this $\implies$ $0 = \partial_{\mathcal{B}}\mathcal{Z}_0 = \partial_{\mathcal{B}} r_U \mathcal{Z} = r_U \partial_{\mathcal{B}} \mathcal{Z}$. So $\partial_{\mathcal{B}}\mathcal{Z} \equiv 0 \pmod{Y}$.

($\hookrightarrow$ in the d.c.): If $\Gamma \in Z^p(\mathcal{X},n)$ has image a boundary $[r_U\Gamma = \partial_{\mathcal{B}} z]$ then the moving lemma $\implies$ $z = r_U\mathcal{Z} + \partial_{\mathcal{B}}\beta$ $\implies$ $r_U\Gamma = \partial_{\mathcal{B}} z = r_U\partial_{\mathcal{B}}\mathcal{Z}$ and so $r_U(\Gamma - \partial_{\mathcal{B}}\mathcal{Z}) = 0$ $\implies$ $\Gamma = \partial_{\mathcal{B}}\mathcal{Z} \pmod{Y}$ [is itself a boundary]. □

REMARK 1.2.7. In particular one could take $\mathcal{X} = X \supset (X \setminus N^iX) = U$ or (as in our applications below) $\mathcal{X} = N^iX \supset (N^iX \setminus N^{i+1}X) = U$.

More concretely, if $\mathcal{Z}_0 \in Z^p(U,n)$ has $\partial_{\mathcal{B}}\mathcal{Z}_0 = r_U\Gamma$, for $\Gamma \in Z^p(\mathcal{X}, n-1)$ [e.g. if $\mathcal{Z}_0$ is $\partial_{\mathcal{B}}$-closed on U], then there are $\beta \in Z^p(U,n+1)$ and $\mathcal{Z} \in Z^p(\mathcal{X},n)$ such that

$$\mathcal{Z}_0 = \partial_{\mathcal{B}}\beta + r_U\mathcal{Z}.$$

Notice that (one may choose $\mathcal{Z}$ so that) the "moved" cycle $r_U\mathcal{Z}$ has admissible closure $\mathcal{Z}$ on $\mathcal{X}$ and the *same $\partial_{\mathcal{B}}$-boundary* as $\mathcal{Z}_0$ (over U). Applying Alt_n to both sides gives immediately an identical moving lemma for $C^p(U,\bullet)$.

REMARK 1.2.8. In fact Bloch [**B4**] proved this lemma for cycles in the simplicial complex, where $(\square^n, \partial\square^n)$ is replaced by

$$\left(\triangle^n_k = \left\{(t_0,\dots,t_n) \in \mathbb{A}^{n+1}_k \mid \sum t_i = 1\right\} \quad , \quad \partial\triangle^n_k = \bigcup_i \{z_i = 0\}\right),$$

but the cubical version also follows from the homotopies constructed there.

We can now use the coniveau filtration and the moving lemma to produce "residues" of γ^0 which will be obstructions to completing its "move" to a higher Chow cycle. Associated to $N^\bullet$ there is a 4^{th}-quadrant[1] spectral sequence with $E_0^{p,-q}(r) = Gr^p_N Z^r(X, q-p)$, $E_1^{p,-q}(r) = H^{p-q}(Gr^p_N Z^r(X,-\bullet))$

[1] For consistency with the literature ([**B3**] in particular) we now switch to cohomological indexing, by simply placing the higher Chow complex in negative degrees; e.g., $CH^5(X,3)$ is now $H^{-3}(Z^5(X,-\bullet))$ instead of $H_3(Z^5(X,\bullet))$.

and converging to $E_\infty^{p,-q}(r) =: Gr_N^p CH^r(X, q-p)$. We sketch how this is constructed. The sequence

$$0 \to N^{i+1}Z^r(X,-\bullet) \to N^i Z^r(X,-\bullet) \to Gr_N^i Z^r(X,-\bullet) \to 0$$

induces an exact triangle (for every i)

$$\begin{tikzcd}
H^*(N^{i+1}Z^r(X,-\bullet)) \arrow[rr, "\alpha_1^i"] & & H^*(N^i Z^r(X,-\bullet)) \arrow[dl, "\beta_1^i"] \\
& E_1^{i,*-i}(r) = H^*(Gr_N^i Z^r(X,-\bullet), d_0 = \partial_{\mathcal{B}}) \arrow[ul, "\gamma_1^i"] &
\end{tikzcd}$$

where the connecting homomorphism γ_1^i again comes from a snake-lemma argument (and we increase $*$ by 1 when applying it, and thus with each trip around the triangle). Then $d_1 : E_1^{i,*-i} \to E_1^{i+1,*-i}$ is given by $\beta_1^{i+1} \circ \gamma_1^i$ and one has a "derived" triangle

$$\begin{tikzcd}
\begin{array}{c} H^*(N^{i+1}Z^r(X,-\bullet)) \\ \cup \\ Im(\alpha_1^{i+1}) \end{array} \arrow[rr, "\alpha_2^i"] & & \begin{array}{c} H^*(N^i Z^r(X,-\bullet)) \\ \cup \\ Im(\alpha_1^i) \end{array} \arrow[dl, "\beta_2^i"] \\
& E_2^{i,*-i}(r) = H^i(E_1^{\bullet,*-i}, d_1) \arrow[ul, "\gamma_2^i"] &
\end{tikzcd}$$

where α_2^i, β_2^i, γ_2^i are induced (respectively) by α_1^i, $\beta_1^{i+1} \circ (\alpha_1^i)^{-1}$, γ_1^i. (Now i itself increases by 1 around the triangle, namely with each application of β_2^i.) So d_2 is essentially $\beta_1^{i+2} \circ (\alpha_1^{i+1})^{-1} \circ \gamma_1^i$ (since we are now only operating on elements which γ_1 take to the image of α_1), taking $E_2^{i,*-i} \to E_2^{i+2,*-i-1}$. Iterating this process gives the successive pages of the spectral sequence (e.g., see [**HS**]), which does *not* degenerate at E_2. (Also this demonstrates that one needs, in addition to (E_1, d_1), some global "patching" information to obtain d_2.)

Invoking the corollary to the moving lemma with (formally) $\mathcal{X} = N^i X$, $U = (N^i X \backslash N^{i+1}X) = \coprod_{x \in X^i} \eta_x$, we have $H^{p-q}(Gr_N^p Z^r(X,-\bullet)) \cong H^{p-q}(Z^{r-p}(U,-\bullet))$ and therefore

$$E_1^{p,-q}(r) \cong \coprod_{x \in X^p} CH^{r-p}(\mathbb{C}(x), q-p).$$

With this replacement, the first triangle laid out flat is a special case of the "localization sequence"; $E_\bullet^{\bullet,\bullet}$ is often called the "local-to-global" spectral sequence. For the 0^{th} page we can also write $\tilde{E}_0^{p,-q}(r) = \coprod_{x \in X^p} Z^{r-p}(\mathbb{C}(x), q-p)$ (with $\tilde{d}_0 = \partial_{\mathcal{B}}$), so that "$E_1^{\bullet,\bullet} \cong \tilde{E}_1^{\bullet,\bullet}$"; this gives the connection with graph cycles, which live in $\ker(\tilde{E}_0) \subset \tilde{E}_0^{0,-n}(n) = Z^n(\mathbb{C}(X), n)$, and it is on these cycles that we now proceed to describe d_1 and d_2 explicitly.

Write r_i for the restrictions to $X \setminus N^{i+1}X$ (r_0 restricts to η_X). As $\gamma^0 \in Z^n(\mathbb{C}(x), n)$ is trivially $\partial_{\mathcal{B}}$-closed, the moving lemma applies to yield $\Gamma^0 \in Z^n(X, n)$ with $r_0\Gamma^0 = \gamma^0 - \partial_{\mathcal{B}}\beta$, well-defined up to $N^1Z^n(X,n) + \partial_{\mathcal{B}}Z^n(X, n+1)$. Again since $\partial_{\mathcal{B}}\gamma^0 = 0$, $\partial_{\mathcal{B}}\Gamma^0$ lands in $N^1Z^n(X, n-1)/\partial_{\mathcal{B}}N^1Z^n(X,n)$; passing to the quotient Gr_N^1 would give $d_1[\Gamma^0]$. In view of the corollary to the moving lemma one obtains an equivalent class by restricting to $(N^1)X \setminus N^2X$; this class in $\coprod_{x \in X^1} CH^{n-1}(\mathbb{C}(x), n-1)$, represented by $r_1\partial_{\mathcal{B}}\Gamma$, is our definition of $\mathrm{Res}^1[\gamma^0]$. Alternatively,

$$\mathrm{Res}^1[\gamma^0] := [r_1\partial_{\mathcal{B}}(\overline{\gamma^0 - \partial_{\mathcal{B}}\beta})].$$

In the event that this is trivial ($\gamma^0 \in \ker(d_1)$), i.e. $\partial_{\mathcal{B}}\Gamma^0 \equiv 0$ in $Gr_N^1 Z^n(X, n-1)/\partial_{\mathcal{B}}N^1Z^n(X,n)$, we have $\partial_{\mathcal{B}}\Gamma^0 \equiv \partial_{\mathcal{B}}\Gamma^1 \pmod{N^2}$ for some $\Gamma^1 \in N^1Z^n(X,n)$. So we have a $\partial_{\mathcal{B}}$-cycle $\Gamma^0 - \Gamma^1 \in \frac{N^0}{N^2}Z^n(X,n)$, well-defined up to ambiguities $\partial_{\mathcal{B}}Z^n(X, n+1) + \{\text{cycles on } \frac{N^1}{N^2}\} + N^2Z^n(X,n)$. Clearly $\partial_{\mathcal{B}}(\Gamma^0 - \Gamma^1)$ lands in $N^2Z^n(X, n-1)$; passing to (a subgroup of) Gr_N^2 modulo the image of the ambiguities should give a class $d_2[G^0]$. In fact, $\partial_{\mathcal{B}}(\text{cycles on } \frac{N^1}{N^2})$ is just a codimension 1 version of the construction in the last paragraph and so $= im(d_1)$, while $\partial_{\mathcal{B}}N^2Z^n(X,n) = im(d_0)$. The corresponding element of

$$\coprod_{y \in X^2} CH^{n-2}(\mathbb{C}(y), n-1) \Big/ \mathrm{Res}^1\left(\coprod_{x \in X^1} CH^{n-1}(\mathbb{C}(x), n)\right),$$

represented by $r_2\partial_{\mathcal{B}}(\Gamma^0 - \Gamma^1)$, is denoted by $\mathrm{Res}^2[\gamma^0]$. In a (not entirely correct) picture:

$$\begin{tikzcd} \Gamma^0 \arrow[r] & \partial_{\mathcal{B}}\Gamma^0 & \\ & \Gamma^1 \arrow[u] \arrow[r] & \partial_{\mathcal{B}}(\Gamma^0 - \Gamma^1) \end{tikzcd} \qquad \begin{tikzcd} \gamma^0 \arrow[r, "\mathrm{Res}"] & \mathrm{Res}^1\gamma^0 & \\ & \mathcal{Z} \arrow[u, "\partial_{\mathcal{B}}"] \arrow[r, "\mathrm{Res}"] & \mathrm{Res}^2\gamma^0 \end{tikzcd}$$

(modcycles$\mathcal{W}$ with$\partial_{\mathcal{B}}\mathcal{W} = 0$) (modRes ofcycles$\mathcal{W}$)

More generally, there is a series of maps

$\mathrm{Res}^i : \ker(\mathrm{Res}^{i-1}) \subset CH^n(\eta_X, n) \to$

$$\coprod_{x \in X^i} CH^{n-i}(\mathbb{C}(x), n-1) \,/\mathrm{im}(\mathrm{Res}^1) + \ldots + \mathrm{im}(\mathrm{Res}^{i-1}),$$

and we define

$$\mathcal{F}^i_{\mathrm{Res}} CH^n(\eta_X, n) := \ker(\mathrm{Res}^i).$$

The process begun above for Res^2 for $\gamma^0 \in \mathcal{F}^1_{\mathrm{Res}}$ tells us how to take Res^{i+1} of $\gamma^0 \in \mathcal{F}^i_{\mathrm{Res}}$. Inductively we obtain $\Gamma^0 - \Gamma^1 + \ldots \pm \Gamma^i$ $\partial_{\mathcal{B}}$-closed in $\frac{N^0}{N^{i+1}} Z^n(X,n)$; we take its $\partial_{\mathcal{B}}$ as an element of $Z^n(X,n)$ and pass to Gr_N^{i+1}. At some point this process must terminate because $\dim X < \infty$; if all Res^i are 0 and $\gamma^0 \in \bigcap \mathcal{F}^i_{\mathrm{Res}}$ then clearly one has $\Gamma^0 - \Gamma^1 + \ldots \pm \Gamma^{\dim X}$ $\partial_{\mathcal{B}}$-closed in $Z^n(X,n)$ restricting to $[\gamma^0]$ on η_X, and we say that $[\gamma^0] \in \mathrm{im} CH^n(X,n)$.

That is, $\bigcap \mathcal{F}^i_{\mathrm{Res}}$ is just an explicit description of the "top-graded piece" of $CH^n(X,n)$ under the coniveau filtration, or equivalently its image under the edge-homomorphism (which is simply restriction to η_X).

We can improve the bound on "termination" of the Res^i. With the reminder that we work $\otimes\mathbb{Q}$ (mod torsion), according to **[MS2]** there is the

CONJECTURE 1.2.9. [Beilinson-Soulé]. *$CH^q(F,m) = 0$ for $2q \leq m$ (where F is a field).*

The version with any smooth (quasi-)projective X replacing F would then follow from the moving lemma and spectral sequence above. We postpone the proof for the known cases $q = 0, 1$ until it is relevant in an example below. An obvious consequence of the full conjecture is triviality of the Res^i on $CH^n(\eta_X, n)$ for $2(n-i) \leq n-1$ or $i \geq (n+1)/2$; in other words we have the

COROLLARY 1.2.10. *If $n \geq 2$, $\mathcal{F}^i_{Res} CH^n(\eta_X, n) = im CH^n(X,n)$ for $i \geq$ $\min\{\frac{n-1}{2}, \dim X + 1\}$. (This is a conjecture for $n \geq 5$.)*

1.2.3. Interpretation of codimension-1 residues via the Tame symbol. For $n \geq 2$ let $K_n^M(F)$ denote the quotient of the abelian group $\otimes^n \mathbb{Z}[\mathbb{P}^1_F \setminus \{0,\infty\}]$ by the Steinberg relations: the subgroup generated by all permutations of

$$\alpha_1 \otimes \alpha_2 \otimes \ldots \otimes \alpha_n \, + \, \beta_1 \otimes \alpha_2 \otimes \ldots \otimes \alpha_n \, - \, \alpha_1\beta_1 \otimes \alpha_2 \otimes \ldots \otimes \alpha_n,$$

$$\alpha_1 \otimes \alpha_2 \otimes \ldots \otimes \alpha_n \, + \, \alpha_2 \otimes \alpha_1 \otimes \ldots \otimes \alpha_n, \quad \alpha_1 \otimes (1-\alpha_1) \otimes \ldots \otimes \alpha_n.$$

(For $n < 2$ just put $K_1^M(F) := F^*$ and $K_0^M(F) := \mathbb{Z}$.) We shall write elements of $\otimes^n \mathbb{Z}[\mathbb{P}^1 \setminus \{0,\infty\}]$ additively (e.g., $\mathbf{a}$ or $\alpha_1 \otimes \ldots \otimes \alpha_n$) but elements of the quotient ("symbols") multiplicatively (e.g. $\{\mathbf{a}\}$ or $\{\alpha_1, \ldots, \alpha_n\}$). A fundamental result of Totaro **[T]** says that the graph homomorphism

$$\gamma : \otimes^n \mathbb{Z}[\mathbb{P}^1 \setminus \{0,\infty\}] \twoheadrightarrow Z^n(\eta_X, n)$$

given by

$$\mathbf{f} = \sum n_\alpha f_{1\alpha} \otimes \ldots \otimes f_{n\alpha} \mapsto \gamma_{\mathbf{f}}$$

descends to a well-defined map on Milnor K-theory and in fact induces an isomorphism:

$$\overline{\gamma} : K_n^M(\mathbb{C}(X)) \xrightarrow[\text{graph}]{\cong} CH^n(\eta_X, n).$$

Pulling the $\mathcal{F}^i_{\text{Res}}$ back to Milnor K-theory along $\overline{\gamma}$, for $n \geq 2$ we write (the first "=" conjecturally for $n \geq 5$)

$$\mathcal{F}^{\left[\frac{n}{2}\right]}_{\text{Res}} K_n^M(\mathbb{C}(X)) = \overline{\gamma}^{-1}\left(\text{im} CH^n(X,n)\right) =: K_n^M(X).$$

Applying the moving lemma to $\gamma_{\mathbf{f}}$ we obtain $\gamma'_{\mathbf{f}} := \gamma_{\mathbf{f}} - \partial_{\mathcal{B}}\beta$ with good closure $\overline{\gamma'_{\mathbf{f}}} \in Z^n(X,n)$. Since γ is surjective, there is a $\mathbf{g} = \sum \ell_\alpha g_{1\alpha} \otimes \ldots \otimes g_{n\alpha}$ with $\gamma_{\mathbf{g}} = \partial_{\mathcal{B}}\beta$, and Totaro's result $\implies$ $\mathbf{g}$ is a Steinberg relation. So *one may move $\{\mathbf{f}\}$ by a Steinberg relation in such a way that the closure of its graph intersects subfaces of $\square^n$ properly* (simultaneously at all divisors of the $f_{i\alpha}$ – though one should note that subtracting $\mathbf{g}$ may both remove and introduce divisors).

EXAMPLE 1.2.11. On $\mathbb{P}^1 = X$ we can modify $x \otimes x$ by $(-1) \otimes x - x \otimes x$. This is a Steinberg relation because $\{x,x\}\{-1,x\}^{-1} = \{-x,x\} = \{\frac{-x}{1-x}, x\} = \{\frac{x-1}{x}, x\}^{-1} = \{1 - \frac{1}{x}, \frac{1}{x}\} = 1$ in $K_2^M(\mathbb{C}(\mathbb{P}^1))$.

We summarize:

PROPOSITION 1.2.12. *For every $\mathbf{f} \in \otimes^n \mathbb{Z}[\mathbb{P}^1_{\mathbb{C}(X)} \setminus \{0,\infty\}]$ there is a Steinberg relation $\mathbf{g}$ ($\{\mathbf{g}\} \equiv 1$ in $K_n^M(\mathbb{C}(X))$) such that, with $\mathbf{f}' := \mathbf{f} - \mathbf{g}$, $\overline{\gamma_{\mathbf{f}'}} \in Z^n(X,n)$. This may be completed to a ($\partial_{\mathcal{B}}$-closed) cycle in $CH^n(X,n)$ exactly when $\{\mathbf{f}\} \in K_n^M(X) \subset K_n^M(\mathbb{C}(X))$.*

In light of this, it would be extremely useful to have an "intrinsic" description of $K_n^M(X)$ (and $\mathcal{F}^i_{\text{Res}} K_n^M(\mathbb{C}(X))$) via kernels of (higher residue) maps defined directly in terms of the functions $\{f_{i\alpha}\} \in \mathbb{C}(X)$ (rather than the associated graph cycles). We can do this immediately for $\mathcal{F}^1_{\text{Res}}$; it turns out to be very difficult beyond this, though we will sketch one possible approach following the examples below.

Using the last proposition it is easy to define the "Tame symbol" (**[BT]**, **[Go2]**, **[L4]**)

$$\text{Tame}^{(1)} : K_n^M(\mathbb{C}(X)) \to \coprod_{x \in X^1} K_{n-1}^M(\mathbb{C}(x)).$$

We can always choose representatives of a class in K_n^M in good position, like $\mathbf{f}'$ above. Fix $x \in X^1$ corresponding to an irreducible divisor component of one (or more) of the $f'_{i\alpha}$. No two functions in the same term ($= \alpha$) of such an $\mathbf{f}'$ share x as a divisor component, unless another function in the same term is $\equiv 1$ there. These latter terms will get mapped to $1 \in K_{n-1}^M(\mathbb{C}(x))$ (that is, thrown away as regards residue at this x). Otherwise we map

$$f_{1\alpha} \otimes \ldots \otimes f_{n\alpha} \mapsto (-1)^i \text{ord}_x(f_{i\alpha}) \cdot f_{1\alpha}|_x \otimes \ldots \otimes \widehat{f_{i\alpha}} \otimes \ldots \otimes f_{n\alpha}|_x \ ,$$

where (for each α) $\mathrm{ord}_x(f_{i\alpha})$ is nonzero for at most one i. Repeating this for all x gives the map; see **[Go2]** for a more algebraic (equivalent) definition.[2] From the form of this definition it is immediate that the diagram

$$\begin{tikzcd} K_n^M(\mathbb{C}(X)) \arrow[r, "Tame^1"] \arrow[d, "\cong"', "\overline{\gamma}"] & \coprod_{x\in X^1} K_n^M(\mathbb{C}(x)) \arrow[d, "\cong"', "\coprod\overline{\gamma}"] \\ CH^n(\eta_X, n) \arrow[r, "\mathrm{Res}^1"] & \coprod_{x\in X^1} CH^{n-1}(\eta_x, n-1) \end{tikzcd}$$

commutes. Combining this with the corollary to Beilinson-Soulé, we have the big non-conjectural point of this section:

PROPOSITION 1.2.13. *For $n = 2$ or 3, $K_n^M(X) = \ker(Tame^1)$: that is, $\gamma_{\mathbf{f}}$ can be moved and completed exactly when $\{\mathbf{f}\}$ has trivial tame symbol.*

We give a series of "examples"; the third one demonstrates this latter proposition.

EXAMPLE 1.2.14. The cycle $\gamma^0 = (f,g,h)-(h,f,g)$ of the introduction is $\gamma_{\mathbf{f}}$ for $\{\mathbf{f}\} = \{f,g,h\}\{h,f,g\}^{-1} = \{f,g,h\}\{f,h,g\} = \{f,g,h\}\{f,g,h\}^{-1} = 1$. Totaro's result says γ^o is automatically trivial (in $\mathrm{im}(\partial_{\mathcal{B}})$), at least on $CH^3(\eta_X, 3)$, or equivalently (since it has good closure) mod N^1, in view of the moving lemma.

EXAMPLE 1.2.15. Let $X/\mathbb{C}$ be a surface and set $n = 4$; choose $f,\ g,\ h,\ k \in \mathbb{C}(X)^*$ and V_1 and V_2 irreducible components of $|(f)_0|$ and $|(g)_0|$ (respectively) with normal crossing, so that $\gamma^0 = (f,g,h,k)$ has good closure Γ^0 on $U = X \setminus \{\text{all other divisor components}\}$. It is assumed V_1 and V_2 are *not* components of $|(h)|$ or $|(k)|$. (As in the previous example we now simply "replace" X by U, V_i by $V_i \cap U$, so that $V_1 \cap V_2 = \{p\}$ is a n.c.) Then we have [representatives] $\mathrm{Res}^1\gamma^0 = (g,h,k)|_{\eta_{V_1}} - (f,h,k)|_{\eta_{V_2}} \in \coprod_{x\in X^1} Z^3(\mathbb{C}(x),3)$, and $d_1\Gamma^0 = i_*^{V_1}\{(g,h,k)|_{V_1}\} - i_*^{V_2}\{(f,h,k)|_{V_2}\}[\in \frac{N^1}{N^2}Z^3(U,3)]$. Suppose next that $g\otimes h\otimes k$ and $f\otimes h\otimes k$ are Steinbergs – say, $g = 1-h$ on V_1 and $f = 1-h$ on V_2. Then (using Totaro) there are cycles $\gamma_1^{V_i} \in Z^3(\eta_{V_i}, 4)$ with $\partial_{\mathcal{B}}\gamma_1^{V_1} = (g,h,k)|_{\eta_{V_1}} = \mathrm{Res}^1_{V_1}\gamma^0$ and $\partial_{\mathcal{B}}\gamma_1^{V_2} = (f,h,k)|_{\eta_{V_2}} = \mathrm{Res}^1_{V_2}\gamma^0$. Since, e.g., $(g,h,k)|_{\eta_{V_1}} = r_{\eta_{V_1}} d_1\Gamma^0$, the moving lemma says we could have chosen

[2] The point is that one can write down a map $T : \bigwedge_{\mathbb{Z}}^n \mathbb{C}(X)^* \to \coprod_{x\in X^1} \bigwedge_{\mathbb{Z}}^{n-1} \mathbb{C}(x)^*$ inducing Tame^1. If $G = 0$ cuts out $\overline{x}$ locally, then it follows from the definitions in **[Go2]** or **[BT]** that

$$T_x\{f_1, \ldots, f_n\} = \prod_{j=1}^{n} \prod_{1\le k_1,\ldots,k_j\le n} \left\{F_1^x, \ldots, \widehat{F_{k_1}^x}, \ldots, -1, \ldots, F_n^x\right\}^{\pm\prod_{i=1}^j \mathrm{ord}_x(f_{k_i})},$$

where the -1 occurs in the spaces $k_2, \ldots, k_j$ and $F_i^x := \frac{f_i}{G^{(\mathrm{ord}_x(f_i))}}\,|_x$.

$\gamma_1^{V_i}$ to have good closure Γ_i^1 (and we assume they do). So $\partial_{\mathcal{B}}(\Gamma^0 - \Gamma_1^1 + \Gamma_2^1) \in \mathrm{N}^2$, which in this simple case means the cycle (which is $\mathrm{Res}^2\gamma^0$) lives over $\{p\}$. Also in this case $\partial_{\mathcal{B}}\Gamma^0$ makes no contribution, so by abuse of notation ($\gamma_1^{V_i}$ are not $\partial_{\mathcal{B}}$-closed) one could write formally $\mathrm{Res}^2_{V_1\cap V_2=p}(\gamma^0) := \mathrm{Res}^1_p(\gamma_1^{V_2}) - \mathrm{Res}^1_p(\gamma_1^{V_1})$, which suggests $\partial_{\mathcal{B}}\mathrm{Res}^2_p\gamma^0 = \partial\mathrm{Res}^1_p\gamma_1^{V_2} - \partial\mathrm{Res}^1_p\gamma_1^{V_1} = \mathrm{Res}^1_p\partial\gamma_1^{V_2} - \mathrm{Res}^1_p\partial\gamma_1^{V_1} = \mathrm{Res}^1_p\mathrm{Res}^1_{V_2}\gamma^0 - \mathrm{Res}^1_p\mathrm{Res}^1_{V_1}\gamma^0 = \mathrm{Res}^1\circ\mathrm{Res}^1\gamma^0 = 0$. This is what the proof that $\mathrm{Res}^2\gamma^0$ is closed would look like in a double complex, and it is essentially the right proof for this example. So we get a class in (a subquotient of) $CH^2(p,3) = CH^2(\mathbb{C},3)$; an example of a nontrivial class in $CH^2(\mathbb{C},3)$ (though unfortunately trivial in the subquotient, being in $\mathrm{im}(\mathrm{Res}^1)$) is $4(z, 1-z, 1-\frac{i}{z}) + (z, \frac{(z-i)^4}{(z-1)^4}, 1-i)$, parametrized by $z \in \mathbb{P}^1$.

EXAMPLE 1.2.16. For X a surface we will be more interested in $n = 3$ and $K_3^M(X)$ than in the $n = 4$ situation above. Let $\mathbf{f} = f_1 \otimes g_1 \otimes h_1 - h_2 \otimes f_2 \otimes g_2$, where $|(f_1)_0|$ and $|(f_2)_0|$ share a divisor component V ($\mathrm{ord}_V f_1 = \mathrm{ord}_V f_2 = 1$). Assume that $\gamma_{\mathbf{f}}$ has good closure and $\mathrm{Tame}_V\{\mathbf{f}\} \equiv 1$; this latter is equivalent to saying that $g_1 \otimes h_1 + h_2 \otimes g_2 =: \partial_V\mathbf{f}$ (temporary notation) is a Steinberg. We will now complete $\overline{\gamma_{\mathbf{f}}}$ along V directly. To take $\partial_{\mathcal{B}}$ of $\overline{\gamma_{\mathbf{f}}}$ along V, write $\overline{\gamma_{\mathbf{f}}} \cap \partial\square_{\eta_V} = (0, g_1, h_1) - (h_2, 0, g_2)$ and pull these back (with correct signs) along the face maps $(z_1, z_2) \mapsto (z_1, 0, z_2)$ and $(z_1, z_2) \mapsto (0, z_1, z_2)$ to obtain $(\partial_{\mathcal{B}}\overline{\gamma_{\mathbf{f}}})|_{\eta_V} = (g_1, h_1) + (h_2, g_2) = \gamma_{\partial_V\mathbf{f}}$. Since $\partial_V\mathbf{f}$ is a Steinberg, Totaro's result immediately hands us a cycle in $Z^2(\mathbb{C}(V), 3)$ that kills $(\partial_{\mathcal{B}}\overline{\gamma_{\mathbf{f}}})|_{\eta_V}$, but we will instead show how to do this slightly more directly. Since $CH^2(\square^2_{\mathbb{C}(V)}, \partial\square^2_{\mathbb{C}(V)}) \cong K_2^M(\mathbb{C}(V))$ (see appendix), we can find a set of meromorphic functions on curves $\{\mathcal{C}_i, F_i\}$ in $\square^2_{\mathbb{C}(V)}$ with $\sum \iota_*^{\mathcal{C}_i}(F_i) = (g_1, h_1) + (h_2, g_2)$ and $F_i \equiv 1$ on $\mathcal{C}_i \cap \partial\square^2$. Recall that π_3 sends $(z_1, z_2, z_3) \mapsto (z_1, z_2)$; pulling back $(\mathcal{C}_i, F_i)$ along π_3 gives a collection $\{\mathcal{C}_i \times \square^1_{\mathcal{Z}_3}, F_i = F_i(z_1, z_2)\}$ on $\square^3_{\mathbb{C}(V)}$. Defining $G_i := \frac{z_3 - F_i}{z_3 - 1}$ and $\gamma_1^V := \sum \iota_*^{\mathcal{C}_i \times \square^1}(G_i) \in Z^2(\mathbb{C}(V), 3)$, one finds that $\partial_{\mathcal{B}}\gamma_1^V = (g_1, h_1) + (h_2, g_2)$. Now one might consider another V' as in the last example, and compute $\mathrm{Res}^2_{p\in V\cap V'}\gamma_{\mathbf{f}} \in CH^1(\mathbb{C}, 2)$. But in fact the following

LEMMA 1.2.17. $CH^1(\mathbb{C}, n) = 0$ $(\forall\, n \geq 2)$.
ensures us that we will have no difficulties in killing this class and finishing the completion of $\gamma_{\mathbf{f}}$ to a higher Chow cycle.

PROOF. For $[\mathcal{Z}] \in CH^1(\mathbb{C}, n)$, recall that $\partial_{\mathcal{B}}\mathcal{Z} = 0 \implies [\mathcal{Z}] \equiv [Alt_n\mathcal{Z}]$, where $Alt_n\mathcal{Z} \cap \partial\square^n = 0$ as a cycle. The point is that we can replace $\mathcal{Z}$ by a divisor $\sum n_i\mathcal{W}_i$ on $\square^n$ with cancelling face-intersections. Now either by Lefschetz $(1,1)$ applied to $\sum n_i\overline{\mathcal{W}_i} + \left\{\begin{array}{c}\text{cycles on } \mathbb{I}^n \text{ added to kill the} \\ \text{homology class in } H^2(\mathbb{P}^1 \times \mathbb{P}^1)\end{array}\right\}$ or by thinking of $\square^n$ as $\mathbb{A}^n$, there is a rational function F cutting out $\sum n_i\mathcal{W}_i = (F)$. Since for each face $0 = \rho^\epsilon_{i*}\square^{n-1} \cap \sum n_i\mathcal{W}_i = \rho_i^{\epsilon*}(F)$, F restricts to a constant function $F \circ \rho_i^\epsilon$ on each face. Were these constants different for distinct faces, the components of $|(F)|$ would not intersect properly the subfaces where these faces meet. So one may assume $F \equiv 1$ on $\partial\square^n$. Let $G := \frac{z_{n+1}-F(z_1,\ldots,z_n)}{z_{n+1}-1}$ on $\square^{n+1}$; then it follows that $\partial_{\mathcal{B}}(G) = (F) = Alt_n\mathcal{Z} = \mathcal{Z} + \partial_{\mathcal{B}}\Gamma$. $\square$

REMARK 1.2.18. It is entirely possible that the curves $\mathcal{C}_i$ intersect corners (say $z_1 = z_2 = 0$) of $\partial\square^2$ in the above "rational equivalence". Since $F_i = 1$ at such an intersection, the construction of γ_1^V then $\implies$ $z_3 = 1$ at its corresponding corner intersection, so it is perfectly admissible.

1.2.4. Interpreting the higher codimension "residues". Now we outline a conjectural strategy for achieving objective (c) of the introduction, namely an alternate description of the $\mathcal{F}^i_{\mathrm{Res}}K^M_*$ generalizing what we did above for $i = 1$. For any field F, the Goncharov complex (see **[Go2]** or **[Go3]**)

$$G^n(F,\bullet) := \mathcal{B}_n(F) \xrightarrow{\delta} \mathcal{B}_{n-1}(F)\otimes F^* \longrightarrow \ \ldots \ \longrightarrow \mathcal{B}_2(F)\otimes \bigwedge\nolimits^{n-2}_{\mathbb{Z}} F^* \xrightarrow{\delta} \bigwedge\nolimits^{n}_{\mathbb{Z}} F^*$$

(where the last term is $G^n(F,n)$, situated in degree $-n$) gives a projective resolution of $K^M_n(F)$. Each $\mathcal{B}_i(F)$ is a quotient of $\mathbb{Z}[\mathbb{P}^1_F]$ by a subgroup of relations $\mathcal{R}_i(F)$ on the i^{th} real (single-valued) polylogarithm $\mathcal{L}_i$ (=generalized Bloch-Wigner functions). For $\{x\}_i \in \mathcal{B}_i(F)$, $\delta(\{x\}_i \wedge y_1 \wedge \ldots \wedge y_j) := \{x\}_{i-1}\otimes x \wedge y_1 \wedge \ldots \wedge y_j$ for $i > 2$ and (for $i = 2$) $\delta(\{x\}_2 \otimes y_1 \wedge \ldots \wedge y_j) = (1-x)\wedge x \wedge y_1 \wedge \ldots \wedge y_j$ is just the standard map st (whose image gives the remainder of the Steinberg relations).

Assuming the Beilinson-Soulé conjecture, the subcomplex

$$S^n(F,\ell) := Alt_\ell \left\{ \sum_{\substack{0<2i\leq j<\ell \\ (i,j)+(i',j')=(n,\ell)}} C^i(F,j)\bigwedge C^{i'}(F,j') \ + \sum_{\substack{0<2i\leq j\leq \ell \\ (i,j)+(i',j')=(n,\ell+1)}} \partial C^i(F,j)\bigwedge C^{i'}(F,j') \right\}$$

of $C^n(F,\bullet)$ (consisting of certain decomposable elements) is acyclic, so that the homology of the quotient complex

$$A^*(F,\bullet) := C^n(F,\bullet)\,/S^n(F,\bullet)$$

still computes the higher Chow groups. For example, for $n = 2$

$$S^2(F,\bullet) = \ldots \to C^1(F,1)\bigwedge C^1(F,3) \to C^1(F,1)\bigwedge C^1(F,2) \to C^1(F,1)\bigwedge \partial_{\mathcal{B}}C^1(F,2).$$

"Extending" work of Gangl, Müller-Stach, and Zhao (**[GM]**, **[Zh]**) we conjecture that the following formulas give well-defined maps of complexes $\bar{\rho}_n(\bullet) : G^n(F,\bullet) \to A^n(F,\bullet)$:

$\bar{\rho}_n(\ell)\left(\{x\}_{\ell-n+1}\otimes y_1 \wedge \ldots \wedge y_{2n-\ell-1}\right) :=$ parametrization by $(\mathbb{P}^1)^{\ell-n}$ of

$$\pm Alt_n\left(1-z_1, 1-\frac{z_2}{z_1}, \ldots, 1-\frac{z_{\ell-n}}{z_{\ell-n-1}}, 1-\frac{x}{z_{\ell-n}}, z_1, \ldots, z_{\ell-n}, y_1, \ldots, y_{2n-\ell-1}\right),$$

$$\text{and} \quad \bar{\rho}_n(n)\,(y_1 \wedge \ldots \wedge y_n) := Alt_n(y_1, \ldots, y_n),$$

where $\bar{\rho}_n(\ell - 1) \circ \delta = \partial_{\mathcal{B}} \circ \bar{\rho}_n(\ell)$. Note that these formulas are really only defined *a priori* on the level of *representatives* $x \otimes y_1 \otimes \ldots \otimes y_{2n-\ell-1} \in \otimes^{2n-\ell}\mathbb{Z}[\mathbb{P}^1_F \setminus \{0, \infty\}]$, so it would be more proper to say that there are maps $\rho_n(\ell)$ *inducing* (conjecturally) well-defined $\bar{\rho}_n(\ell)$; the reader may check directly that ρ commutes with differentials. We also note that the map $\bar{\rho}_n(n)$ gives the same class in $CH^n(\eta_X, n)$ as the graph cycle when $F = \mathbb{C}(X)$.

REMARK 1.2.19. It's quite impossible to get a map on the level of complexes $G^n(F, \bullet) \to Z^n(F, \bullet)$ (even $\bar{\rho}_n(n)$ would fail to be well-defined because relations $y_1 \otimes y_2 + y_1' \otimes y_2 - y_1 y_1' \otimes y_2$ do not go to zero – they go to $C^1(F, 1) \bigwedge \partial_{\mathcal{B}} C^1(F, 2)$).

More conjectures arise in the attempt to employ a local-global argument to induce out of this a global picture on X. According to Goncharov ([**Go2**], [**Go3**]) we want ideally to work with a double complex

$$G^n(\mathbb{C}(X), \bullet) \xrightarrow{T} \coprod_{x \in X^1} G^{n-1}(\mathbb{C}(x), \bullet - 1) \xrightarrow{T} \coprod_{y \in X^2} G^{n-2}(\mathbb{C}(y), \bullet - 2) \longrightarrow \ldots,$$

– e.g., for $n = 4$ and X a 3-fold,

$$\begin{tikzcd}
\bigwedge^4 \mathbb{C}(X)^* \arrow[r, "T"] & \boxed{\coprod_{x \in X^1} \bigwedge^3 \mathbb{C}(x)^*} \arrow[r, "T"] & \coprod_{y \in X^2} \bigwedge^2 \mathbb{C}(y)^* \arrow[r, "T"] & \coprod_{z \in X^3} \mathbb{C}(z)^* \\
\mathcal{B}_2(\mathbb{C}(X)) \otimes \bigwedge^2 \mathbb{C}(X)^* \arrow[u, "\delta"] \arrow[r, "T"] & \coprod_{x \in X^1} \mathcal{B}_2(\mathbb{C}(x)) \otimes \mathbb{C}(x)^* \arrow[u, "\delta"] \arrow[r, "T"] & \boxed{\coprod_{y \in X^2} \mathcal{B}_2(\mathbb{C}(y))} \arrow[u, "\delta"] & \\
\mathcal{B}_3(\mathbb{C}(X)) \otimes \mathbb{C}(X)^* \arrow[u, "\delta"] \arrow[r, "T"] & \coprod_{x \in X^1} \mathcal{B}_3(\mathbb{C}(x)) \arrow[u, "\delta"] & & \\
\mathcal{B}_4(\mathbb{C}(X)) \arrow[u, "\delta"] & & &
\end{tikzcd}$$

– and take $G^n(X, \bullet)$ to be the associated simple complex[3] with differential $\delta \pm T$ and natural filtration "by columns" $'N^*$. For example, entries in the boxed terms would give an element of $'N^1 G^4(X, 3)$. $'N^*$ would then induce

[3] To fix intuition here, $H^{-i}(G^n(X, -\bullet))_{\mathbb{Q}}$ is then supposed to be isomorphic to $gr^{\gamma}_n K_i(X)_{\mathbb{Q}} \cong H^{2n-i}_{\mathcal{M}}(X, \mathbb{Q}(i)) \cong CH^n(X, i)_{(\mathbb{Q})}$.

the spectral sequence $('E_\bullet^{\bullet,\bullet})$ of this double complex, which has $d_1 = T$, d_2 given by the dotted arrow, and more generally

$$\mathrm{Tame}^i := d_i : \left\{ker(d_{i-1}) \subset K_n^M(\mathbb{C}(X))\right\} \to \coprod_{x\in X^i} G^{n-i}(\mathbb{C}(x), n-1) \Bigg/ \bigcup_{j<i} \mathrm{im}(d_j) .$$

We want to compare the filtration $\mathcal{F}^i_{\mathrm{Tame}} K_n^M(\mathbb{C}(X)) := \ker(\mathrm{Tame}^i)$ with $\mathcal{F}^i_{\mathrm{Res}}$, by showing $\mathrm{Res}^i \circ \bar{\rho}_n(n) = \bar{\rho}_{n-i}(n-1) \circ \mathrm{Tame}^i$ in the appropriate sense. This would follow from the existence of a (commuting) map of exact triangles (and thus of spectral sequences):

$$\begin{tikzcd}[column sep=small]
H^\bullet('N^{i+1}G^*(X,-\bullet)) \arrow[rr] & & H^\bullet('N^iG^*(X,-\bullet)) \arrow[dl] & & H^\bullet(N^{i+1}Z^*(X,-\bullet)) \arrow[rr] & & H^\bullet(N^iZ^*(X,-\bullet)) \arrow[dl] \\
& {'E_1^{\bullet,\bullet} = \coprod_{x\in X^i} H^\bullet(G^*(\mathbb{C}(x),-\bullet))} \arrow[ul] & & \longrightarrow & & {E_1^{\bullet,\bullet} = \coprod_{x\in X^i} CH^*(\mathbb{C}(x),-\bullet)} \arrow[ul] &
\end{tikzcd}$$

Here the map on E_1's is supposed to be induced by the maps $\bar{\rho}$. So the problem reduces essentially to constructing maps: $H^\bullet('N^iG^*(X,-\bullet)) \to H^\bullet(N^iZ^*(X,-\bullet))$. We first indicate very explicitly how this should go and then point out the (rather serious) technical obstacles concerning the definition of the maps T and thus of $G^*(X,\bullet)$.

We begin with a cycle [$=(\delta\pm T)$-closed] in $'N^iG^p(X,n)$ consisting of representatives ξ^i, ξ^{i+1}, ... of classes on diagonal terms $\coprod_{x\in X^i} G^{p-i}(\mathbb{C}(x),n)$, $\coprod_{y\in X^{i+1}} G^{p-i-1}(\mathbb{C}(y),n)$, ... in the double complex. Since $\delta\xi^i = 0$, $\partial_{\mathcal{B}}(\rho(\xi^i)) \equiv 0$ in $\coprod_{x\in X^i} A^{p-i}(\mathbb{C}(x),n-1)$ and so $\partial_{\mathcal{B}}(\rho(\xi^i)) = s^i \implies \partial_{\mathcal{B}} s^i = 0$, which (with acyclicity of S^*) $\implies \exists\ S^i$ so that $\partial_{\mathcal{B}}(\rho\xi - S^i) = 0$; applying the moving lemma gives $\eta^i := \rho\xi^1 - S^i + \partial_{\mathcal{B}}\mathcal{C}^i$ with $\overline{\eta^i} \in N^iC^p(X,n)$. We conjecture a moving lemma now for "fractional linear cycles" ($=\mathrm{im}(\rho)$), saying that we may choose $\mathcal{C}^i$ and S^i so that $\eta^i = \rho('\xi^i)$ for some $'\xi^i = \xi^i - \delta\zeta^i +$ {relations in $\coprod G^{p-i}(\mathbb{C}(x),n)$}. Set $j = 2(p-i)-n-1$, $k = n-p+i+1$, and assume for simplicity that $\overline{x}$ is smooth. If $'\xi^i_x = \{f\}_k \otimes g_1 \wedge \ldots \wedge g_j$, then on $X \setminus N^{i+2}X$, $\partial_{\mathcal{B}}\overline{\eta^i}|_{\overline{x}} = \partial_{\mathcal{B}}\overline{\rho('\xi^i_x)}$

$$= \partial_{\mathcal{B}} Alt\left(1-z_1,\ldots,1-\frac{z_{k-1}}{z_{k-2}}, 1-\frac{f}{z_{k-1}}, z_1,\ldots,z_{k-1},g_1,\ldots,g_j\right)_{\overline{x}}$$

$$= Alt\left(1-z_1,\ldots,1-\frac{f}{z_{k-2}}, z_1,\ldots,z_{k-2},f,g_1,\ldots,g_j\right)_{\overline{x}}$$

$$+\sum_{y\in\overline{x}}\sum_{l=1}^{j} \mathrm{ord}_y(g_l) Alt\left(1-z_1,\ldots,1-\frac{f|_y}{z_{k-1}}, z_1,\ldots,z_{k-1},g_1,\ldots,\widehat{g_l},\ldots,g_j\right)_y$$

$= \rho\{(\delta+T)('\xi^i_x)\}$, and since we knew the generic part over x was 0 to begin with from our choice of $'\xi^i$ above (ξ^i won't do), this is just $\rho T('\xi^i_x)$. More generally $\partial_{\mathcal{B}}\overline{\eta^i} = \partial_{\mathcal{B}}\overline{\rho('\xi^i)} = \rho T('\xi^i)$ (in an ideal world where all $\overline{x}$ are smooth at all y); the main point is that ρ commutes with residues.

Next let $'\xi^{i+1} := \xi^{i+1} \pm T\zeta^i$, and consider $\rho('\xi^{i+1}) \in \coprod_{y\in X^{i+1}} C^{p-i-1}(\mathbb{C}(y), n)$. Since $\delta('\xi^{i+1}) \pm T('\xi^i) \in$relations "in" $\coprod_{y\in X^{i+1}} G^{p-i-1}(\mathbb{C}(y), n-1)$, we have $\partial_{\mathcal{B}}\rho('\xi^{i+1}) = \rho\delta('\xi^{i+1}) = s^{i+1}\pm\rho T('\xi^i) = s^{i+1}\pm\partial_{\mathcal{B}}\overline{\rho('\xi^i)}$, so that $\partial_{\mathcal{B}}(\pm\overline{\rho('\xi^i)}\pm \rho('\xi^{i+1})) = s^{i+1}$ and we once more use acyclicity of S^* and the moving lemma to write an element $\eta^{i+1} = \rho('\xi^{i+1}) - S^{i+1} + \partial_{\mathcal{B}}\mathcal{C}^{i+1}$ with good closure, and $\partial_{\mathcal{B}}(\pm\overline{\eta^i} \pm \eta^{i+1}) = 0$. Again, assuming η^{i+1} can also be chosen so that $\eta^{i+1} = \rho(''\xi^{i+1}) := \rho('\xi^{i+1} - \delta\xi^{i+1} + \text{relations})$, one should obtain on $X \setminus N^{i+3}$ $\partial_{\mathcal{B}}(\overline{\eta^i \pm \eta^{i+1}}) = \rho T(''\xi^{i+1})$, and so on. If not from the previous argument, it is clear from the course of this one that ρ should induce a map of spectral sequences commuting with all the d_i (as well as homomorphisms from $H^{-n}('N^iG^p(X,-\bullet))_{\mathbb{Q}} \to H^{-n}(N^iZ^p(X,-\bullet))$ for all i). Consequently we would have $\mathcal{F}^i_{\text{Tame}} \subset \mathcal{F}^i_{\text{Res}}$ (if not $\cong$, which was shown for $i=1$); although at present the theoretical picture is unsound, this is still a working principle for doing computations.

Referring back to the picture of the weight 4 "double complex", the first problem arises in defining the map

$$T : \coprod_{x\in X^1} \bigwedge^3 \mathbb{C}(x)^* \to \coprod_{y\in X^2} \bigwedge^2 \mathbb{C}(y)^*.$$

Suppose $\overline{x}$ contains the codimension 2 point y and is singular there; writing $\mathcal{N} : \tilde{\bar{x}} \to \bar{x}$ for the normalization, the components of $\mathcal{N}^{-1}(y) =: \cup y_i$ give coverings $y_i \to y$. In order to take the residue $T_y(f_1\wedge f_2\wedge f_3)$, one first computes T_{y_i} on $\tilde{\overline{x}}$ and pushes the results down to y using norm maps. But these are only available on the level of Milnor K-theory: in the appendix we will show explicitly how to construct homomorphisms

$$\mathcal{N}_{\mathbb{C}(y_i)/\mathbb{C}(y)} : K_2^{(M)}(\mathbb{C}(y_i)) \to K_2^{(M)}(\mathbb{C}(y));$$

this also follows abstractly from the existence of a transfer as in [**BT**]. In general (for the whole "diagram") it is *expected* that T is defined on at least vertical cohomology; Goncharov (in [**Go2**]) hints that one can then still proceed with the above constructions in the derived category, but we have not pursued this. The much more serious problem is the conjectural status of norm maps on the remainder of the complex $G^*(F,\bullet)$ (even in the derived category, again see Goncharov), so that e.g.

$$T : \coprod_{x\in X^1} \mathcal{B}_2(\mathbb{C}(x)) \otimes \mathbb{C}(x)^* \xrightarrow{T} \coprod_{y\in X^2} \mathcal{B}_2(\mathbb{C}(y))$$

is not defined as such. However, we emphasize that for purposes of intution, as well as computation in specific examples (where $\overline{x}$ is smooth at y for all codimension 1 and 2 points concerned), this picture is still quite useful (e.g., see the end of §2.4).

1.3. Abel-Jacobi for Relative Cycles

1.3.1. An excision lemma for relative cycles. Define the subgroup $Z^p(X \times (\square^n, \partial\square^n)) \subset Z^p(X,n)$ of *relative* (*algebraic*) *cycles* $\mathcal{Z}$ by the requirement that $\mathcal{Z} \cap \partial\square^n = 0$ as a cycle (sometimes written $\mathcal{Z} \cdot \partial\square^n = 0$); that is to say, the face intersections cancel (and $\mathcal{Z}$ intersects all subfaces properly). Besides 1.1.1 we have the following

EXAMPLE 1.3.1. (a) For any higher Chow ($\partial_{\mathcal{B}}$-)cycle $\mathcal{Z} \in Z^p(X,n)$, $Alt_n\mathcal{Z}$ provides a relative cycle representing the same class in $CH^p(X,n)$. (For instance $\mathcal{Z}$ might come from moving and completing a graph cycle if the Res^i vanish.)

(b) More specific: if $\{\mathbf{f}\} \in \ker(\text{Tame}) \subset K_2^M(\mathbb{C}(X))$, for X a curve, Bloch **[B1]** showed how to complete $\gamma_{\mathbf{f}}$ to a relative cycle (in a slightly expanded sense) *without* moving. In fact we lose nothing by moving (or for that matter a more restrictive definition of relative cycles), which gives us $\overline{\gamma_{\mathbf{f}'}} \in Z^2(X,2)$, which we complete to a $\partial_{\mathcal{B}}$-closed Γ. Now add degenerate cycles ($\in d^2(X,2)$) to "transfer" face intersections from ∞ to 0. Calling the result Γ', $\partial_{\mathcal{B}}\Gamma' = 0$ combined with $\rho_i^{\infty *}(\Gamma') = 0 \implies \rho_1^{0*}(\Gamma' \cap (\{p\} \times \square^2)) = \rho_2^{0*}(\Gamma' \cap (\{p\} \times \square^2))$; that is, the face intersections at $z_1 = 0$ and $z_2 = 0$ are identical over each $p \in X$. For a particular p, say we have $\Gamma' \cap (\{p\} \times \partial\square^2) = \{(a,0) + (0,a)\} \times \{p\}$; then subtracting $\{p\} \times (z, \frac{a-z}{1-z}) = \{p\} \times \{(1-z_1)(1-z_2) = (1-a)\}$ cancels this intersection. We'll have more to say about this technique toward to end of the section.

NOTATION 1.3.2. ("The Lefschetz-dual perspective.") Let

$$\hat{\square}^n = (\mathbb{P}^1, \{1\})^n := ((\mathbb{P}^1)^n, \mathbb{I}^n),$$

$$\partial\hat{\square}^n := \bigcup_{i,e} \rho^e_{i\,*}\hat{\square}^{n-1}$$

(warning: ∂ in what follows usually means topological boundary; the above formula is an exception!). We will want to recast relative algebraic cycles as limits of topological (∂-closed) cycles with compact support on $X \times (\hat{\square}^n \setminus \partial\hat{\square}^n)$. These topological cycles can still have boundary at $X \times \mathbb{I}^n$; all differential forms we shall use pull back to 0 there. (Note also: sometimes $\hat{\square}^n$ will be written in lieu of $(\mathbb{P}^1)^n$.)

MORE NOTATION. Let $\epsilon > 0$ be "small" and

$$N_\epsilon(\partial\hat{\square}^n) = \left\{ (z_1, \dots, z_n) \in \hat{\square}^n \;\middle|\; |z_i| < \epsilon \text{ or } > \frac{1}{\epsilon} \text{ for some } i \right\}.$$

Now set

$$\hat{\square}^n_\epsilon = \hat{\square}^n \setminus \overline{N_\epsilon(\partial\hat{\square}^n)};$$

this is an important bit of notation since $\mathcal{Z}\cap(X\times\hat{\square}^n_\epsilon)$ indicates throughout the "excision" of $\mathcal{Z}$ ($\mathcal{Z}$ minus its intersection with a closed $\epsilon-$neighborhood of the faces). We also write

$$\overline{\hat{\square}^n_\epsilon}=\hat{\square}^n\setminus N_\epsilon(\partial\hat{\square}^n) \quad \text{and}$$

$$\mathcal{T}^1_\epsilon := \mathcal{T}^1_\epsilon(\partial\hat{\square}^n) := \partial\overline{N_\epsilon(\partial\hat{\square}^n)} = \partial\overline{\hat{\square}^n_\epsilon}.$$

This is a union of tubes around faces: set $p(e)=\begin{cases} 1, & e=0 \\ -1, & e=\infty \end{cases}$. Then

$$\mathcal{T}^1_\epsilon=\cup\mathcal{T}^1_\epsilon\begin{pmatrix} e\\ i\end{pmatrix} \quad \text{where}$$

$$\mathcal{T}^1_\epsilon\begin{pmatrix} e\\ i\end{pmatrix}=\left\{(z_1,\ldots,z_n)\in\hat{\square}^n\,\middle|\,|z_i|^{p(e)}=\epsilon,\text{ and } \epsilon\le|z_j|\le\frac{1}{\epsilon}\text{ for } j\neq i\right\}$$

looks like $S^1_\epsilon\times\hat{\square}^{n-1}_\epsilon$. Their intersections look like 2-tori $(S^1_\epsilon)^2\times\hat{\square}^{n-2}_\epsilon$ over the subfaces:

$$\mathcal{T}^2_\epsilon\begin{pmatrix} e_1 & e_2\\ i_1 & i_2\end{pmatrix}=\mathcal{T}^1_\epsilon\begin{pmatrix} e_1\\ i_1\end{pmatrix}\cap\mathcal{T}^1_\epsilon\begin{pmatrix} e_2\\ i_2\end{pmatrix};\ \text{define } \mathcal{T}^2_\epsilon=\cup\mathcal{T}^2_\epsilon\begin{pmatrix} \mathbf{e}\\ \mathbf{i}\end{pmatrix},\quad\text{etc.}$$

Notice that $(\mathcal{T}^i_\epsilon,\mathcal{T}^{i+1}_\epsilon)\sim((S^1_\epsilon)^i\times(\hat{\square}^{n-i},\partial\hat{\square}^{n-i}))$. Finally, define

$$\overset{\circ}{N^1_\epsilon}\begin{pmatrix} e\\ i\end{pmatrix}:=\left\{(z_1,\ldots,z_n)\in\hat{\square}^n\,\middle|\,0<|z_i|^{p(e)}\le\epsilon,\text{ and } \epsilon\le|z_j|\le\frac{1}{\epsilon}\text{ for } j\neq i\right\}$$

$$=\bigcup_{0<\epsilon_0\le\epsilon}\mathcal{T}^1_{\epsilon_0}\begin{pmatrix} e\\ i\end{pmatrix}\quad\text{and}\quad \overset{\circ}{N^1_\epsilon}:=\cup\overset{\circ}{N^1_\epsilon}\begin{pmatrix} e\\ i\end{pmatrix}\neq N_\epsilon(\partial\square^n)\setminus\partial\square^n,$$

$$\overset{\circ}{N^2_\epsilon}\begin{pmatrix} e_1 & e_2\\ i_1 & i_2\end{pmatrix}:=\left\{(z_1,\ldots,z_n)\in\hat{\square}^n\,\middle|\,0<|z_{i_1}|^{p(e_1)}=|z_{i_2}|^{p(e_2)}\le\epsilon,\text{ etc.}\right\}$$

$$=\bigcup_{0<\epsilon_1\le\epsilon}\mathcal{T}^2_{\epsilon_1}\begin{pmatrix} e_1 & e_2\\ i_1 & i_2\end{pmatrix}\neq\overset{\circ}{N^1_\epsilon}\begin{pmatrix} e_1\\ i_1\end{pmatrix}\cap\overset{\circ}{N^1_\epsilon}\begin{pmatrix} e_2\\ i_2\end{pmatrix},$$

with $\overset{\circ}{N^2_\epsilon}$ their union, and so on. We write π_X and $\pi_\square$ for the obvious projections from $X\times\hat{\square}^n$ (possibly composed with the inclusion of $\mathcal{Z}$); d will always denote $\dim X$.

LEMMA 1.3.3. *Let $\mathcal{Z}\in Z^p(X\times(\square^n,\partial\square^n))$ be a relative cycle, where X is compact. Then for $\epsilon_0>0$ sufficiently small there is a continuous family $\{\mathcal{Z}_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$ of topological cycles such that*
(a) $\mathcal{Z}_\epsilon$ has support on $X\times\overline{\hat{\square}^n_\epsilon}$ and is ∂-closed as an object on $X\times(\overline{\hat{\square}^n\setminus\partial\hat{\square}^n})$.
(b) $\mathcal{Z}\equiv\mathcal{Z}_\epsilon$ on $X\times\hat{\square}^n_\epsilon$ (i.e. modulo chains with support on $X\times\overline{N_\epsilon(\partial\hat{\square}^n)}$).

(c) writing (by (b)) $\mathcal{Z}_\epsilon = \mathcal{Z} \cap (X \times \overline{\hat{\Box}^n_\epsilon}) + \mathcal{W}_\epsilon$*,* $\int_{\mathcal{W}_\epsilon} \bigwedge^n dlog z_i \wedge \pi_X^* \alpha \to 0$ *as* $\epsilon \to 0$ *for all* C^∞*-forms* $\alpha \in \Omega^{2d-(2p-n)}_{X^\infty}(X)$.

Sketch of Proof. The idea is to construct the chain $\mathcal{W}_\epsilon \in C_*(X \times \mathcal{T}^1_\epsilon)$ so that $\partial \mathcal{W}_\epsilon = \partial(\mathcal{Z} \cap [X \times \overline{\hat{\Box}^n_\epsilon}]) = \mathcal{Z} \cap (X \times \mathcal{T}^1_\epsilon) =: R^1_\epsilon$, i.e. with boundary just the intersection of $\mathcal{Z}$ with the tube. (Note that this is reasonable since R^1_ϵ is at least a topological *cycle* – as $\mathcal{Z}$ and $(X \times \mathcal{T}^1_\epsilon)$ both are.)

For $\epsilon_0 > 0$ "sufficiently small" we give an heuristic procedure for "shrinking" $R^1_{\epsilon_0}$ to zero. Filter the chains $C_*(X \times \mathcal{T}^1_{\epsilon_0})$ by subgroups $C_*(X \times \mathcal{T}^i_{\epsilon_0})$ so that the graded pieces are $C_*(X \times (\mathcal{T}^i_{\epsilon_0}, \mathcal{T}^{i+1}_{\epsilon_0}))[:= C_*(X \times \mathcal{T}^i_{\epsilon_0})/C_*(X \times \mathcal{T}^{i+1}_{\epsilon_0})]$. Since $\mathcal{Z}$ intersects the subfaces of $\Box^n$ properly, all intersections with subfaces $\rho_* \Box^{n-i}$ come from first intersecting with some face $\rho_* \Box^{n-1} \subset \partial \Box^n$ (no "corners" are allowed). By taking $\epsilon_0 > 0$ sufficiently small we can ensure the same thing for intersecting with the $\mathcal{T}^i_{\epsilon_0}$. Therefore it is enough to kill $R^1_{\epsilon_0}$ as a cycle $\bar{R}^1_{\epsilon_0}$ on $X \times (\mathcal{T}^1_{\epsilon_0}, \mathcal{T}^2_{\epsilon_0})$, and deal with any "residues" that arise from additional boundary (in $X \times \mathcal{T}^2_{\epsilon_0}$) of the chain we use to kill it. Since $\bar{R}^1_{\epsilon_0} \sim S^1_{\epsilon_0} \times \{\text{cycle}\}$ on each "component" of $X \times (\mathcal{T}^1_{\epsilon_0}, \mathcal{T}^2_{\epsilon_0})$, the resulting "residue" cycle in $X \times \mathcal{T}^2_{\epsilon_0}$ will be homologous to $S^1_{\epsilon_0} \times S^1_{\epsilon_0} \times \{\text{cycle}\}$, and so on. Since the "toric" factors of the resulting classes are uninteresting for all i (this is essentially a consequence of proper intersection), all that is at issue are the "$X \times \Box^{n-i}$" factors, the support of which will tend to $\mathcal{Z} \cap (X \times \Box^{n-i})$ as we let $\epsilon \to 0$. So it makes sense to try to "project" this shrinking back out to $\mathcal{T}^i_{\epsilon_0}$ and use this to produce the bounding chain that "pushes" our cycle into the next $\mathcal{T}^{i+1}_{\epsilon_0}$.

More precisely, there are maps (for $0 < \epsilon_1 \leq \epsilon_0$)

$$P^1_\epsilon \begin{pmatrix} e \\ i \end{pmatrix} : (0, \epsilon_1] \times \mathcal{T}^1_{\epsilon_1} \begin{pmatrix} e \\ i \end{pmatrix} \to \overset{\circ}{N^1_{\epsilon_1}} \begin{pmatrix} e \\ i \end{pmatrix}$$

defined by

$$\epsilon, \ \underbrace{(z_1, \ldots, z_i, \ldots, z_n)}_{|z_i|^{p(e)} = \epsilon_1} \mapsto (z_1, \ldots, \left(\frac{\epsilon}{\epsilon_1}\right)^{p(e)} z_i, \ldots, z_n)$$

which we summarize by

$$P^1_{\epsilon_1} : (0, \epsilon_1] \times \mathcal{T}^1_{\epsilon_1} \to \overset{\circ}{N^1_{\epsilon_1}} .$$

Let

$$\mathcal{W}^1_{\epsilon_1} := \overline{{P^1_{\epsilon_1}}^* (\mathcal{Z} \cap (X \times \overset{\circ}{N^1_{\epsilon_1}}))};$$

since supp $\left\{\mathcal{Z} \cap (X \times \mathcal{T}^1_{\epsilon_1})\right\}$ tends to $S^1_\epsilon \times \{\mathcal{Z} \cap (X \times \partial \Box^n)\}$ as $\epsilon \to 0$ (bringing $+$ and $-$ components together) we must have $\partial \mathcal{W}^1_{\epsilon_1} = R^1_{\epsilon_1} + R^2_{\epsilon_2}$, where $R^2_{\epsilon_1} \in C_*(X \times \mathcal{T}^2_{\epsilon_1})$. This is a cycle since both $\partial \mathcal{W}^1_{\epsilon_1}$ and $R^1_{\epsilon_1}$ are; also, as

$\epsilon_1 \to 0$ its support $\to S^1_{\epsilon_1} \times S^1_{\epsilon_1} \times \left\{\mathcal{Z} \cap (X \times \cup \rho_* \square^{n-2})\right\}$, which is of lower dimension (this is in lieu of the $+/-$ pair annihilation above). Writing (for $0 < \epsilon_2 \leq \epsilon_0$)

$$R^2_{\leq \epsilon_2} := \bigcup_{0<\epsilon_1 \leq \epsilon_2} R^2_{\epsilon_1},$$

define

$$P^2_{\epsilon_2} : (0, \epsilon_2] \times \mathcal{T}^2_{\epsilon_2} \to \overset{\circ}{N^2_{\epsilon_2}}$$

by analogy to $P^1_{\epsilon_1}$, and let

$$\mathcal{W}^2_{\epsilon_2} := \overline{{P^2_{\epsilon_2}}^*(R^2_{\leq \epsilon_2})};$$

then $\partial \mathcal{W}^2_{\epsilon_2} = R^2_{\epsilon_2} + R^3_{\epsilon_2}$. If, say, $R^3 = 0$, then we put $\mathcal{W}_\epsilon := \mathcal{W}^1_\epsilon - \mathcal{W}^2_\epsilon$, so $\partial \mathcal{W}^1_\epsilon = \partial(\mathcal{W}^1_\epsilon - \mathcal{W}^2_\epsilon) = R^1_\epsilon = \mathcal{Z} \cap (X \times \mathcal{T}^1_\epsilon)$ exactly, for any $0 < \epsilon \leq \epsilon_0$. In general $\mathcal{W}_\epsilon = \mathcal{W}^1_\epsilon - \mathcal{W}^2_\epsilon + \mathcal{W}^3_\epsilon - \ldots$, proving (a) and (b).

Now let V be any (irreducible) algebraic subvariety of $X \times \square^m$ intersecting all subfaces properly, and let $V_\epsilon = V \cap (X \times \hat{\square}^m_\epsilon)$. Then

$$\lim_{e \to 0} \left| \int_{V_\epsilon} \bigwedge^m \mathrm{dlog} z_i \wedge \pi_X^* \beta \right| < \infty$$

for β a C^∞-form on X, because the proper intersection condition prevents $\bigwedge^m \mathrm{dlog} z_i$ from having worse than log poles (since no two z_i's share a divisor component). (One can then use a polar $\int$ argument locally to show convergence.) From the previous discussion, $\mathcal{W}^i_\epsilon$ may be treated as $(S^1_\epsilon)^i \times \tilde{V}_\epsilon$, where $\tilde{V}_\epsilon$ is essentially the product of such a V_ϵ with a narrow band (of width bounded by a positive power of ϵ). Therefore

$$\frac{1}{(2\pi\sqrt{-1})^i} \int_{\mathcal{W}^i_\epsilon} \bigwedge^n \mathrm{dlog} z_i \wedge \pi_X^* \alpha \approx \epsilon^q \int_{V_\epsilon} \omega \to 0$$

where ω has at worst dlog poles by essentially the same argument as above.

Finally we sketch how to deal with (sub)face intersections at $\mathbb{I}^n$, which may not be proper. In the proper case, a component Y of $\mathrm{supp}(Z)$ intersects a face $\square^{n-1}$, e.g. $z_2 = 0$, at say $z_1 = 1$. let $\pi_1 Y$ be the projection of Y to $\{z_1 = 1\} \subset (\mathbb{P}^1)^n$; we construct a chain $\mathcal{W}^0_\epsilon$ by letting z_1 vary slightly on $\mathcal{T}^1_\epsilon$ between $Y \cap \mathcal{T}^1_\epsilon$ and $\pi_1 Y \cap \mathcal{T}^1_\epsilon$. Integrating $\bigwedge^n \mathrm{dlog} z_i \wedge \alpha$ first in the z_1-direction then on $S^1_\epsilon(z_2)$ we get $(2\pi\sqrt{-1})\epsilon \times \{\text{convergent } \int\}$ as before. If Y intersects (improperly) a subface $\square^{n-2}$, e.g. $z_2 = z_3 = 0$, at $z_1 = 1$, then we have no residue because the S^1_ϵ enclose a double pole. $\square$

In fact the sublemma we used above on V_ϵ is so important we restate it for reference in subsequent sections.

LEMMA 1.3.4. *If $\mathcal{Z} \in Z^p(X,n)$ is a higher Chow cycle and X is compact, then* $\lim_{\epsilon\to 0}\int_{\mathcal{Z}\cap(X\times\hat{\square}^n_\epsilon)} \bigwedge^n dlogz_i \wedge \pi_X^*\alpha$ *is convergent for any C^∞-form α.*

REMARK 1.3.5. If X is quasiprojective then we must take α to be compactly supported. (We may also be able to get by with "holomorphic vanishing at the boundary": if $X = \bar{X} - Y$ then these are the $\Omega_{\bar{X}}(\mathrm{null}Y)$ forms of King.) Intuitively, this is because we must treat the original $\mathcal{Z}$ as a "relative cycle": to understand the cohomology class

$$[\mathcal{Z}] \in H^{2p}((\bar{X}-Y)\times(\square^n,\partial\square^n))$$

we compute the dual homology class

$$[\mathcal{Z}_\epsilon] \in H_{2(d+n-p)}((\bar{X},Y)\times(\hat{\square}^n\setminus\partial\hat{\square}^n));$$

in order to do this we must integrate forms representing classes in $H^{2p-n}(\bar{X},Y)\otimes H^n(\hat{\square}^n\setminus\partial\hat{\square}^n)$ on $\mathcal{Z}_\epsilon$ (which now involves an extra "residue" component around Y). We'll do this carefully in the next section for graph cycles over η_X.

1.3.2. The basic homotopy. Now we establish the basic dictionary for what will be essentially a reworking and generalization of Bloch's regulator construction in Chapter 8 of his book [**B1**], which corresponds to the case $n = 2$, $p = 2$, $d = 1$ (X a curve).

...AND MORE NOTATION. The lemma gives us an homology class

$$[\mathcal{Z}_\epsilon] \in H_{2(d+n-p)}(X\times(\hat{\square}^n\setminus\partial\hat{\square}^n)) \cong H_{2d-(2p-n)}(X)\otimes H_n((\mathbb{C}^*,\{1\})^n);$$

clearly

$$\mathcal{Z}_\epsilon \sim \mathrm{cycle}_X \times (S^1)^n + \text{stuff on } \mathbb{I}^n,$$

and this homological equivalence will be provided by an explicit homotopy below. Let T denote the branch cut in $\log z$ on $\mathbb{C}^*$ along the negative real axis $\mathbb{R}^-$ (the point is to avoid $\{1\}$). Sometimes T will also indicate its closure, which defines the generator of $H_1(\mathbb{P}^1-\{1\},\{0,\infty\})$ dual to $[S^1]\in H_1(\mathbb{C}^*,\{1\})$. Define a "topological normalization"

$$\mathcal{N} : \mathbb{D} \to \mathbb{C}^*$$

where $\mathbb{D}$ is just the closure $\mathbb{C}^*\setminus T$ with the left and right-hand limits *not* identified, so that $\tilde{z} := \mathcal{N}^{-1}(z\notin T)$ is one point[4] while $\mathcal{N}^{-1}(z\in T) =$

[4]Generally speaking, $\tilde{z}$ always denotes *one* point $\in \partial_{\mathcal{B}}$ with $\mathcal{N}(\tilde{z}) = z$, even for $z\in T$ ($\tilde{z}$ would mean z^+ or z^-); this $\tilde{z}$ is our notation for definitions. The "tilde" notation generally means *one lift*; the main exception is $\tilde{T} = T^+ - T^-$. However when one has a lift $\widetilde{\gamma_f}$ of a graph and specializes to the part of the lift over, say, those $x\in T_f$, it becomes necessary to write $\widetilde{f(x)} = f(x)^+ - f(x)^-$, breaking the above rule. This is just to forewarn the reader.

$\{z^+, z^-\}$; that is, $\mathcal{N}^{-1}(T)$ consists of the two disjoint open segments T^+ and T^-. We write

$$\tilde{T} := \partial\mathbb{D} = T^+ - T^- \overset{\mathcal{N}}{\longmapsto} 0.$$

In this simple case ($n = 1$) the homotopy is given by choosing any C^∞ map

$$\theta : [0,1] \times \mathbb{D} \to \mathbb{C}^*$$

sending

$$0, \tilde{z} \mapsto z \quad \text{and} \quad 1, \tilde{z} \mapsto 1$$

so that $\theta(z) = \theta(t,z)$ gives a path from z to 1 in $\mathbb{C}^*$. Also observe that the chain $\theta(z^+) - \theta(z^-) =: \theta^+(z) - \theta^-(z)$ is a circle (S^1), where we have defined maps

$$\theta^\pm : [0,1] \times T \to \mathbb{C}^*.$$

Most importantly, we write formally "$\partial\theta(\mathbb{D}) = \theta(\partial[0,1]\times\mathbb{D}) + \theta([0,1]\times\partial\mathbb{D}) = \mathbb{C}^* + \theta^+(T) - \theta^-(T)$" (ignoring stuff at $\{1\}$as we shall usually do) which means that for a topological (∂-closed) cycle Γ with compact support on $\mathbb{C}^*$

$$\partial\theta(\tilde{\Gamma}) = \Gamma + \theta^+(\Gamma \cap T) - \theta^-(\Gamma \cap T)$$

where $\tilde{\Gamma} := \mathcal{N}^{-1}(\Gamma)$ is its lift to $\mathbb{D}$.

More generally (working on $(\mathbb{C}^*)^n$ for $n > 1$) let T_{z_i} be branch cuts for $\log z_i$ along $\mathbb{C}^* \times \ldots \times \mathbb{R}^- \times \ldots \times \mathbb{C}^*$. Writing $\mathbb{D}^n$ for the formal closure of $(\mathbb{C}^*)^n \setminus \cup T_{z_i}$ as above (this should be thought of as a sort of fundamental domain for $(\mathbb{C}^*)^n$), we have maps

$$\mathcal{N} : \mathbb{D}^n \to (\mathbb{C}^*)^n$$

taking

$$T^+_{z_i}, T^-_{z_i} \overset{\mathcal{N}}{\longmapsto} T_{z_i}$$

so that

$$\partial\mathbb{D}^n = \sum(-1)^i(T^+_{z_i} - T^-_{z_i}) =: \sum(-1)^i\tilde{T}_{z_i}$$

which (as a chain) is taken to 0 by $\mathcal{N}$. We define various maps/chains

$$\theta_{i_1 \ldots i_k} : [0,1]^k \times \mathbb{D}^n \to (\mathbb{C}^*)^n$$

by formulas

$$\theta_1(\tilde{z}_1, \ldots, \tilde{z}_n) = (\theta(t_1, \tilde{z}_1), z_2, \ldots, z_n),$$

$$\theta_{12}(\tilde{z}_1, \ldots, \tilde{z}_n) = (\theta(t_1, \tilde{z}_1), \theta(t_2, \tilde{z}_2), z_3, \ldots, z_n),$$

and so on (note that we often omit the parameters t_i in the argument). Again certain restrictions descend to subsets of $(\mathbb{C}^*)^n$, e.g. by restricting θ_{12} to $T^+_{z_1} \cap T^-_{z_2}$we have

$$\theta^{+-}_{12} : [0,1]^2 \times (T_{z_1} \cap T_{z_2}) \to (\mathbb{C}^*)^n$$

and more generally $\theta^{s_1\dots s_k}_{i_1\dots i_k} =: \theta^{\mathbf{s}}_{\mathbf{i}}$. For $z \in T_{z_{i_1}} \cap \ldots \cap T_{z_{i_k}}$, the formal sum (considered as a chain)

$$\sum_{s_1,\ldots,s_k=+,-} (-1)^{\prod s_j} \theta^{\mathbf{s}}_{\mathbf{i}}(z)$$

yields a topological k-torus $(S^1)^k$. Since all these constructions may be crossed with X, it makes sense to take the preimage $\tilde{\mathcal{Z}}_\epsilon$ of $\mathcal{Z}_\epsilon$ under $\mathcal{N}$ and make the following

CLAIM 1.3.6. $\partial\theta_1(\tilde{\mathcal{Z}}_\epsilon) = \mathcal{Z}_\epsilon + \theta_1^+(\mathcal{Z}_\epsilon \cap T_{z_1}) - \theta_1^-(\mathcal{Z}_\epsilon \cap T_{z_1})$.

We just have to interpret the terms of

$$\partial\theta_1(\tilde{\mathcal{Z}}_\epsilon) = \theta_1(\tilde{\mathcal{Z}}_\epsilon \cap \partial\mathbb{D}^n[\times[0,1]]) + \theta_1(\tilde{\mathcal{Z}}_\epsilon[\cap\mathbb{D}^n] \times (\{0\} - \{1\})),$$

the second obviously being $\mathcal{Z}_\epsilon$ (mod $\mathbb{I}^n$). The first term is

$$\sum_i (-1)^i \theta_1(\tilde{\mathcal{Z}}_\epsilon \cap \tilde{T}_{z_i}) = \sum_i (-1)^i \left\{ \theta_1(\tilde{\mathcal{Z}}_\epsilon \cap T^+_{z_i}) - \theta_1(\tilde{\mathcal{Z}}_\epsilon \cap T^-_{z_i}) \right\}.$$

for $i \neq 1$ the $\tilde{z}_1$-coordinates (on $\mathbb{D}^n$) of $\tilde{\mathcal{Z}}_\epsilon \cap T^+_{z_i}$ and $\tilde{\mathcal{Z}}_\epsilon \cap T^-_{z_i}$ are the same: so $(\theta(t,\tilde{z}_1), z_2 \ldots, z_n)$ is the same on the two (z_i annihilates the difference between z_i^+ and z_i^-). On the other hand for $i = 1$ this is $\theta_1^+(\mathcal{Z}_\epsilon \cap T_{z_1}) - \theta_1^-(\mathcal{Z}_\epsilon \cap T_{z_1})$ as promised.

Similarly, one writes formally (mod $\mathbb{I}^n$) "$\partial\theta_{12}(\mathbb{D}^n) = \theta_{12}(\partial\mathbb{D}^n) + \theta_{12}(\mathbb{D}^n \times \partial[0,1]^2) = \sum_i (-1)^i \{\theta_{12}(T^+_{z_i}) - \theta_{12}(T^-_{z_i})\} + \theta_1(\mathbb{D}^n) - \theta_2(\mathbb{D}^n)$". Again, this only makes sense when arguments are intersected with a compactly supported topological cycle. As above, the $i \neq 1, 2$ terms of the sum are zero. Now the cycle we "intersect with" is not $\tilde{\mathcal{Z}}_\epsilon$ but $\tilde{\mathcal{Z}}_\epsilon \cap \tilde{T}_{z_1}$ and similarly $\theta_2(\tilde{\mathcal{Z}}_\epsilon \cap \tilde{T}_{z_1}) = \theta_2(\tilde{\mathcal{Z}}_\epsilon \cap T^+_{z_1}) - \theta_2(\tilde{\mathcal{Z}}_\epsilon \cap T^-_{z_2}) = 0$. Therefore

$$\partial\theta_{12}(\tilde{\mathcal{Z}}_\epsilon \cap \tilde{T}_{z_1}) = \theta_1(\tilde{\mathcal{Z}}_\epsilon \cap \tilde{T}_{z_1}) + \theta_{12}(\tilde{\mathcal{Z}}_\epsilon \cap \tilde{T}_{z_1} \cap \tilde{T}_{z_2}),$$

and similar reasoning yields

$$\begin{aligned}\partial\theta_{123}(\tilde{\mathcal{Z}}_\epsilon \cap \tilde{T}_{z_1} \cap \tilde{T}_{z_2}) &= \theta_{12}(\tilde{\mathcal{Z}}_\epsilon \cap \tilde{T}_{z_1} \cap \tilde{T}_{z_2}) - \theta_{23}(\tilde{\mathcal{Z}}_\epsilon \cap \tilde{T}_{z_1} \cap \tilde{T}_{z_2}) \\ &\quad + \theta_{13}(\tilde{\mathcal{Z}}_\epsilon \cap \tilde{T}_{z_1} \cap \tilde{T}_{z_2}) + \theta_{123}(\tilde{\mathcal{Z}}_\epsilon \cap \tilde{T}_{z_1} \cap \tilde{T}_{z_2} \cap \tilde{T}_{z_3}) \\ &= \theta_{12}(\tilde{\mathcal{Z}}_\epsilon \cap \tilde{T}_{z_1} \cap \tilde{T}_{z_2}) + \sum_{\mathbf{s}} (-1)^{\prod s_j} \theta^{\mathbf{s}}_{123}(\mathcal{Z}_\epsilon \cap T_{z_1} \cap T_{z_2} \cap T_{z_3}).\end{aligned}$$

If $n = 3$, then adding

$$\theta_1(\tilde{\mathcal{Z}}_\epsilon) - \theta_{12}(\tilde{\mathcal{Z}}_\epsilon \cap \tilde{T}_{z_1}) + \theta_{123}(\tilde{\mathcal{Z}}_\epsilon \cap \tilde{T}_{z_1} \cap \tilde{T}_{z_2}) =: \theta(\tilde{\mathcal{Z}}_\epsilon),$$

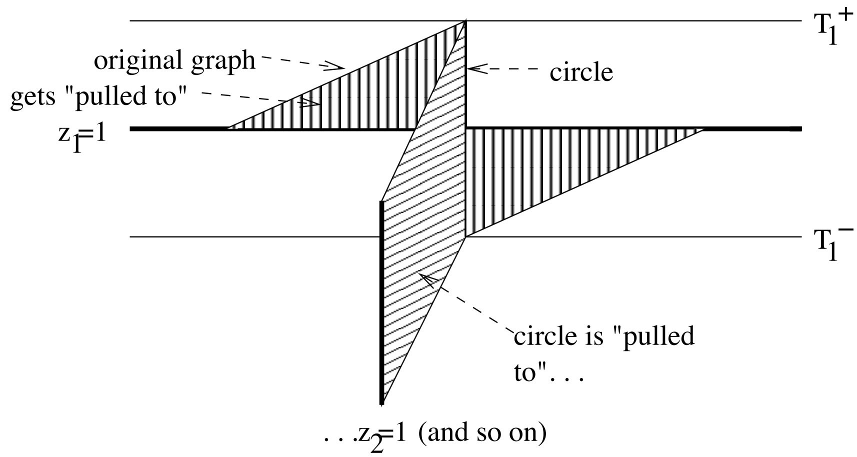
we have

$$\partial\theta(\tilde{\mathcal{Z}}_\epsilon) = \mathcal{Z}_\epsilon + \sum_{\mathbf{s}} (-1)^{\prod s_j} \theta^{\mathbf{s}}_{123}(\mathcal{Z}_\epsilon \cap T_{z_1} \cap T_{z_2} \cap T_{z_3})$$

Let $T_n = T_{z_1} \cap \ldots \cap T_{z_n}$, so that $[T_n]$ represents the one nontrivial class $\in H_*(\square^n, \partial\square^n) = H_n([\mathbb{P}^1 - \{1\}, \{0, \infty\}]^n)$. Then we have established the following general formula (modulo $\mathbb{I}^n$):

$$\boxed{\begin{array}{c}\partial\theta(\tilde{\mathcal{Z}}_\epsilon) = \mathcal{Z}_\epsilon \pm \sum_{\mathbf{s}}(-1)^{\mathbf{s}}\theta^{\mathbf{s}}_{1\ldots n}(\mathcal{Z}_\epsilon \cap T_n), \text{ where} \\ \theta(\tilde{\mathcal{Z}}_\epsilon) := \theta_1(\tilde{\mathcal{Z}}_\epsilon) + \sum_{i=1}^n (-1)^i\, \theta_{1\ldots(i+1)}(\tilde{\mathcal{Z}}_\epsilon \cap \tilde{T}_{z_1} \cap \ldots \cap \tilde{T}_{z_i}).\end{array}}$$

In words, θ pushes $\mathcal{Z}_\epsilon$ down to $z_1 = 1$, with discrepancy arising from $\mathcal{Z}_\epsilon \cap T_{z_1}$ which we then push down to $z_2 = 1$, and so on ($\mathbb{I}^n$ serves as a sort of topological trashcan). In a picture over $\mathbb{R}$,



We'll record what exactly the boundary is at $\mathbb{I}^n$ for some specific examples later.

1.3.3. Triviality of the cycle-class map. We now enter the Hodge-theoretic part of the section. Topologically,

$$\sum_{\mathbf{s}}(-1)^{\mathbf{s}}\theta^{\mathbf{s}}_{1\ldots n}(\mathcal{Z}_\epsilon \cap T_n) \sim \pi_X(\mathcal{Z}_\epsilon \cap T_n) \times (S^1)^n =: T_{\mathcal{Z}_\epsilon} \times (S^1)^n,$$

where $T_{\mathcal{Z}_\epsilon}$ is clearly a (topological) cycle on X ($\mathcal{Z}_\epsilon$ avoids "∂T_n" as the latter is contained inside $\partial\square^n$). Specializing now to the case $p = n$, if $\mathcal{Z}_\epsilon = \Gamma_\epsilon$ comes from a graph completion $\Gamma \in Z^n(X \times (\square^n, \partial\square^n))$, then we will show $T_{\mathcal{Z}_\epsilon}$ is in fact a topological *boundary*. We do this by proving that Γ_ϵ has trivial cycle class on $X \times (\hat{\square}^n \setminus \partial\hat{\square}^n)$, i.e. that for all *closed* C^∞ forms $\alpha \in \Omega^{2d-n}_{X^\infty}(X)$,

$$\int_{\Gamma_\epsilon} \bigwedge^n \mathrm{dlog} z_i \wedge \pi_X^*\alpha \, = \, 0.$$

In fact it is sufficient to show

$$\int_{\Gamma\cap(X\times\hat{\square}^n_\epsilon)} \bigwedge^n \mathrm{dlog} z_i \wedge \pi_X^*\alpha \, \to \, 0$$

as $\epsilon \to 0$ since then

$$\int_{\Gamma_\epsilon} = \int_{\Gamma\cap(X\times\hat{\square}^n_\epsilon)} + \int_{\mathcal{W}_\epsilon} \to 0$$

by the lemma, and we have the following simple rigidity argument: as the homology class of Γ_ϵ is not changing, neither is the integral, so it must be 0 for small $\epsilon > 0$.

Now we split α into types: $\alpha = \alpha_0 + \alpha_1$ (both d-closed), where

$$\alpha_0 \in \Omega^{d-n,d}_{X^\infty}(X) \quad \text{and} \quad \alpha_1 \in F^{d-n+1}\Omega^{2d-n}_{X^\infty}(X).$$

Since all pieces of [the support of] $\Gamma \cap (X \times \hat{\square}^n_\epsilon)$ are algebraic,

$$\int_{\Gamma\cap(X\times\hat{\square}^n_\epsilon)} \bigwedge^n \mathrm{dlog} z_i \wedge \pi_X^* \alpha_1 = 0$$

by Hodge type. On the other hand, Γ itself splits into "two" parts:

$$\Gamma = \overline{\gamma_{\mathbf{f}}} + \sum_{i\geq 1} \gamma^i$$

where the γ^i are localized over codimension $i \geq 1$ points $x \in X^i$. However,

$$i_x^* \alpha_0^{(d-n,d)} = 0$$

(since $d = \dim X$) by Hodge type and so

$$\int_{(\sum_{i\geq 1}\gamma^i)\cap(X\times\hat{\square}^n_\epsilon)} \bigwedge^n \mathrm{dlog} z_i \wedge \pi_X^* \alpha_0 = 0.$$

Therefore, writing

$$\bigwedge^n \mathrm{dlog}\mathbf{f} := \sum m_j \mathrm{dlog} f_{1j} \wedge \ldots \wedge \mathrm{dlog} f_{nj}$$

and

$$X^\epsilon := X \setminus \mathrm{supp}\left\{\bar{\gamma}_{\mathbf{f}} \cap \left(X \times N_\epsilon(\partial\hat{\square}^n)\right)\right\}$$

(note all the aforementioned $x \in X^i$ are contained in the compliment $X \backslash X^\epsilon$), it suffices to show[5]

$$\int_{\gamma_{\mathbf{f}}\cap(X\times\hat{\square}^n_\epsilon)} \bigwedge^n \mathrm{dlog} z_i \wedge \pi_X^* \alpha_0 \approx \boxed{\int_{X^\epsilon} \wedge^n \mathrm{dlog}\mathbf{f} \wedge \alpha_0 \to 0}$$

for all ∂-closed α_0 of pure type $(d-n, d)$.

Now recall that any $\mathbf{h} = \sum m_j h_{1j} \otimes \ldots \otimes h_{nj}$ may be brought into general position $\mathbf{h}'$ by addition of a Steinberg relation $\mathbf{g}$ (our $\mathbf{f}$ already is but it doesn't hurt to develop the addtional theory now). For such an $\mathbf{h}'$, $\bigwedge^n \mathrm{dlog}\mathbf{h}' = \sum m_j \mathrm{dlog} h'_{1j} \wedge \ldots \wedge \mathrm{dlog} h'_{nj}$ has log poles along codimension 1

[5]The "$\approx$" means we are throwing out integrals over pieces of $\gamma_{\mathbf{f}}$ which are supported over $X \setminus X^\epsilon$ but not in $X \times N_\epsilon(\partial\hat{\square}^n)$. This only happens when the graph has multiple components, and these irrelevant integrals obviously go to 0 with ϵ. Anyway, what we now show is the thing in the box.

points $x \in X^1$, while $\bigwedge^n \mathrm{dlog}\mathbf{g} = 0$ since $\mathrm{dlog}g \wedge \mathrm{dlog}(1-g) = \frac{1}{g(g-1)}\mathrm{d}g \wedge \mathrm{d}g = 0$. So without even having to choose our representatives (or move $\mathbf{h}$), one may define a dlog map on the level of $\otimes^n \mathbb{Z}[\mathbb{P}^1_{\mathbb{C}(X)} \setminus \{0, \infty\}]$ that descends to a map on Milnor K-theory

$$\bigwedge^n \mathrm{dlog} : K_n^M(\mathbb{C}(X)) \to \varinjlim_{D \subset X} H^0(\Omega_X^n(\log D)) =: H^0(\Omega_X^n(\log))$$

(abbreviated "dlog") and commutes with Tame:

$$\begin{tikzcd} K_n^M(\mathbb{C}(X)) \arrow[r, "\mathrm{dlog}"] \arrow[d, "\mathrm{Tame}"] & H^0(\Omega_X^n(\log)) \arrow[d, "\mathrm{Res}"] \\ \coprod_{x \in X^1} K_{n-1}^M(\mathbb{C}(x)) \arrow[r, "\mathrm{dlog}"] & \coprod_{x \in X^1} H^0(\Omega_{\tilde{x}}^{n-1}(\log)). \end{tikzcd}$$

Since Γ above was a graph completion of $\gamma_{\mathbf{f}}$, $\{\mathbf{f}\}$ was already in $\ker(\mathrm{Tame})$ and so $\bigwedge^n \mathrm{dlog}\mathbf{f}$ has zero residues. Therefore by duality it defines a class

$$[\wedge^n \mathrm{dlog}\mathbf{f}] \in H^{n,0}(X, \mathbb{C}) \cong H^{d-n,d}(X, \mathbb{C})^\vee.$$

In particular, if $\alpha = \mathrm{d}\beta \in \Omega_{X^\infty}^{d-n,d}(X)$ is exact, then

$$\lim_{\epsilon \to 0} \int_{X^\epsilon} \wedge^n \mathrm{dlog}\mathbf{f} \wedge \mathrm{d}\beta \ = \ -\lim_{\epsilon \to 0} \int_{\partial X^\epsilon} \wedge^n \mathrm{dlog}\mathbf{f} \wedge \beta$$

$$= -\lim_{\epsilon \to 0} \sum_{x \in X^1} \int_{x^\epsilon} \mathrm{Res}_x(\wedge^n \mathrm{dlog}\mathbf{f}) \wedge i_x^* \beta \ = \ 0.$$

Now obviously any form in the image of dlog has, on η_X, periods in $\mathbb{Z}(n) = (2\pi\sqrt{-1})^n \mathbb{Z}$; in fact we can verify the stronger statement that the class

$$[\wedge^n \mathrm{dlog}\mathbf{f}] \in H^{n,0}(X, \mathbb{C}) \cap H^n(X, \mathbb{Z}(n)),$$

as follows. If $\mathrm{d}\alpha = 0$, then (working away from the singularities of $\wedge^n \mathrm{dlog} z_i$)

$$0 \ = \ \int_{\theta(\tilde{\Gamma}_\epsilon)} \wedge^n \mathrm{dlog} z_i \wedge \mathrm{d}(\pi_X^* \alpha) \ = \ \int_{\partial\theta(\tilde{\Gamma}_\epsilon)} \wedge^n \mathrm{dlog} z_i \wedge \pi_X^* \alpha$$

$$= \int_{\Gamma_\epsilon} \wedge^n \mathrm{dlog} z_i \wedge \pi_X^* \alpha \ \pm \ \int_{(S^1) \times T_{\Gamma_\epsilon}} \wedge^n \mathrm{dlog} z_i \wedge \pi_X^* \alpha$$

$$= \int_{X^\epsilon} \wedge^n \mathrm{dlog}\mathbf{f} \wedge \alpha \ \pm \ (2\pi\sqrt{-1})^n \int_{T_{\Gamma_\epsilon}} \alpha.$$

So the class $[\bigwedge^n \mathrm{dlog}\mathbf{f}]$ defined by integrating against $\bigwedge^n \mathrm{dlog}\mathbf{f}$, is equivalent to integration over an integral chain (times $(2\pi\sqrt{-1})^n$).[6]

Now consider the periods of a (d-closed) holomorphic form representing this class: $[\omega] = [\bigwedge^n \mathrm{dlog}\mathbf{f}] \in H^{n,0}(X,\mathbb{C}) \cap H^n(X,\mathbb{Z}(n))$, where $\omega \in \Omega^n(X)$. If we assume $[\omega] \neq 0$ then $\omega_0 = \frac{1}{(2\pi\sqrt{-1})^n}\omega$ and its conjugate $\bar{\omega}_0$ represent nontrivial classes in $H^n(X,\mathbb{C})$ contained, respectively, in the subspaces $H^{n,0}(X,\mathbb{C})$ and $H^{0,n}(X,\mathbb{C})$. Since $H^{n,0} \cap H^{0,n} = \{0\}$, these classes are distinct. But the periods of ω_0 are integral, so $\bar{\omega}_0$'s (conjugate) periods are the same, and so they represent the same class in $H^n(X,\mathbb{C})$, a contradiction $\implies$ $[\omega] = 0 = [\bigwedge^n \mathrm{dlog}\mathbf{f}]$. So its class on η_X must also be 0, and since $H^0(\Omega^n_X(\log)) \hookrightarrow H^n(\eta_X,\mathbb{C})$ it follows that the form $\bigwedge^n \mathrm{dlog}\mathbf{f}$ is identically zero. (Note that for $d = \dim X < n$ this was clear to begin with.) This completes the proof that T_{Γ_ϵ} is a boundary (which implies, incidentally, that its limit T_Γ is one as well).

In fact we might have showed above that $\bigwedge^n \mathrm{dlog}\mathbf{f}$ had to extend to a holomorphic form as follows: otherwise,

$$i^*_{\eta_X}\omega - \bigwedge^n \mathrm{dlog}\mathbf{f} \in H^0(\Omega^n_X(\log)) \hookrightarrow H^n(\eta_X,\mathbb{C})$$

gives a nontrivial class, which is impossible since on X we have $[\omega] = [\bigwedge^n \mathrm{dlog}\mathbf{f}]$, which $\implies$ $[i^*_{\eta_X}\omega]_{(\eta_X)} = [\bigwedge^n \mathrm{dlog}\mathbf{f}]_{(\eta_X)}$.

For later purposes (in §1.4)we give a proof (see the corollary below) that $\bigwedge^n \mathrm{dlog}\mathbf{f} = 0$ starting from the weaker hypothesis that $\{\mathbf{f}\} \in \ker(\mathrm{Tame})$ (so that $\bigwedge^n \mathrm{dlog}\mathbf{f} \in \ker(\mathrm{Res})$), i.e. $\gamma_{\mathbf{f}} \in \ker(\mathrm{Res}^1)$. (Above we've assumed more, since the existence of Γ_ϵ, and $\therefore$ of T_{Γ_ϵ}, assumes completability of $\gamma_{\mathbf{f}}$, i.e. $\gamma_{\mathbf{f}} \in \ker(\mathrm{Res}^i)$ for all i.) It will help to keep in mind the following localization exact sequence for cohomology (where V is an arbitrary union of divisors):

$$\begin{tikzcd}
{} \arrow[r] & \varinjlim H^n_V(X) \arrow[r, "Gy"] & H^n(X) \arrow[r, "i^*_{\eta_X}"] & H^n(\eta_X) \arrow[r, "\mathrm{Res}"] & \varinjlim H^{n+1}_V(X) \arrow[r] & {} \\
 & & F^nH^n(X) \arrow[u, phantom, "\cup"] & F^nH^n(\eta_X) \arrow[u, phantom, "\cup"] & & \\
 & & H^0(\Omega^n_X) \arrow[u, "\cong"] \arrow[r, hook] & \varinjlim H^0(\Omega^n_X(\log V)) \arrow[u, "(*)"] \arrow[r, "\mathrm{Res}"] & \left\{\mathrm{im}(\mathrm{Res}) \subseteq \coprod_{x\in X^1} H^0(\Omega^{n-1}_{\tilde{\tilde{x}}}(\log))\right\} \arrow[uu, hook] &
\end{tikzcd}$$

Since we don't assume normal crossings (where it is well -known that $(*)$ is an isomorphism) in the divisors we remove, we explain why $(*)$ has to at least be injective, a fact we need for 1.3.8 below. In fact we may blow

[6] We repeat that we are working under the assumption that $\{\mathbf{f}\} \in K^M_n(X)$, so that $\gamma_{\mathbf{f}}$ comes essentially from restricting a relative cycle Γ over all of X to the generic point. In particular, this is the only reason we can work with α supported on all of X (and not just a compact subset of X^ϵ).

up to obtain from $V \subset X$ a divisor $\tilde{V} \subset \tilde{X}$ with normal crossings, via a holomorphic map $\mathcal{F} : \tilde{X} \to X$ restricting to a biholomorphism

$$\tilde{X} - \tilde{V} \overset{\sim}{\longrightarrow} X - V.$$

Now by a result of King [**Ki**], $\Omega^\bullet_X(\log V)$ forms pull back to $\Omega^\bullet_{\tilde{X}}(\log \tilde{V})$ forms under F^*, and so the diagram

$$\begin{tikzcd}
H^0(\Omega^n_X(\log V)) \arrow[r] \arrow[d, hook, "\mathcal{F}^*"'] & H^n(X - V) \arrow[d, "\mathcal{F}^*"', "\cong"] \\
H^0(\Omega^n_{\tilde{X}}(\log \tilde{V})) \arrow[r, hook] & H^n(\tilde{X} - \tilde{V})
\end{tikzcd}$$

assures us that the top map has to be injective too.

LEMMA 1.3.7. *For X compact, let $\xi \in F^n H^n(X, \mathbb{C})$ be a class whose restriction $i^*_{\eta_X}\xi \in F^n H^n(\eta_X, \mathbb{C}) \cap H^n(\eta_X, \mathbb{Z}(n))$. Then $\xi \sim 0$.*

PROOF. Assume $\xi \nsim 0$. If $(2\pi\sqrt{-1})^n\omega$ is a representative of ξ, d-closed of pure type $(n, 0)$, then $\bar{\omega}$ and ω give distinct classes in $H^{n,0}(X) \oplus H^{0,n}(X) \subset H^n(X, \mathbb{C})$. Since the image of the Gysin map in the localization sequence is contained in $H^{n-1,1} \oplus \ldots \oplus H^{1,n-1}$, it follows that

$$H^{n,0}(X) \oplus H^{0,n}(X) \overset{i^*_{\eta_X}}{\hookrightarrow} H^n(\eta_X)$$

and so $[i^*_{\eta_X}\omega] \neq [i^*_{\eta_X}\bar{\omega}]$. On the other hand, since $i^*_{\eta_X}\omega$ has integral periods, $i^*_{\eta_X}\bar{\omega}$ has the same periods, which is a contradiction. □

REMARK. Note that $\mathbb{Z}(n)$ in the above Lemma could just as well have been replaced by $\mathbb{Q}(n)$ or $\mathbb{R}(n)$.

COROLLARY 1.3.8. *If $\eta \in H^0(\Omega^n_X(\log))$ has class $[\eta] \in \{\ker(Res) \subset H^n(\eta_X, \mathbb{C})\} \cap H^n(\eta_X, \mathbb{Z}(n))$ then $\eta = 0$.*

PROOF. By the localization sequence, $\ker(\mathrm{Res}) = \mathrm{im}(i^*_{\eta_X})$ and so $[\eta] = i^*_{\eta_X}\xi$, but it does not immediately follow that ξ can be chosen in $F^n H^n(X, \mathbb{C})$. To force this we once again argue by duality that $\lim_{\epsilon\to 0}\int_{X^\epsilon} \eta \wedge \alpha$ kills coboundaries ($\mathrm{Res}\eta = 0$) as well as $F^{d-n+1}H^{2d-n}(X, \mathbb{C})$, so that integration against η gives the desired class (and one can even argue as before that η extends to a holomorphic form on X). The lemma then shows $\xi \sim 0$ so that $[\eta] = i^*_{\eta_X}0 = 0$; injectivity of the vertical map into $H^n(\eta_X)$ (sending $\eta \mapsto [\eta]$) then shows $\eta = 0$. □

REMARK 1.3.9. We offer a brief interpretation from the more sophisticated point of view we develop in §2.4 (where we write down maps from

$Z^p(X,\bullet)$ to the complex computing Deligne [co]homology, descending to regulator maps from $CH^p(X,n)$). The integrals

$$\int_{\mathcal{Z}} \bigwedge^n \mathrm{dlog} z_i \wedge \pi_X^* \alpha := \lim_{\epsilon \to 0} \int_{\mathcal{Z} \cap (X \times \hat{\square}^n_\epsilon)} \bigwedge^n \mathrm{dlog} z_i \wedge \pi_X^* \alpha,$$

interpreted in this section as functionals in

$$\left(H^{2d-(2p-n)}(X,\mathbb{C}) / F^{d-p+1} H^{2d-(2p-n)} \right)^\vee \cong F^p H^{2p-n}(X,\mathbb{C}),$$

correspond to the cycle-class map $[\,\cdot\,]$:

$$\begin{tikzcd} & H^{2p-n}_{\mathcal{D}}(X,\mathbb{Q}(p)) \arrow[dr] & \\ CH^p(X,n) \arrow[ur, "c_{\mathcal{D}}"] \arrow[rr, "{[\cdot]}"'] & & F^p H^{2p-n}(X,\mathbb{C}) \cap H^{2p-n}(X,\mathbb{Q}(p)) \end{tikzcd}$$

though later we will interpret this map as pushing $\bigwedge^n \mathrm{dlog} z_i$ down (via π_{X*} applied to its pullback to $\mathcal{Z}$) to a [singular] current of pure type $(p,p-n)$ on X. In the lemma, (c) says roughly that this class corresponds to the homology class of $\mathcal{Z}_\epsilon$, which we computed to be $(2\pi\sqrt{-1})^n$ times $[T_{\mathcal{Z}_\epsilon} \to] T_{\mathcal{Z}} = \pi_X(\mathcal{Z} \cap T_n)$. Though we only work with relative cycles here, for $\partial_{\mathcal{B}} \mathcal{Z} = 0$ these are always the same class. If $n = p$, our last argument says that *this class is zero for X compact* (or if we can otherwise determine that the cycle class has trivial residue, as in the case of $\gamma_{\mathbf{f}}$ on η_X, $\mathbf{f} \in \ker(\mathrm{Tame})$). This is not surprising since the target space above is 0 in that case.

1.3.4. Defining the Abel-Jacobi map. One of course expects that a cycle-class map (if zero) should be followed by an "Abel-Jacobi" type map, and returning to our simple point of view we explain this in the case $n = p$ at hand. Since the topological cycle $T_{\mathcal{Z}_\epsilon}$ (with real dimension $2d-n$) is a boundary on X, we may choose a bounding chain $\partial_X^{-1}(T_{\mathcal{Z}_\epsilon})$ ambiguous up to a $(2d-n+1)$cycle on X. We also choose a chain $\mathcal{B}$ on $X \times (\hat{\square}^n \setminus \partial\hat{\square}^n)$ with

$$\pm \partial \mathcal{B} = \sum_{\mathbf{s}} (-1)^{\mathbf{s}} \theta^{\mathbf{s}}_{1\ldots n}(\mathcal{Z}_\epsilon \cap T_n) - T_{\mathcal{Z}_\epsilon} \times (S^1)^n$$

and $\pi_X(\mathcal{B})$ supported over $T_{\mathcal{Z}_\epsilon}$, and S^1 say the unit circle. Writing

$$\partial_\epsilon^{-1} \mathcal{Z} := \theta(\tilde{\mathcal{Z}}_\epsilon) \mp \partial_X^{-1}(T_{\mathcal{Z}_\epsilon}) \times (S^1)^n - \mathcal{B}$$

(notice that its boundary really is $\mathcal{Z}_\epsilon$), we are interested in the integrals

$$\lim_{\epsilon \to 0} \int_{\partial_\epsilon^{-1} \mathcal{Z}} \bigwedge^n \mathrm{dlog} z_i \wedge \pi_X^* \omega$$

for all d-closed C^∞ $(2d-n+1)$-forms ω on X. First off, we have $\int_{\mathcal{B}} \bigwedge^n \mathrm{dlog} z_i \wedge \pi_X^* \omega = 0$ just by noting that $\dim \mathrm{supp} \pi_X \mathcal{B} < \deg \omega$ by 1 (so $\mathcal{B}$ can in fact be ignored in future computations).

Next, for fixed $\partial_X^{-1}(T_{\mathcal{Z}_\epsilon})$ we claim this defines (by duality) a class in $H^{n-1}(X, \mathbb{C})$. To begin with, if $\omega = \mathrm{d}\eta_0$, we can (by Hodge theory) substitute for η_0 a form $\eta \in F^{d-n+1}\Omega^{2d-n}_{X^\infty}(X)$ (that is, writing both as sums of forms of pure type, η_0 may have a $(d-n, d)$ component while η does not). Of course this does nothing (nor is it needed) in the case $d < n$. Now since $\partial_\epsilon^{-1}\mathcal{Z}$ avoids faces $\partial\hat{\square}^n$,

$$\int_{\partial_\epsilon^{-1}\mathcal{Z}} \bigwedge^n \mathrm{dlog} z_i \wedge \pi_X^* \mathrm{d}\eta \;=\; \int_{\partial_\epsilon^{-1}\mathcal{Z}} \mathrm{d}\left(\bigwedge^n \mathrm{dlog} z_i \wedge \pi_X^* \eta\right) =$$

$$\int_{\mathcal{Z}_\epsilon} \bigwedge^n \mathrm{dlog} z_i \wedge \pi_X^*\eta = \int_{\mathcal{Z}\cap(X\times\hat{\square}^n_\epsilon)} \bigwedge^n \mathrm{dlog} z_i \wedge \pi_X^* \eta \;+\; \int_{\mathcal{W}_\epsilon} \bigwedge^n \mathrm{dlog} z_i \wedge \pi_X^*\eta.$$

The last integral goes to zero as $\epsilon \to 0$ by the lemma. On the other hand, $\bigwedge^n \mathrm{dlog} z_i \wedge \eta \in F^{d+1}\Omega^{2d}_{X^\infty}(X)$ and so its pullback to $\mathcal{Z} \cap (X \times \hat{\square}^n_\epsilon)$ (which is algebraic) must vanish. So for exact ω, the limits of interest $\lim_{\epsilon\to 0} \int_{\partial_\epsilon^{-1}\mathcal{Z}} \bigwedge^n \mathrm{dlog} z_i \wedge \pi_X^* \omega$ are 0, establishing our claim.

Finally, the ambiguity in the choice of $\partial_X^{-1}(T_{\mathcal{Z}_\epsilon})$ (and $\therefore$ of $\partial_\epsilon^{-1}\mathcal{Z}$) corresponds to

$$\int_{(S^1)^n\times\Delta} \bigwedge^n \mathrm{dlog} z_i \wedge \omega = (2\pi\sqrt{-1})^n \int_\Delta \omega$$

where $[\Delta] \in H_{2d-n+1}(X, \mathbb{Z})$, i.e. (as a functional) an element of $H^{n-1}(X, \mathbb{Z}(n))$. In summary, our relative cycle $\mathcal{Z} \in Z^n(X \times (\square^n, \partial\square^n))$ has trivial cycle class and so gives rise to a class

$$[AJ(\mathcal{Z})] \in H^{n-1}(X, \mathbb{C}/\mathbb{Z}(n)) := H^{2d-n+1}(X, \mathbb{C})^\vee / \mathrm{im} H_{2d-n+1}(X, \mathbb{Z}(n))$$

via the above limit of integrals. The target space is essentially an intermediate Jacobian. (We do not yet know that $[AJ(\cdot)]$ is well-defined on the level of Chow groups or respects relative rational equivalence.)

Remark 1.3.10. One may adopt an even simpler point of view by substituting for ω singular homology $(n-1)$-cycles γ, so that the AJ-map is given by

$$\lim_{\epsilon\to 0} \int_{(\partial_\epsilon^{-1}\mathcal{Z})\cap\pi_X^{-1}(\gamma)} \bigwedge^n \mathrm{dlog} z_i \;=\; \lim_{\epsilon\to 0} \int_{\partial_\epsilon^{-1}(\mathcal{Z}\cap\pi_X^{-1}(\gamma))}$$

$$\equiv \boxed{\lim_{\epsilon\to 0} \int_{\theta(\tilde{\mathcal{Z}}_\epsilon\cap\pi_X^{-1}(\gamma))} \bigwedge^n \mathrm{dlog} z_i} \mod \mathbb{Z}(n),$$

which defines functionals in $H^{n-1}(X, \mathbb{C}/\mathbb{Z}(n)) := Hom\,(H_{n-1}(X, \mathbb{Z}), \mathbb{C}/\mathbb{Z}(n))$ on homology.

Now we ask in anticipation of later work: what equivalence relation on cycles $\mathcal{Z}$ can $[AJ(\mathcal{Z})]$ be expected to kill? As we've hinted there are two points of view here: relative cycles (modulo the equivalence relation of "relative rational equivalence") and higher Chow cycles (modulo $\mathrm{im}(\partial_{\mathcal{B}})$), and in fact these coincide in a sense we now make plain. As usual $\mathcal{X}$ denotes a quasiprojective variety.

First we say what relative rational equivalence is. The usual notion of rational equivalence on $Z^p(\mathcal{X},0) = Z^p(\mathcal{X})$ is given by taking for all $\mathcal{W} \in Z^p(\mathcal{X},1)$, $\partial_{\mathcal{B}}\mathcal{W} = \mathcal{W}\cap(\mathcal{X}\times\{0\}) - \mathcal{W}\cap(\mathcal{X}\times\{\infty\})$. Now suppose we replace $\mathcal{X}$ by a relative variety $(\mathcal{X},\mathcal{Y} = \cup D_i)$, restricting to the case where the $\{D_i\}$ are smooth with normal crossings. Then cycles in $Z^p(\mathcal{X},\mathcal{Y})$ must intersect D_i, $D_i \cap D_j$, etc. properly and have $\mathcal{Z}\cap D_i = 0$ *as a cycle* for all i. To produce the equivalence relation via $\partial_{\mathcal{B}}\mathcal{W}$, we require $\mathcal{W} \subset \mathcal{X}\times\square^1$ to meet once again all (intersections of) faces $\mathcal{X}\times\{0\}$, $\mathcal{X}\times\{\infty\}$, $D_i\times\square^1$ properly and cancel at the $D_i\times\square^1$. (Continuing in this manner one could easily define higher Chow groups for $(\mathcal{X},\mathcal{Y})$). More concretely, collections (V_i,f_i) produce rational equivalences on $\mathcal{X}$ by $\sum i_*^{V_i}(f_i)$; the same goes for $(\mathcal{X},\mathcal{Y})$ if $f_i \equiv 1$ on $V_i\cap\mathcal{Y}$ (proper intersection is unnecessary because $\{1\}\notin\square^1$), but note that these are not the only possibilities.

In particular, if $\mathcal{X} = X\times\square^n$ and $\mathcal{Y} = X\times\partial\square^n$, then the boundary map "coincides" with $\partial_{\mathcal{B}}$ on $X\times\square^{n+1}$: that is, the diagram

$$\begin{tikzcd}
{} \arrow[r] & Z^p(X,n+1) \arrow[r, "\partial_{\mathcal{B}}"] & Z^p(X,n) \\
{} \arrow[r] & Z^p((X\times\square^n, X\times\partial\square^n),1) \arrow[r, "\partial_{\mathcal{B}}"] \arrow[u, hook, "j"] & Z^p(X\times\square^n, X\times\partial\square^n) \arrow[u, hook, "j"']
\end{tikzcd}$$

commutes. In fact the resulting maps

$$j_* : CH^p(X\times\square^n, X\times\partial\square^n) \to CH^p(X,n)$$

are isomorphisms. Clearly j_* is surjective, since it hits all classes containing an alternating cycle (which is to say, all classes). To approach the injectivity issue, it's convenient to slightly modify the bottom complex to $Z^p(X\times\square^n, X\times\partial^+\square^n)$, where $\partial^+\square^n$is the union of all the faces but $z_n = 0$; this has $CH^p(X\times\square^n, X\times\partial\square^n)$ as its homology groups for all n. Then we try to invert j_* in the derived category by adding special cycles to cancel intersections of $\mathcal{Z}$ with all faces but $z_n = 0$. We can use degenerate chains

$$\pi_i^*(\mathcal{Z}\cap\rho_i^{\infty}{}_*\square^{n-1}) \in d^p(X,n)$$

to "move" intersections at $z_i = \infty$ to $z_i = 0$, then move them from $z_i = 0$ to $z_n = 0$ with chains of the form[7]

$$\{ \ (z_1 \ldots , z_n) \,\big|\, (z_1, \ldots , z_{i-1}, 0, z_{i+1}, \ldots , z_{n-1}, a) \in \mathcal{Z} \cap \rho^0_{i\,*} \square^{n-1}, \qquad (1 - z_i)(1 - z_n) = (1 - a) \ \}$$

Combining this with the more sophisticated point of view we will develop in §2.4 gives as a composite

$$\begin{tikzcd}
 & & Hom_{MH}(\mathbb{Q}(0), H^{2p-n}(X, \mathbb{Q}(p))) = W_p F^p H^{2p-n}(X, \mathbb{C}) \cap H^{2p-n}(X, \mathbb{Q}(p)) \\
CH^p(X \times \square^n, X \times \partial\square^n) \arrow[r, "j_*", "\cong"'] & CH^p(X, n) \arrow[ur, "{[\cdot]}"] \arrow[r, "\mathcal{R}"] \arrow[dr, "AJ(\text{if } [.]=0)"'] & H^{2p-n}_{\mathcal{H}}(X, \mathbb{Q}(p)) \arrow[u] \\
 & & Ext^1_{MH}(\mathbb{Q}(0), H^{2p-n-1}(X, \mathbb{Q}(p))) = \dfrac{W_p H^{2p-n-1}(X, \mathbb{C})}{F^p(\text{num}) + \text{num} \cap H^{2p-n-1}(X, \mathbb{Q}(p))} \arrow[u]
\end{tikzcd}$$

a generalized version of the maps defined in this section. It seems tempting to extend the Bloch-Beilinson conjecture to $\Psi \circ j_*$ (that is, if X is defined over $\mathbb{Q}$, this should be injective on the subgroup of relative cycles defined over $\bar{\mathbb{Q}}$).[8]

In the next section we will consider the case $X = \eta_X$, $n = p$, where

$$CH^n(\square^n_{\mathbb{C}(X)}, \partial\square^n_{\mathbb{C}(X)}) \cong CH^n(\eta_X, n) \cong K^M_n(\mathbb{C}(X))$$

maps (according to the above) to $H^n_{\mathcal{H}}(\eta_X, \mathbb{Q}(n))$ which is an extension of

$$Hom_{MH}\left(\mathbb{Q}(0), H^n(X, \mathbb{Q}(n))\right) = F^n H^n(\eta_X, \mathbb{C}) \cap H^n(\eta_X, \mathbb{Q}(n))$$

and

$$Ext^1_{MH}\left(\mathbb{Q}(0), H^{n-1}(X, \mathbb{Q}(n))\right) = H^{n-1}(\eta_X, \mathbb{C}/\mathbb{Q}(n))$$

[7] There are technical problems here (it's not as easy as it sounds) but one can get around them with a result of Levine **[Le]**. The point is that since this "back map" π is also a map of complexes, we should now have means of using either type of equivalence relation to construct the other.

[8] In lieu of defining absolute Hodge cohomology (see **[L2]** for this), we have indicated the short exact sequence in which it sits. It replaces Deligne cohomology when X itself is quasiprojective (a situation we will have some use for below).

Formally, one would expect the target space of a "Deligne-class" map for the relative Chow group, to be $H^{2p}_{\mathcal{H}}(X \times (\square^n, \partial\square^n), \mathbb{Q}(p))$ (if such an object is defined), and indeed by a completely formal computation with weights this comes out to $H^n(\square^n, \partial\square^n) \otimes H^{2p-n}_{\mathcal{H}}(X, \mathbb{Q}(p))$; since $H^n(\square^n, \partial\square^n)$ is a trivial Hodge structure (in particular its weight must be 0 to compensate for $H^n(\hat{\square}^n \setminus \partial\hat{\square}^n)$, which has weight $2n$), this actually coincides with the target space in the diagram.

(no difference with Deligne cohomology in this case) and the proposed relative Bloch-Beilinson conjecture reduces to injectivity of regulator maps on $K_n^M(\mathbb{Q}(X))$ (which brings us back to the world of more celebrated conjectures).

1.4. Abel-Jacobi for Graphs

1.4.1. Motivation and setup; f-substrata. We return to the simple case of a "function"

$$\mathbf{f} = \sum_j m_j f_{1j} \otimes \ldots \otimes f_{nj} \in \otimes^n \mathbb{Z}[\mathbb{P}^1_{\mathbb{C}(X)} \setminus \{0, \infty\}]$$

giving rise to the associated data

$$\gamma_{\mathbf{f}} \in Z^n(\eta_X \times \square^n),$$

$$[\gamma_{\mathbf{f}}] \in CH^n(\eta_X, n) \;\leftrightarrow\; \{\mathbf{f}\} = \prod \{f_{1j}, \ldots, f_{nj}\}^{m_j} \in K_n^M(\mathbb{C}(X)),$$

$$\Omega_{\mathbf{f}} = \wedge^n \mathrm{dlog}\mathbf{f} = \sum m_j \frac{df_{1j}}{f_{1j}} \wedge \ldots \wedge \frac{df_{nj}}{f_{nj}} \in H^0(\Omega^n_X(log)) = F^n H^n(\eta_X, \mathbb{C}).$$

For this section we make the assumption that either $n > d = \dim X$ or $\{\mathbf{f}\} \in \ker(\mathrm{Tame})$, in order that $\Omega_{\mathbf{f}} = 0$.

For $f = f_{ij}$, define

$$V_f = |(f)| = f^{-1}(\{0, \infty\}),\; T_f = f^{-1}(\mathbb{R}^-)$$

where the latter denotes the branch cut in $\log f$ (which by definition always denotes the branch with $\arg \in (-\pi, \pi]$). Since it bounds on V_f, it yields a class in

$$\varinjlim_{D \subset X} H_*(X, D; \mathbb{Z}) =: H_*(X_{rel}, \mathbb{Z})$$

(where one may think of X_{rel} as a sort of Lefschetz dual to η_X). Also

$$T_{\mathbf{f}} := \sum m_j T_{f_{1j}} \cap \ldots \cap T_{f_{nj}} = \pi_*^X \{\overline{\gamma_{\mathbf{f}}} \cap (X \times T_n)\}$$

(recall $T_n = \bigcap_{i=1}^n T_{z_i} \subset \hat{\square}^n$) gives a class

$$[T_{\mathbf{f}}] \in H_{2d-n}(X_{rel}, \mathbb{Z}).$$

It is convenient for our geometric computations here to use Steinbergs to move $\mathbf{f}$ into "good" position: this means

(1) moving the divisors so that $\overline{\gamma_{\mathbf{f}}} \in Z^n(X, n)$ (i.e., its complex codimension 1 intersections with the faces do not hit corners) and, special to this section,

(2) moving by certain Steinbergs $(fe^{i\xi} - f - e^{i\xi}) \otimes \ldots$ [i.e. replacing $f \mapsto fe^{i\xi} - e^{i\xi}$] which do *not* affect divisors, then taking a limit and ignoring $e^{i\xi}$. (We want T_f to hit only V_f, and not the other components of $|\mathbf{f}|$, in real codimension 1.)

Though strictly speaking one needs (2) for $T_{\mathbf{f}}$ to make sense as an intersection product, it will become clear in the next chapter that these assorted assumptions are in fact unnecessary. We now write down some symbols associated with the **f**-substrata of X and the system of neighborhoods associated to them.

NOTATION 1.4.1. Generalizing "$\partial\hat{\square}^n$" we define for the codimension r faces of $\hat{\square}^n$

$$\partial^r\hat{\square}^n := \bigcup_{\mathbf{i},\mathbf{e}} \rho_{\mathbf{i}\,*}^{\mathbf{e}}\hat{\square}^{n-r}$$

where $\mathbf{i}$ ranges over sets $\{(i_1,\ldots,i_k)\,|1 \leq i_1 < \ldots < i_k \leq n\}$; clearly $\partial^1\hat{\square}^n$ is just $\partial\hat{\square}^n$. We can pull these back to X to get the codimension r **f**-substrata:

$$V_{\mathbf{f}}^r := \mathbf{f}^{-1}(\partial^r\hat{\square}^n)$$

where for a set $\mathcal{S} \subset \hat{\square}^n$

$$\mathbf{f}^{-1}(\mathcal{S}) := \cup_j(id, f_{1j}, \ldots, f_{nj})^{-1}(X \times \mathcal{S});$$

we will write frequently

$$V = V_{\mathbf{f}}^1 = \cup|(f_{ij})|, \;\; W = V_{\mathbf{f}}^2.$$

Extending "$N_\epsilon(\partial\hat{\square}^n)$" in a different direction from that used above, let

$$N_\epsilon^r := N_\epsilon(\partial^r\hat{\square}^n) = \bigcup_{\mathbf{i},\mathbf{e}} \left\{(z_1,\ldots,z_n) \in \hat{\square}^n \,\middle|\, |z_{i_k}|^{p(e_k)} < \epsilon \;\forall\, k = 1,\ldots,r\right\};$$

N_ϵ^1 is just $N_\epsilon(\partial\hat{\square}^n)$, so that $\partial\overline{N_\epsilon^1} = \mathcal{T}_\epsilon^1$ and $\hat{\square}_\epsilon^n = \hat{\square}^n \setminus \overline{N_\epsilon^1}$. The desired "system of neighborhoods" is given by

$$N_\epsilon(V_{\mathbf{f}}^r) := \mathbf{f}^{-1}(N_\epsilon^r).$$

Sometimes it is convenient to work with only the generic parts of successive substrata: recalling $X_\epsilon = X \setminus N_\epsilon(V)$, we write

$$V_{\mathbf{f},\epsilon}^r = V_{\mathbf{f}}^r \setminus V_{\mathbf{f}}^r \cap N_\epsilon(V_{\mathbf{f}}^{r+1}),$$

(and thus V_ϵ, W_ϵ). We can take "neighborhoods" of these also, in the sense of

$$N_{\epsilon_1}(V_{\epsilon_2}) := N_{\epsilon_1}(V) \setminus N_{\epsilon_1}(V) \cap N_{\epsilon_2}(W)$$

and its obvious generalizations.

REMARK 1.4.2. The motivating idea for this section is this: we would like to apply the "homotopy" θ to $\overline{\gamma_{\mathbf{f}}}$, but we're in trouble if the $\mathrm{Res}^i(\gamma_{\mathbf{f}})$ are nontrivial. In that case $\overline{\gamma_{\mathbf{f}}}$ is not completable to a higher Chow cycle on X, let alone a relative cycle; applying Alt_n does not change the situation. So the conclusions of section 1.3, which assumed a compact base, do not apply directly. Even if the Res^i are trivial, the completion is ambiguous by

$$im\left\{CH^{n-1}(V,n) \to CH^n(X,n)\right\},$$

and so its AJ-image is ambiguous (see Chapter 2) by

$$im\left\{Gy:\; H_{\mathcal{D}}^{n-2}(V,\mathbb{Z}(n-1)) \to H_{\mathcal{D}}^n(X,\mathbb{Z}(n))\right\}.$$

Everything tells us to repeat the AJ-construction directly over the generic point (and we can actually compute the result; see eqn. 1.4.1). According to remark 1.3.5, in order to do this we should view $\gamma_{\mathbf{f}}$ as giving an homology class in

$$H_{2d}\left((X,\overline{N_\epsilon(V)})\times(\hat{\square}^n\setminus\partial\hat{\square}^n)\right)\cong H_{2d-n}(X,V)\otimes H_n(\hat{\square}^n\setminus\partial\hat{\square}^n)$$

$$\to H_{2d-n}(X_{rel})\otimes H_n(\hat{\square}^n\setminus\partial\hat{\square}^n)$$

and integrate $\int_{\gamma_{\mathbf{f}}}\bigwedge^n \mathrm{dlog} z_i\wedge\pi_X^*\alpha$ to determine this class. Here α is d-closed and compactly supported on $X\setminus\overline{N_\epsilon(V)}$, i.e. represents a class in

$$H_c^{2d-n}(X\setminus V)\cong H^{2d-n}(X,V)\to H^{2d-n}(X_{rel}).$$

The intuition for the AJ map will be similar. Note that under the assumption (of this section) that $\Omega_{\mathbf{f}}=0$, the above integral (and therefore the cycle-class from this point of view) is trivially zero.

1.4.2. An integral formula for $\mathbf{AJ}(\gamma_{\mathbf{f}})$. Now set $\gamma_{\mathbf{f}}^\epsilon:=\gamma_{\mathbf{f}}\cap(X\times\overline{\hat{\square}_\epsilon^n})$, so that $\partial\gamma_{\mathbf{f}}^\epsilon\subset X\times\mathcal{T}_\epsilon^1$. As before we have

$$\theta(\tilde{\gamma_{\mathbf{f}}^\epsilon}):=\theta_1(\tilde{\gamma_{\mathbf{f}}^\epsilon})-\theta_{12}(\tilde{\gamma_{\mathbf{f}}^\epsilon}\cap\tilde{T}_{z_1})+\ldots\pm\theta_{1\ldots n}(\tilde{\gamma_{\mathbf{f}}^\epsilon}\cap\tilde{T}_{z_1}\cap\ldots\cap\tilde{T}_{z_{n-1}});$$

the only difference is the extra boundary term:

$$\partial\theta(\tilde{\gamma_{\mathbf{f}}^\epsilon})=\gamma_{\mathbf{f}}^\epsilon\pm\sum_{\mathbf{s}}(-1)^{\mathbf{s}}\theta_{1\ldots n}^{\mathbf{s}}(\gamma_{\mathbf{f}}^\epsilon\cap T_n)+\theta(\widetilde{\partial\gamma_{\mathbf{f}}^\epsilon})$$

where the middle term is supported over $T_{\mathbf{f}}$, and may be replaced by $T_{\mathbf{f}}\times(S^1)^n$ for purposes of integration as before. The last term is supported over $\mathbf{f}^{-1}(\mathcal{T}_\epsilon^1)\subset\overline{N_\epsilon(V)}$ (note that in general $\mathbf{f}^{-1}(\mathcal{T}_\epsilon^1)\neq\partial\overline{N_\epsilon(V)}$).

First we get some abstract nonsense out of the way. For $\alpha\in\Omega^{2d-n}_{X^\infty,c}(X\setminus V)$ d-closed and compactly supported on [a closed subset of] $X\setminus\overline{N_\epsilon(V)}$, we have that

$$0=\int_{\theta(\tilde{\gamma_{\mathbf{f}}^\epsilon})}\wedge^n\mathrm{dlog}z_i\wedge\mathrm{d}\pi_X^*\alpha=\int_{\partial\theta(\tilde{\gamma_{\mathbf{f}}^\epsilon})}\wedge^n\mathrm{dlog}z_i\wedge\pi_X^*\alpha$$

$$=\int_{\mathcal{X}^\epsilon}\wedge^n\mathrm{dlog}\mathbf{f}\wedge\alpha+\int_{\theta(\widetilde{\partial\gamma_{\mathbf{f}}^\epsilon})}\wedge^n\mathrm{dlog}z_i\wedge\pi_X^*\alpha+\int_{T_{\mathbf{f}}\times(S^1)^n}\wedge^n\mathrm{dlog}z_i\wedge\pi_X^*\alpha$$

$$=(2\pi\sqrt{-1})^n\int_{T_{\mathbf{f}}}\alpha$$

(where $\bigwedge^n\mathrm{dlog}\mathbf{f}$ in the first term and $\pi_X^*\alpha$ in the second term are zero), so that as before $\Omega_{\mathbf{f}}=0\implies[T_{\mathbf{f}}]\sim 0$. This means we may choose a relative bounding chain $\partial^{-1}_{(X,V)}(T_{\mathbf{f}})$ ambiguous up to a $(2d-n+1)$-cycle on (X,V), so that ($\mathcal{B}$ a "trivial" chain exactly as before uniformizing the loops generated by θ),

$$\partial_\epsilon^{-1}\gamma_{\mathbf{f}} := \theta(\tilde{\gamma_{\mathbf{f}}^\epsilon}) \mp \partial_{(X,V)}^{-1}(T_{\mathbf{f}}) \times (S^1)^n - \mathcal{B}$$

has boundary

$$\partial(\partial_\epsilon^{-1}\gamma_{\mathbf{f}}) = \gamma_{\mathbf{f}}^\epsilon + \theta(\widetilde{\partial\gamma_{\mathbf{f}}^\epsilon}) \mp \{V \cap \partial_X \partial_{(X,V)}^{-1}(T_{\mathbf{f}})\} \times (S^1)^n$$

plus trivial "residues" from $\mathcal{B}$. The latter two terms above are supported on $\overline{N_\epsilon(V)}$ and constitute *nontrivial* residues: they don't count topologically as far as $(X, \overline{N_\epsilon(V)}) \times (\hat{\square}^n \setminus \partial\hat{\square}^n)$ is concerned, but will matter immensely for our computations.

It makes sense at this point to define the AJ image of $\gamma_{\mathbf{f}}$ as a functional on forms $\omega \in \Omega^{2d-n+1}_{X^\infty,c}(X \setminus V)$ supported on a compact subset of $X \setminus N_\epsilon(V)$, via the integral

$$\int_{\partial_\epsilon^{-1}\gamma_{\mathbf{f}}} \wedge^n \mathrm{dlog} z_i \wedge \pi_X^* \omega.$$

It follows immediately from the foregoing that if $\omega = \mathrm{d}\alpha$ (α also c.s. away from $\overline{N_\epsilon(V)}$) then the integral vanishes. Therefore it gives a cohomology class in $H^{n-1}(X-V) \leftarrow H^{n-1}(\eta_X, \mathbb{C})$ by duality. Factoring in the ambiguity of $H_{2d-n+1}(X, V; \mathbb{Z})$ generated in choosing $\partial_{(X,V)}^{-1}(T_{\mathbf{f}})$, we have defined

$$[AJ_X(\gamma_{\mathbf{f}})] \in H^{n-1}(\eta_X, \mathbb{C}/\mathbb{Z}(n)).$$

We will show AJ kills "trivial graphs", i.e. descends to a map on $CH^n(\eta_X, n) \cong K_n^M(\mathbb{C}(X))$, in the next chapter.

Toward that end we simplify the expression, by computing $\int_{\theta(\tilde{\gamma_{\mathbf{f}}^\epsilon})} \bigwedge^n \mathrm{dlog} z_i \wedge \pi_X^* \omega$ for ω as above. The idea is that since, e.g., $\theta_1(\tilde{\gamma_{\mathbf{f}}^\epsilon})$ "pulls the graph" to $z_1 = 1$, we have

$$\begin{aligned}\int_{\theta_1(\tilde{\gamma_{\mathbf{f}}^\epsilon})} \wedge^n \mathrm{dlog} z_i \wedge \pi_X^*\omega \;&=\; \int_{\gamma_{\mathbf{f}}^\epsilon} \log z_1 \mathrm{dlog} z_2 \wedge \ldots \wedge \mathrm{dlog} z_n \wedge \pi_X^* \omega \\ &= \sum m_j \int_X \log f_{1j} \mathrm{dlog} f_{2j} \wedge \ldots \wedge \mathrm{dlog} f_{nj} \wedge \omega,\end{aligned}$$

and more generally

$$\int_{\theta_{1\ldots k}(\tilde{\gamma_{\mathbf{f}}^\epsilon} \cap \tilde{T}_{z_1} \cap \ldots \cap \tilde{T}_{z_{k-1}})} \wedge^n \mathrm{dlog} z_i \wedge \pi_X^* \omega \;=\; \int_{(\tilde{\gamma_{\mathbf{f}}^\epsilon} \cap \tilde{T}_{z_1} \cap \ldots \cap \tilde{T}_{z_{k-1}}) \times [0,1]^k} \theta_{1\ldots k}^* (\wedge^n \mathrm{dlog} z_i \wedge \pi_X^* \omega)$$

$$= \int_{(\tilde{\gamma_{\mathbf{f}}^\epsilon} \cap \tilde{T}_{z_1} \cap \ldots \cap \tilde{T}_{z_{k-1}}) \times [0,1]^k} \theta^*(\mathrm{dlog} z_1) \cdot \ldots \cdot \theta^*(\mathrm{dlog} z_k) \mathrm{dlog} z_{k+1} \wedge \ldots \wedge \mathrm{dlog} z_n \wedge \pi_X^* \omega$$

$$= (2\pi\sqrt{-1})^{k-1} \int_{\gamma_{\mathbf{f}}^\epsilon \cap T_{z_1} \cap \ldots \cap T_{z_{k-1}}} \log z_k \mathrm{dlog} z_{k+1} \wedge \ldots \wedge \mathrm{dlog} z_n \wedge \pi_X^* \omega$$

$$= (2\pi\sqrt{-1})^{k-1} \sum_j m_j \int_{T_{f_{1j}} \cap \ldots T_{f_{(k-1)j}}} \log f_{kj} \mathrm{dlog} f_{(k+1)j} \wedge \ldots \wedge \mathrm{dlog} f_{nj} \wedge \omega,$$

where $\log f_{kj}$ is understood as having argument $\in (-\pi, \pi]$. Therefore the whole integral $[AJ(\gamma_{\mathbf{f}})](\omega) =$

$$\int_{\partial_\epsilon^{-1}\gamma_{\mathbf{f}}} \wedge^n \mathrm{dlog} z_i \wedge \pi_X^* \omega \ = \sum m_j \left\{ \int_X \log f_{1j} \mathrm{dlog} f_{2j} \wedge \ldots \wedge \mathrm{dlog} f_{nj} \wedge \omega \right.$$

$$- 2\pi\sqrt{-1} \int_{T_{f_{1j}}} \log f_{2j} \mathrm{dlog} f_{3j} \wedge \ldots \wedge \mathrm{dlog} f_{nj} \wedge \omega \ + \ldots$$

$$\left. \pm (2\pi\sqrt{-1})^{n-1} \int_{T_{f_{1j}} \cap \ldots \cap T_{f_{(n-1)j}}} (\log f_{nj})\omega \right\} \mp (2\pi\sqrt{-1})^n \int_{\partial^{-1}_{(X,V)}(T_{\mathbf{f}})} \omega$$

$$=: \int_X R_{\mathbf{f}} \wedge \omega \ \ \mp (2\pi\sqrt{-1})^n \int_{\partial^{-1}(T_{\mathbf{f}})} \omega =: \int_X R'_{\mathbf{f}} \wedge \omega. \tag{1.4.1}$$

1.4.3. Codimension 1 residues. We describe the residues first in the case $n = 2$, $d = 1$ (X a smooth curve), $\mathbf{f} = f \otimes g$. We assume that this $\mathbf{f}$ is already in "good" position, with the intersections $|(f)| \cap |(g)|$, $T_f \cap |(g)|$, $T_g \cap |(f)|$ all empty. Let ω be a d-closed 1-form which is zero on the closure of $N_\epsilon(|(f)| \cup |(g)|)$. Then

$$\int_X R_{\mathbf{f}} \wedge \omega = \int_X \log f \mathrm{dlog} g \wedge \omega - 2\pi\sqrt{-1} \int_{T_f} (\log g)\omega$$

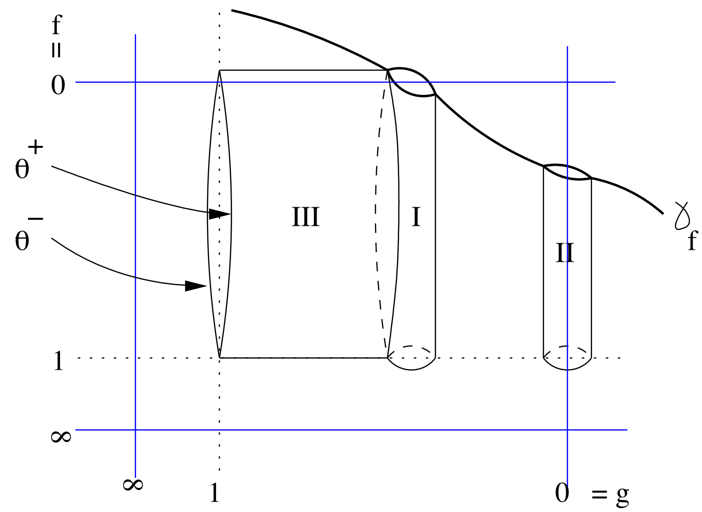
and $\partial^{-1}_{(X,V)} T_{\mathbf{f}}$ simply consists of paths connecting the points $T_f \cap T_g$ to $V = |(f)| \cup |(g)|$. Write $\Xi = V \cap \partial_X \partial^{-1}_{(X,V)} T_{\mathbf{f}}$ for the function from $V \to \mathbb{Z}$ describing the sum of their boundaries on V (this is a sort of relative homology connecting homomorphism for singular chains). Now if we let S_f^ϵ and S_g^ϵ consist of circles (with appropriate orientation) around $|(f)|$ and $|(g)|$, then $\mathbf{f}^{-1}(\mathcal{T}_\epsilon^1)$ is simply their union. Writing $\theta(z)$ for the path from z to 1, we have

$$\theta(\widetilde{\partial \gamma_{\mathbf{f}}^\epsilon}) \ = \ \theta_1(\widetilde{\partial \gamma_{\mathbf{f}}^\epsilon}) - \theta_{12}(\widetilde{\partial \gamma_{\mathbf{f}}^\epsilon} \cap \tilde{T}_{z_1})$$

$$= \left\{ x \in S_f^\epsilon \,\middle|\, (x, \theta(\widetilde{f(x)}), g(x)) \right\} + \left\{ x \in S_g^\epsilon \,\middle|\, (x, \theta(\widetilde{f(x)}), g(x)) \right\}$$

$$- \left\{ x \in S_f^\epsilon \cap T_f \,\middle|\, (x, \theta(\widetilde{f(x)}), \theta(\widetilde{g(x)})) \right\}$$

where $x \in S_f^\epsilon \cap T_f$ is the same as $f(x) = -\epsilon$ (or $-\frac{1}{\epsilon}$ for a pole), and so in the 3^{rd} term (only) we may rewrite $\theta(\widetilde{f(x)}) = \theta^+(-\epsilon) - \theta^-(-\epsilon) \sim S^1$. In a picture, these three terms look like:



To compute residues, let β be a class in $H^0(V)$, i.e. a complex number on each point of $|(f)| \cup |(g)|$. (In particular, one might choose $\beta = 1$ on one point and $\beta = 0$ on the rest.) If we take $\epsilon_1 > \epsilon$ and extend this to a C^∞ function $\tilde{\beta}$ on X, supported on $\overline{N_{2\epsilon_1}(V)}$ and with $\mathrm{d}\tilde{\beta} = 0$ on $\overline{N_{\epsilon_1}(V)}$, then $\mathrm{d}\tilde{\beta}$ is c.s.[=compactly supported] away from $\overline{N_\epsilon(V)}$ and gives a class in $H^1(X, V)$. (This is just a picky realization of the connecting homomorphism $\Delta : H^0(V) \to H^1(X, V)$.) Then $[AJ_X(\gamma_{\mathbf{f}})](\mathrm{d}\tilde{\beta}) =$

$$\int_X R'_{\mathbf{f}} \wedge \mathrm{d}\tilde{\beta} := \int_{\partial_\epsilon^{-1}\gamma_{\mathbf{f}}} \mathrm{dlog} z_1 \wedge \mathrm{dlog} z_2 \wedge \mathrm{d}\pi_X^* \tilde{\beta}$$

$$= \left[\int_{\gamma_{\mathbf{f}}^\epsilon} + \int_{\theta(\widetilde{\partial\gamma_{\mathbf{f}}^\epsilon})} + \int_{(S^1)^2 \times \Xi} \right] (\mathrm{dlog} z_1 \wedge \mathrm{dlog} z_2) \cdot \pi_X^* \tilde{\beta}$$

$$= \int_{X^\epsilon} (\mathrm{dlog} f \wedge \mathrm{dlog} g) \cdot \tilde{\beta} + \left\{ \int_{S_f^\epsilon} \tilde{\beta} \log f \mathrm{dlog} g + \int_{S_g^\epsilon} \tilde{\beta} \log f \mathrm{dlog} g \right.$$

$$\left. -2\pi\sqrt{-1} \sum_{x \in S_f^\epsilon \cap T_f} \tilde{\beta}(x) \log g(x) \right\} - 4\pi^2 \sum_{\Xi} \beta,$$

where the first term is 0 by Hodge type. Since $\log f$ jumps by $2\pi\sqrt{-1}$ at each $x \in T_f \cap S_f^\epsilon$, the 1^{st} and 3^{rd} terms in braces (corresponding to the "tubes" above) get combined into $-\int_{S_f^\epsilon} \tilde{\beta} \log g \mathrm{dlog} f =: (*)$ using $\mathrm{d}\tilde{\beta} = 0$ on $N_\epsilon(V)$ and the integration by parts:

$$2\pi\sqrt{-1} \sum_{x \in S_f^\epsilon \cap T_f} \log g(x) = \int_{S_f^\epsilon \setminus S_f^\epsilon \cap T_f} \mathrm{d}\{\log f \log g\} = \int_{S_f^\epsilon} \log f \mathrm{dlog} g + \int_{S_f^\epsilon} \log g \mathrm{dlog} f.$$

An alternate point of view here is that taking $\epsilon \to 0$ forces the 1^{st} term in braces to 0 as $\epsilon \log \epsilon$, i.e. the 1^{st} tube does not contribute to the $\int$ in the limit. Since according to the expression (1.4.1) for $\int_X R'_{\mathbf{f}} \wedge \mathrm{d}\tilde{\beta}$, shrinking ϵ cannot affect the total answer, the 3^{rd} term must then approach the original difference $(*)$. This conclusion is obvious here (for $n = 2$), but for larger n it's useful to be able to just throw away some terms and take a limit. Either way we have as the result

$$(2\pi\sqrt{-1}) \sum_{p \in V} \beta(p) \cdot \left\{ \nu_p(g) \log f(p) - \nu_p(f) \log g(p) + (2\pi\sqrt{-1})\Xi(p) \right\}$$

which is precisely $2\pi\sqrt{-1}$ times an AJ-map where the data $(d\tilde{\beta}, X, \mathbf{f})_{n=2}$ is replaced by $(\beta, V, \partial(\mathbf{f}))_{n=1}$ and

$$\partial_p(\mathbf{f}) = \begin{cases} \nu_p(g) \cdot f(p), & p \in |(g)| \\ -\nu_p(f) \cdot g(p), & p \in |(f)| \end{cases} \quad \in \mathbb{Z}[\mathbb{P}^1_{\mathbb{C}(X)} \setminus \{0, \infty\}]$$

(on good $\mathbf{f}$) induces the tame symbol on Milnor K-theory. Note that the third term $(= 2\pi\sqrt{-1}\,\Xi)$ in the braces may be ignored since it corresponds to the "integral" functionals we work modulo in

$$H^0(V, \mathbb{C}/\mathbb{Z}(1)) :\cong \frac{H^0(V, \mathbb{C})^\vee}{im\{H_0(V, \mathbb{Z}(1))\}} \ni [AJ_V(\gamma_{\partial(\mathbf{f})})](\beta).$$

Note that if $\mathrm{Tame}\{\mathbf{f}\} \in \coprod_{p \in X} \mathbb{C}^*$ is trivial, so that the image of $\partial(\mathbf{f})$ in the quotient (e.g., $\nu_p(g) \cdot f(p)$ becomes $f(p)^{\nu_p(g)}$) is 1 for all p, then the first two terms in braces (and therefore the residues) are also trivial in this sense.

In fact one can write geometrically

$$\theta(\widetilde{\partial\gamma^\epsilon_{\mathbf{f}}}) = \sum_{x \in |(g)|} S^1_{z_2} \times \theta_{z_1}(\tilde{\gamma}_{\partial_x(\mathbf{f})}) - \sum_{x \in |(f)|} S^1_{z_1} \times \theta_{z_2}(\tilde{\gamma}_{\partial_x(\mathbf{f})})$$

plus terms that don't contribute to the integral $(\epsilon \to 0)$, reflecting the equality

$$[AJ_X(\gamma_{\mathbf{f}})](\mathrm{d}\tilde{\beta}) = 2\pi\sqrt{-1} \sum_{x \in [V \subset] X^1} [AJ_x(\gamma_{\partial_x \mathbf{f}})](\beta).$$

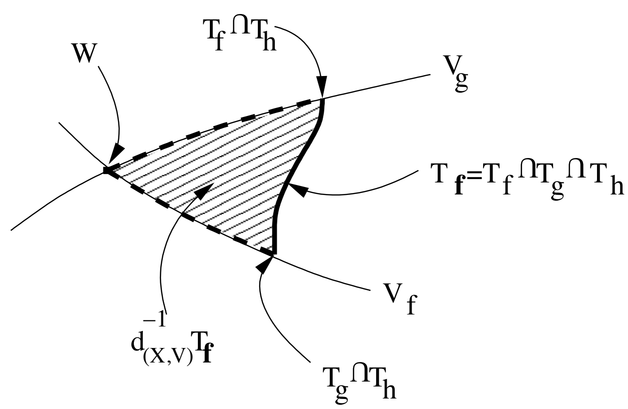
For larger n and d we can prove a generalization of this working modulo a large enough neighborhood of the codimension-2 substrata. For simplicity assume $\mathbf{f} = f_1 \otimes \ldots \otimes f_n$ so that (writing $V_i \subset X^1$ for the set of components of V_{f_i}) one may take simply

$$\partial_i(\mathbf{f}) := \sum_{x \in V_i} (-1)^i \nu_x(f_i) \cdot f_1 |_x \otimes \ldots \otimes \widehat{f_i} \otimes \ldots \otimes f_n |_x .$$

Also set $\epsilon_2 \gg 2\epsilon_1 \gg \epsilon > 0$ (all understood to be very small!); we shall work modulo $N_{\epsilon_2}(W)[\times\hat{\square}^n$ where appropriate].[9] Then in this sense (ignoring codimension 2) the following beautiful formula is true even when $\Omega_{\mathbf{f}}$ and $T_{\mathbf{f}}$ are nontrivial:

$$\partial\theta(\tilde{\gamma_{\mathbf{f}}^{\epsilon}}) = \gamma_{\mathbf{f}}^{\epsilon} \pm (S^1)^n \times T_{\mathbf{f}} + \sum_i S^1_{z_i} \times \theta(\tilde{\gamma}^{\epsilon}_{\partial_i \mathbf{f}}) + \ldots \tag{1.4.2}$$

(where the $\ldots$ accounts for trivial terms from uniformizing loops and terms whose contributions to the $\int \to 0$ as $\epsilon \log \epsilon$). This says that the codimension 1 residues are *always* the $(n-1)$-AJ maps on the [components of][10] V_{f_i}. Moreover in case $\Omega_{\mathbf{f}} = 0$ one may view $\{\partial_X(\partial^{-1}_{(X,V)} T_{\mathbf{f}}) - T_{\mathbf{f}}\} \subset V$ as providing bounding chains on (V_{f_i}, W) for $T_{\partial_i \mathbf{f}} \subset V_{f_i}$ since $\sum T_{\partial_i \mathbf{f}} = \partial_X T_{\mathbf{f}}$. In a picture, for $n = 3$ and X a complex surface, $\mathbf{f} = f \otimes g \otimes h$ in "good" position:



Therefore we have over $X \setminus \overline{N_{\epsilon_2}(W)}$

$$\partial(\partial_\epsilon^{-1}\gamma_{\mathbf{f}}) = \gamma_{\mathbf{f}}^{\epsilon} + \sum_i S^1_{z_i} \times (\partial_\epsilon^{-1}\gamma_{\partial_i \mathbf{f}}) + \ldots$$

and the construction is "telescopic" in codimension 1.

We briefly indicate the "computation" for (1.4.2): generically on the i^{th} face, writing $\partial_i \gamma_{\mathbf{f}}^{\epsilon} = \gamma_{\mathbf{f}} \cap \mathcal{T}^1_\epsilon \begin{pmatrix} 0, \infty \\ i \end{pmatrix}$, we have that in $\theta(\widetilde{\partial_i \gamma_{\mathbf{f}}^{\epsilon}})$ the term $\theta_{12\ldots i}(\widetilde{\partial_i \gamma_{\mathbf{f}}^{\epsilon}} \cap \tilde{T}_{z_1} \cap \ldots \cap \tilde{T}_{z_{i-1}})$ gives rise to one of our $\epsilon \log \epsilon$ integrals while

$$\sum_{k=1,\ldots,\hat{i},\ldots,n} \pm\theta_{12\ldots k}(\widetilde{\partial_i \gamma_{\mathbf{f}}} \cap \tilde{T}_{z_1} \cap \ldots \cap \tilde{T}_{z_{k-1}}) \to S^1_{z_i} \times \theta(\widetilde{\gamma^{\epsilon}_{\partial_i \mathbf{f}}})$$

plus trivial stuff as $\epsilon \to 0$.

[9] in this we also tacitly include neighborhoods of the remaining intersections of components of V.

[10] we shall sometimes write formulas as if V_{f_i} has only one component to avoid bizarre notation.

1.4.4. Local-global picture, higher codimension residues. Now suppose $\omega = \mathrm{d}\tilde{\beta} + \mathrm{d}\alpha$, with β c.s. (and d-closed) on V_{ϵ_2}, $\tilde{\beta}$ c.s. on $N_{2\epsilon_1}(V_{\epsilon_2})$ such that $\mathrm{d}\tilde{\beta} = 0$ on $N_{\epsilon_1}(V_{\epsilon_2})$ and $\iota_V^*\tilde{\beta} = \beta$, and α (& of course ω) c.s. on X_ϵ. By our computation a few pages ago $\int_X R'_{\mathbf{f}} \wedge \mathrm{d}\alpha = 0$, so assuming as usual $\wedge^n \mathrm{dlog}\mathbf{f} = 0$ and using (1.4.2),

$$\int_X R'_{\mathbf{f}} \wedge \omega = \int_{\theta(\tilde{\gamma}^\epsilon_{\mathbf{f}})} \wedge^n \mathrm{dlog} z_i \wedge \mathrm{d}\pi_X^* \tilde{\beta} \mp (2\pi\sqrt{-1})^n \int_{\partial^{-1}_{(X,V)}(T_{\mathbf{f}})} \mathrm{d}\tilde{\beta}$$

$$= \int_{\partial\theta(\tilde{\gamma}^\epsilon_{\mathbf{f}})} \wedge^n \mathrm{dlog} z_i \wedge \pi_X^* \tilde{\beta} \mp (2\pi\sqrt{-1})^n \int_{\partial_X \partial^{-1}_{(X,V)}(T_{\mathbf{f}})} \tilde{\beta}$$

$$= \left\{ \int_{X^\epsilon} \wedge^n \mathrm{dlog}\mathbf{f} \wedge \pi_X^* \tilde{\beta} \pm (2\pi\sqrt{-1})^n \int_{T_{\mathbf{f}}} \tilde{\beta} - 2\pi\sqrt{-1} \sum_i \int_{\theta(\tilde{\gamma}^\epsilon_{\partial_i \mathbf{f}})} \wedge^{n-1} \mathrm{dlog} z_i \wedge \pi_X^* \beta \right\}$$

$$\mp (2\pi\sqrt{-1})^n \left\{ \int_{T_{\mathbf{f}}} \tilde{\beta} \mp \sum_i \int_{\partial^{-1}_{(V_i,W)}(T_{\partial_i \mathbf{f}})} \beta \right\}$$

$$= -2\pi\sqrt{-1} \sum_i \int_{V_i} R'_{\partial_i \mathbf{f}} \wedge \beta_{V_i}.$$

In the context of Chapter 2, thinking of R' as a current, this will be immediate. It will also be apparent later on that $\int R'_{\mathrm{Tame}_i \mathbf{f}} \wedge (\cdot)$ is trivial as an element of $H^{n-2}(V_i - W; \mathbb{C}/\mathbb{Z}(n-1))$ if $\mathrm{Tame}_i \mathbf{f}$ is trivial as an element of $K^M_{n-1}(\mathbb{C}(V_i))$.

The c.s. forms ω we consider span

$$\mathrm{im}\left(H^{2d-n+1}(X,V) \to H^{2d-n+1}(X) \right);$$

we would like to know when $\int_X R'_{\mathbf{f}} \wedge (\cdot)$ is a well-defined functional (modulo $\mathbb{Z}(n)$-functionals) on this subspace of $H^{2d-n+1}(X)$. For example, is it enough to check that it is "trivial" on $\mathrm{d}\tilde{\beta} + \mathrm{d}\alpha$ in the sense described above – can all boundaries be written as such?[11] The answer is *no* for $n \geq 4$ (but yes for $n = 3$, e.g. on a complex surface), as we can see with the aid of a local-global spectral sequence for $H^*(X)$ which uses c.s. forms on substrata of X (of all codimensions). We need to change the $\mathbf{f}$-substrata $V^i_{\mathbf{f}}$ slightly to include all intersections (and self-intersections) of all components of V in V^2, all $\cap$'s of components of V^2 in V^3, and so on; denote this version by $'V^i_{\mathbf{f}}$ (of course $V = V^1_{\mathbf{f}} =' V^1_{\mathbf{f}}$). They are needed to get the spectral sequence right but not

[11] Indeed, this would seem to be only part of the problem: as "trivial" in this sense only means that the map $H^*(X,V)^\vee \to H^*(V)^\vee$ takes $\int R'_{\mathbf{f}} \wedge (\cdot)$ to the image of $H_*(V,\mathbb{Z}) \subset H^*(V)^\vee$, i.e. "the functional has integral boundary on V" (and so is not *a priori* a well-defined functional on $H^*(X)$). However one can just normalize the functional by modifying $\partial^{-1}_{(X,V)}(T_{\mathbf{f}})$ by a cycle on (X,V) killing this boundary, and so this is not an issue. The real issue is still that we've only examined the residue (or "boundary") of the functional in $H^*(V_i,W)^\vee$, and this isn't in general enough.

for the (higher codimension) residues. A reference for some of what follows is the last section of [**F**].

Set ${}'V_{\mathbf{f}}^0 = X$ and let[12] $\iota_{(i)} : {}'V_{\mathbf{f}}^{(i)} \hookrightarrow X$,

$$\jmath^{(i)} : X -' V_{\mathbf{f}}^i \hookrightarrow X \text{ and } \jmath^{(i-1,i)} :' V_{\mathbf{f}}^{i-1} -' V_{\mathbf{f}}^i \hookrightarrow X$$

be the inclusions; we have the exact sequence of extension-by-zero sheaves

$$\begin{array}{ccccccc} 0 \to \jmath_!^{(i-1)}\mathbb{C} \to \jmath_!^{(i)}\mathbb{C} \to & \jmath_!^{(i-1,i)}\mathbb{C} & \to 0. \\ & \iota_{(i)}^* \downarrow\cong & \\ & \jmath_!^{(i,i+1)}\mathbb{C} & \end{array}$$

We take resolutions of each term by acyclic sheaves, e.g.

$$\begin{array}{cc} \jmath_!^{(i)}\mathbb{C} \to \jmath_!^{(i)}\Omega^\bullet_{(X\setminus' V_{\mathbf{f}}^i)^\infty}, \ \jmath_!^{(i+1)}\mathbb{C}/\jmath_!^{(i)}\mathbb{C} \to & \jmath_!^{(i+1)}\Omega^\bullet_{(X\setminus' V_{\mathbf{f}}^{i+1})^\infty} \Big/ \jmath_!^{(i)}\Omega^\bullet_{(X\setminus' V_{\mathbf{f}}^i)^\infty} \\ & \iota_{(i)}^* \downarrow\simeq \\ & \jmath_!^{(i,i+1)}\Omega^\bullet_{('V_{\mathbf{f}}^i\setminus V_{\mathbf{f}}^{i+1})^\infty} \end{array}$$

obtaining localization long-exact sequences for relative cohomology

$$\to H_c^*(X -' V_{\mathbf{f}}^i) \to H_c^*(X -' V_{\mathbf{f}}^{i+1}) \to H_c^*('V_{\mathbf{f}}^i -' V_{\mathbf{f}}^{i+1}) \to$$

which become our initial exact triangles. Define a descending filtration ($F_p^N \supseteq F_{p+1}^N$) on the sheaf $\mathbb{C}$ (on X) by

$$F_p^N\mathbb{C} = \jmath_!^{(d-p+1)}\mathbb{C}, \ \ Gr_p^N\mathbb{C} = \jmath_!^{(d-p,d-p+1)}\mathbb{C}$$

so that the associated spectral sequence

$$E_1^{p,q} = H^{p+q}(X, Gr_p^N\mathbb{C}) \cong H_c^{p+q}('V_{\mathbf{f}}^{d-p} -' V_{\mathbf{f}}^{d-p+1}) \implies E_\infty^{p,q} = Gr_p^N H^{p+q}(X)$$

gives a filtration on cohomology by niveau.

The right-most column is

$$E_1^{d,q} \cong H_c^{d+q}(X - V) \cong H^{d+q}(X, V);$$

if we replace it by zero then we have

$$'E_1^{p,q}(V) := \begin{cases} E_1^{p,q} & p \neq d \\ 0 & p = d \end{cases} \implies {}'E_\infty^{p,q}(V) = Gr_p^N H^{p+q}(V),$$

and in fact this is a good definition for the cohomology of V (it inherits a MHS, etc.). An easy algebraic argument shows that these fit together in the localization exact sequence

$$\to H^{*-1}(V) \xrightarrow{\Delta} H_c^*(X - V) \xrightarrow{\jmath_*^{(1)}} H^*(X) \to H^*(V) \to$$

where $\jmath_*^{(1)}$ is given by including the right-hand column into the spectral sequence. So for a functional *a priori* defined on $H_c^{2d-n+1}(X - V)$ to be

[12]roughly, in what follows $\jmath$ is for inclusion of Zariski open sets, ι is for inclusion of (open or closed) analytic subsets of lower dimension, the index i is for codimension and p for dimension.

well-defined on $\mathrm{im}\,\{H_c^*(X-V)\to H^*(X)\}$, it must kill $\mathrm{im}(\Delta)$, which is to say the image of $[Gr^N_{d-i}H^*(V)$ by$]$ successive d_i in the right-hand column. However, $\mathrm{d}\alpha$ and $\mathrm{d}\tilde{\beta}$ only represent the images of d_0 and d_1, respectively. Since

$$\mathrm{im}(\mathrm{d}_i)\subset \frac{H_c^{2d-n+1}(X-V)}{\cup_{k<i}\mathrm{im}(\mathrm{d}_k)}\cong E_i^{d,d-n+1}$$

comes from

$$E_i^{d-i,d-n+i} = \text{subquotient of } E_1^{d-i,d-n+i}\cong H_c^{2d-n}('V_{\mathbf{f}}^i -' V_{\mathbf{f}}^{i+1}),$$

it can be nontrivial exactly for $2d-n\le \dim('V_{\mathbf{f}}^i)=2(d-i)$, or $i\le \frac{n}{2}$. As we shall see, this corresponds to having to check that "higher (i^{th}) residues of $R'_{\mathbf{f}}$" vanish for $i\le\frac{n}{2}$, and that these can be seen as arising from the Res^i of $\gamma_{\mathbf{f}}$, which (conjecturally with Beilinson-Soulé) become trivial at exactly the same i (see 1.2.10).

For $n=4$, and say $d=3$ (X a threefold) already we must consider $\mathrm{im}(\mathrm{d}_2)$: using the above "exact triangles" and writing $'W(\supseteq W)$ for $'V_{\mathbf{f}}^2$, $'Y$ for the point set $'V_{\mathbf{f}}^3$,

$$\begin{tikzcd}
 & {} \arrow[r] & H_c^2('W -' Y) \arrow[r, "\delta"] \arrow[d, bend right, "\mathrm{d}_2"'] \arrow[drr, bend left, "\mathrm{d}_1"] & H_c^3(X -' W) \arrow[r] \arrow[d, equal] & H_c^3(X -' Y) \arrow[r] & {} \\
{} \arrow[r] & H_c^2(V -' W) \arrow[r, "\delta", "\mathrm{d}_1"'] & H_c^3(X - V) \arrow[r] & H_c^3(X -' W) \arrow[r] & H_c^3(V -' W) \arrow[r, "\delta"] & {}
\end{tikzcd}$$

where d_2 is defined on $\ker(\mathrm{d}_1)$ and we may describe these maps as follows in terms of our system of neighborhoods. The bottom δ says: if β is d-closed and c.s. on V_{ϵ_2} then there is β supported on $\overline{N_{2\epsilon_1}(V_{\epsilon_2})}\subset X$ extending β ($\iota_V^*\tilde{\beta}=\beta$) such that $\mathrm{d}\tilde{\beta}=0$ on $\overline{N_{\epsilon_1}(V_{\epsilon_2})}$, exactly as before. So $\mathrm{d}\tilde{\beta}$ on X gives a form c.s. on X^ϵ (and so on $X-V$).

The top δ says similarly: if η is d-closed and c.s. on $W-p$ (we don't use all of $'W$), then we may "extend" it to $\tilde{\eta}$ supported on $\overline{N_{2\epsilon_2}(W)}\subset X$ with $\iota_W^*\tilde{\eta}=\eta$ and $\mathrm{d}\tilde{\eta}=0$ on $\overline{N_{\epsilon_2}(W)}$. So $\mathrm{d}\tilde{\eta}$ gives a class in $H_c^3(X-'W)$, and we assume $\iota_V^*\eta=\mathrm{d}\beta$ for some β c.s. on V_{ϵ_2} (and obviously *not* d-closed) so that the long d_1 is zero on $[\eta]$. Now β extends to $\tilde{\beta}$ supported on $\overline{N_{2\epsilon_1}(V_{\epsilon_2})}\subset X$ such that $\mathrm{d}(\tilde{\eta}-\tilde{\beta})=0$ on $N_{\epsilon_1}(V_{\epsilon_2})$ and thus on $\overline{N_{\epsilon_1}(V)}$. So $\mathrm{d}(\tilde{\eta}-\tilde{\beta})$ gives a class in $H_c^3(X-V)$.

We would like to see what "codimension 2 residues" arise in the integral $\int_X R'_{\mathbf{f}}\wedge\omega$ for $\omega=\mathrm{d}(\tilde{\eta}-\tilde{\beta})$, assuming say normal crossings at W and $\mathbf{f}=f_1\otimes f_2\otimes f_3\otimes f_4$. Instead of arguing with chains we ask the reader's indulgence in pretending $R'_{\mathbf{f}}$ is a degree 3 form with residue $2\pi\sqrt{-1}R'_{\partial_i\mathbf{f}}$ (a degree 2 form) at V_{f_i}. Since $\mathrm{d}(\tilde{\eta}-\tilde{\beta})$ is zero on $N_\epsilon(V)$, and we can assume $\tilde{\eta}$ pulls back to

0 on $\partial N_\epsilon(W)$, the integral is

$$\int_{X^\epsilon} R'_{\mathbf{f}} \wedge \mathrm{d}(\tilde{\eta} - \tilde{\beta}) \;=\; \int_{\partial X^\epsilon} R'_{\mathbf{f}} \wedge (\tilde{\eta} - \tilde{\beta}) \;=$$

$$2\pi\sqrt{-1}\sum_i \int_{V^\epsilon_{f_i}} R'_{\partial_i \mathbf{f}} \wedge (\iota_V^* \tilde{\eta} - \beta) \;=:\; 2\pi\sqrt{-1}\int_{V_\epsilon} R'_{\partial \mathbf{f}} \wedge (\iota_V^* \tilde{\eta} - \beta).$$

To proceed further we essentially need to assume that the codimension 1 residues are "trivial", i.e. on V_ϵ, $R'_{\partial \mathbf{f}} = \mathrm{d}\mathcal{L}$ for some "1-form" $\mathcal{L}$.[13] (Under this condition, $\int_V R'_{\partial \mathbf{f}} \wedge \beta = 0$ for β d-closed with c.s. away from W as in the previous computation.) We then get

$$\int_{V_\epsilon} \mathrm{d}\mathcal{L} \wedge (\iota_V^* \tilde{\eta} - \beta) \;=\; \int_{\partial V_\epsilon} \mathcal{L} \wedge \iota_V^* \tilde{\eta} \;=\; \int_W \eta \cdot \mathrm{Res}(\mathcal{L})$$

where $\mathrm{Res}(\mathcal{L})$ will be closed (but not necessarily exact). We would like to write $\mathrm{Res}\mathcal{L} = \mathrm{Res}^2 R(\gamma_{\mathbf{f}}) = {}'R(\mathrm{Res}^2\gamma_{\mathbf{f}})$ but we don't know what ${}'R$ should be ($\mathrm{Res}^2\gamma_{\mathbf{f}}$ is no graph cycle); all of this will be worked out (correctly) with currents in §2.4.

1.4.5. Abel-Jacobi for "boxes". We conclude the present chapter on a somewhat lighter note. By analogy with Bloch's grading (in [**B5**]) on the Chow groups of an abelian variety it is not hard (see Chapter 5) to put a suitable grading on the relative Chow group[14]

$$CH^n((\mathbb{P}^1_F, \{0,\infty\})^n) := CH^n\left((\mathbb{P}^1_F)^n, \cup_{i,e}\rho^i_{e*}(\mathbb{P}^1_F)^{n-1}\right)$$

so that the last graded piece $Gr^n CH^n((\mathbb{P}^1_F, \{0,\infty\})^n)$ is spanned by the cycles of the form

$$B_{\mathbf{a}} := (a_1, \dots, a_n) - \sum_i (a_1, \dots, 1, \dots, a_n)$$
$$+ \sum_{i_1 < i_2} (a_1, \dots, \underset{i_1}{1}, \dots, \underset{i_2}{1}, \dots a_n) - \;\dots\; \pm (1, \dots, 1).$$

EXAMPLE 1.4.3. $Gr^2 CH^2(\mathbb{P}^1 \times \mathbb{P}^1, \#)$ is spanned by all $B_{a,b} = (a,b) - (a,1) - (1,b) + (1,1)$ where $a, b \in \mathbb{C}^*$ and $\# = \cup_{i=1,2}^{e=0,\infty}\{z_i = e\}$. Here $F^1 CH^2$ consists of cycles of degree 0, $F^2 CH^2$ is the Albanese kernel, and $F^3 = 0$.

We will be interested in the case $F = \mathbb{Q}(X)$ so that $B_{\mathbf{f}}$ can be considered as an element either of $Gr^n CH^n((\mathbb{P}^1_{\mathbb{C}}, \{0,\infty\})^n)$, by embedding $F \hookrightarrow \mathbb{C}$, or as a graph cycle $B(\gamma_{\mathbf{f}}) \in Z^n(\eta_X \times (\mathbb{P}^1, \{0,\infty\})^n(\mathbb{Q}))$. Applying θ to this and taking boundary (but ignoring the "residues" near V) we have

$$\partial\theta(\widetilde{B(\gamma^\epsilon_{\mathbf{f}})}) = B(\gamma^\epsilon_{\mathbf{f}}) + \sum_{\mathbf{s}} (-1)^{\mathbf{s}} \theta^{\mathbf{s}}_{1\dots n}(\gamma^\epsilon_{\mathbf{f}} \cap T_n) \qquad (**)$$

[13] We use this notation because $\mathcal{L}$ will be essentially dilogarithmic in nature.

[14] basically like $CH^n(\square^n, \partial\square^n)$ but without throwing out $\mathbb{I}^n$ (which means topologically we are no longer going *relative* $\mathbb{I}^n$)

on the nose (= without ignoring boundary at $\mathbb{I}^n$), where the latter term is equivalent to $(S^1)^n \times T_{\mathbf{f}}$. We merely indicate how things work for $n = 2$; it's the same for all n, with more cancellations. First,

$$\begin{aligned}\theta(\widetilde{B(\gamma^\epsilon_{f\otimes g})}) &= \theta(\tilde{\gamma}^\epsilon_{f\otimes g}) - \theta(\tilde{\gamma}^\epsilon_{f\otimes 1}) - \theta(\tilde{\gamma}^\epsilon_{1\otimes g}) + \theta(\tilde{\gamma}^\epsilon_{1\otimes 1}) \\ &= \left\{\theta_1(\tilde{\gamma}^\epsilon_{f\otimes g}) - \theta_{12}(\tilde{\gamma}^\epsilon_{f\otimes g} \cap \tilde{\mathcal{T}}_{z_1})\right\} - \theta_1(\tilde{\gamma}^\epsilon_{f\otimes g})\end{aligned}$$

(all other terms are trivial – i.e. of lower dimension). Taking ∂ (and ignoring residues) gives

$$\left\{(f,g)_{X^\epsilon} - (1,g)_{X^\epsilon} + (S^1_f, g)_{T^\epsilon_f}\right\} - \left\{(S^1_f, g)_{T^\epsilon_f} - (S^1_f, 1)_{T^\epsilon_f} + (S^1_f, S^1_g)_{T^\epsilon_f \cap T^\epsilon_g}\right\}$$
$$- \left\{(f,1)_{X^\epsilon} - (1,1)_{X^\epsilon} + (S^1_f, 1)_{T^\epsilon_f}\right\}$$

and after the obvious cancellations (and uniformizing the loops)

$$\{(f,g) - (1,g) - (f,1) + (1,1)\}_{X^\epsilon} - (S^1_f, S^1_g)_{T_f \cap T_g} = B(\gamma^\epsilon_{\mathbf{f}}) - (S^1)^2 \times T_{\mathbf{f}},$$

verifying $(**)$.

So we have justified using[15]

$$\int_{\theta(\widetilde{B(\gamma^\epsilon_{\mathbf{f}})})} \wedge^n \mathrm{dlog} z_i \wedge \omega \, - (2\pi\sqrt{-1})^n \int_{\partial^{-1}_{(X,V)} T_{\mathbf{f}}} \omega \qquad (\omega \text{ d-closed c.s.})$$

as the AJ-map on n-box graphs. The important point here is that only the $\theta(\tilde{\gamma}^\epsilon_{\mathbf{f}})$ term in $\theta(\widetilde{B(\gamma^\epsilon_{\mathbf{f}})})$ contributes to the first integral; $\theta(\widetilde{\gamma^\epsilon_{f\otimes 1}})$ and so on (and their equivalents for larger n) fail to contribute, basically because $\bigwedge^n \mathrm{dlog} z_i$ pulls back to 0 on $\mathbb{I}^n$. So we have the same computational result $\int_X R'_{\mathbf{f}} \wedge \omega$ for the Abel-Jacobi map as above in (1.4.1).

[15]provided the (relative) cycle-class of $B(\gamma_{\mathbf{f}})$ is zero; by arguments similar to those in preceding sections, this is computed by integrals $\int_X (\wedge^n \mathrm{dlog}\mathbf{f}) \wedge \alpha$.

CHAPTER 2

Properties of the Milnor Regulator

2.1. Equivalence Relations

2.1.1. Abel-Jacobi annihilates the Steinberg graphs. In the previous chapter we showed how to associate to a graph cycle $\gamma_{\mathbf{f}}$ over η_X a "$\mathbb{C}/\mathbb{Z}(n)$-functional" on relative cohomology:

$$\otimes^n \mathbb{Z}[\mathbb{P}^1_{\mathbb{C}(X)} \setminus \{0,\infty\}] \to Z^n(\eta_X, n) \to \frac{Hom(H^{n-1}(X_{rel}), \mathbb{C})}{\mathrm{im}\{H_{n-1}(X_{rel}, \mathbb{Z}(n))\}} =: H^{n-1}(\eta_X, \mathbb{C}/\mathbb{Z}(n))$$

$$\text{via} \quad \mathbf{f} \longmapsto \gamma_{\mathbf{f}} \longmapsto \int_{\theta(\tilde{\gamma}^{\epsilon}_{\mathbf{f}})} \wedge^n \mathrm{dlog} z_i \wedge \pi_X^*(\cdot) \pm (2\pi\sqrt{-1})^n \int_{\partial^{-1}_{(X,V_{\mathbf{f}})}(T_{\mathbf{f}})} (\cdot) =: \int_X R'_{\mathbf{f}} \wedge (\cdot)$$

where X_{rel} is the "dual" of η_X. (We do not *work* in this limit; for each $\mathbf{f}$ we work with $X - V_{\mathbf{f}}$ and $(X, V_{\mathbf{f}})$, where $V_{\mathbf{f}}$ is the union of supports of divisors of all functions "in" $\mathbf{f}$, rather than with η_X and X_{rel}.) For this map to deserve the name "Abel-Jacobi" it must descend to the level of $CH^n(\eta_X, n) \cong CH^p(\eta_X \times \hat{\square}^n, \eta_X \times \partial\hat{\square}^n)$ and so respect (relative) rational equivalence, or equivalently to $K_n^M(\mathbb{C}(X))$, which is what we now show in three different ways (in §2.1, §2.2, and §2.4). From this latter point of view the map is called a "Milnor regulator".[1]

Our present approach is naively geometric, and it will help to assume $\dim X = d < n$ (we mainly have in mind $d = n-1$, $n = 2, 3, 4$). The aim is of course to show the Steinbergs, by which we *moved* $\mathbf{f}$ in §1.4 to get the AJ-map, correspond to "integral"($= \mathbb{Z}(n)$) functionals.[2] In fact what we will show is that for an obvious *choice* of $\partial^{-1}_{(X,V)} T_{\mathbf{f}}$ they correspond to the zero functional.

There are three kinds of Steinbergs in $\otimes^n \mathbb{Z}[\mathbb{P}^1_{\mathbb{C}(X)} \setminus \{0,\infty\}]$; we do $\mathbf{f} = f \otimes (1-f) \otimes \mathbf{g}$ (for $\mathbf{g} \in \otimes^{n-2} \mathbb{Z}[\mathbb{P}^1_{\mathbb{C}(X)} \setminus \{0,\infty\}]$) first. As motivation we note that $\gamma_{\mathbf{f}}$ is "degenerate" in the sense that it is the inage of $\gamma_{\mathbf{f}_0}$, $\mathbf{f}_0 = f \otimes \mathbf{g}$, under the map $st : \hat{\square}^{n-1} \to \hat{\square}^n$ given by $(z_1, z_2, \ldots, z_{n-1}) \mapsto (z_1, 1 - z_1, z_2, \ldots, z_{n-1})$. So $\theta(\tilde{\gamma}^{\epsilon}_{\mathbf{f}})$ ought to be homotopic to something like $st(\theta(\tilde{\gamma}^{\epsilon}_{\mathbf{f}_0}))$ (in fact only the first term of $\theta(\tilde{\gamma}^{\epsilon}_{\mathbf{f}_0})$ is correct here); since $st^* \bigwedge^n \mathrm{dlog} z_i = 0$ the AJ-map is then trivial.

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[1] which is to say, [the pullback of] a secondary characteristic class for higher K-theory.

[2] Recall that this does *not* mean $\mathbb{Z}(n)$-valued functionals; it means $(2\pi\sqrt{-1})^n \times$integration over representatives of $H_{n-1}((X,V), \mathbb{Z})$.

2.1.2. How the dilogarithm arises from a homotopy. We redefine the basic retraction-to-$\{1\}$ map

$$\theta : \mathbb{D} \times [0,1] \to \hat{\square}^1$$

as well as maps

$$\Psi : (\mathbb{D} \setminus \{0,\infty\}) \times [0,1]^2 \to \hat{\square}^2, \qquad \Upsilon : (\partial\mathbb{D} \setminus \{0,\infty\}) \times [0,1]^3 \to \hat{\square}^2$$

that will be used in generating the homotopy.[3] First let $\tilde{\theta} : \partial\mathbb{D} \times [0,1] \to \mathbb{D}$ be a continuous map with $\tilde{\theta}(\partial\mathbb{D} \times [0,1]) \subset \mathbb{D} \setminus \partial\mathbb{D}$, $\tilde{\theta}(\partial\mathbb{D} \times \{1\}) = \{\tilde{1}\}$, and $\tilde{\theta}(\partial\mathbb{D} \times \{0\}) = \partial\mathbb{D}$ the identity. We also assume that only the path starting at $\{\infty\}$ intersects $\{T_{1-z} = (1,\infty)\} \subset \mathbb{R}^+$, so that $\widetilde{1 - \theta(t,\tilde{z})}$ is defined unambiguously for all $\tilde{z} \in \partial\mathbb{D} \setminus \{0,\infty\}$ and $t \in [0,1]$ (no $\pm$ issues). Extend this $\tilde{\theta}$ to $\mathbb{D}$ by the rule

$$\tilde{\theta}(1 - t_1, \tilde{\theta}(1 - t_2, \tilde{z})) = \tilde{\theta}(1 - t_1 t_2, \tilde{z})$$

for any point $\tilde{z} \in \partial\mathbb{D}$ and write θ for the composition $\mathcal{N} \circ \tilde{\theta}$ with $\mathcal{N} : \mathbb{D} \to \mathbb{P}^1$. (Note in particular that we get out of this $\theta(t,1) = 1\,\forall t$, which we used tacitly above in the closing pages of Chapter 1.)

The basic building-block for the homotopy mentioned above will now be defined by

$$\Psi(\tilde{z}) := \Psi(\tilde{z}, t_1, t_2) := \left(\theta(t_2, \tilde{\theta}(t_1, \tilde{z}), 1 - \theta(t_1, \tilde{z})\right),$$

since for each fixed $\tilde{z} \in \mathbb{D} \setminus \{0,\infty\}$ the real 2-chain $\Psi(\tilde{z})$ has boundary (by setting $t_1 = 0,1$ and $t_2 = 0,1$)

$$\partial\Psi(\tilde{z}) = (\theta(t_1,\tilde{z}), 1 - \theta(t_1,\tilde{z})) + (\theta(t_2,\tilde{z}), 1 - z) - (1, 1 - \theta(t_1,\tilde{z}))$$

$$= st(\theta_{(1)}(\tilde{z})) + \theta_1(\tilde{z}, \widetilde{1 - z}) \qquad \text{mod chains with support } \subset \mathbb{I}^2.$$

Finally define the "solid" (for each point $\tilde{\mathcal{Z}} \in \partial\mathbb{D} \setminus \{0,\infty\}$)

$$\Upsilon(\tilde{z}) := \Upsilon(\tilde{z}, t_1, t_2, t_3) := \theta_{12}(\widetilde{st}(\tilde{\theta}(\tilde{z}))$$

$$= \left(\theta(t_2, \tilde{\theta}(t_1,\tilde{z}), \theta(t_3, \widetilde{1 - \theta(t_1,\tilde{z})})\right)$$

with boundary components (at $t_1 = 0,1$, $t_2 = 0,1$, $t_3 = 0,1$ in that order)

$$\partial\Upsilon(\tilde{z}) = \left(\theta(t_2,\tilde{z}), \theta(t_3, \widetilde{1 - z})\right) - (1, \theta(t_3, 0))$$

$$+ \left(\theta(t_1,\tilde{z}), \theta(t_3, \widetilde{1 - \theta(t_1,\tilde{z})})\right) - \left(1, \theta(t_3, \widetilde{1 - \theta(t_1,\tilde{z})})\right)$$

$$+ \left(\theta(t_2, \tilde{\theta}(t_1,\tilde{z}), \widetilde{1 - \theta(t_1,\tilde{z})}\right) - \left(\theta(t_2, \tilde{\theta}(t_1,\tilde{z})), 1\right)$$

[3] It is desirable now to include $\{0,\infty\}$ in $\partial\mathbb{D}$ and T, as the definition of Ψ and Υ require these to be in the domain of θ.

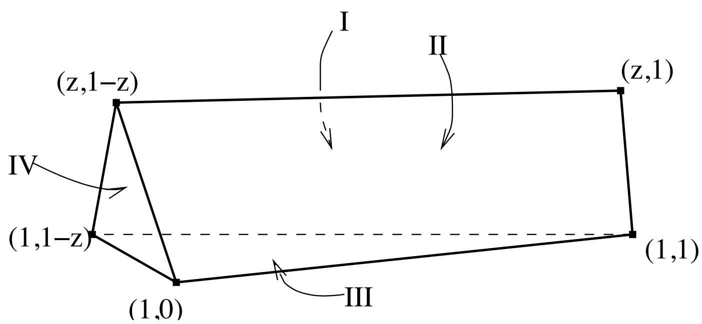
where the 2^{nd} and 6^{th} terms are 1-dimensional (in particular, $\theta(t_2, \tilde{\theta}(t_1, \tilde{z})) = \theta(1-(1-t_1)(1-t_2), \tilde{z})$). We rewrite the remaining terms

$$\Psi_I(\tilde{z}) + \Psi_{II}(\tilde{z}) - \Psi_{III}(\tilde{z}) + \Psi_{IV}(\tilde{z})$$

where

$$\Psi_I(\tilde{z}) = \theta_{12}(\tilde{z}, \widetilde{1-z}), \;\; \Psi_{IV}(\tilde{z}) = \Psi(\tilde{z})$$

and $\Psi_{III}(\tilde{z})$ is essentially trivial (its support $\subset \mathbb{I}^2$). In a picture,



All of the above maps extend to $[X\times]\mathbb{D}^{n-1}$ or $[X\times]\partial\mathbb{D}\times\mathbb{D}^{n-2}$ (e.g., Ψ_1, Υ_1) simply by taking $\tilde{z}_i \mapsto z_i$ in the remaining $(n-2)$ coordinates.

Returning to the subject of $\theta(\tilde{\gamma}^{\epsilon}_{\mathbf{f}}) = \theta_1(\tilde{\gamma}^{\epsilon}_{\mathbf{f}}) - \theta_{12}(\tilde{\gamma}^{\epsilon}_{\mathbf{f}} \cap \tilde{T}_{z_1}) + \ldots$, we remark that since $T_f \cap T_{1-f} = (f)_\infty \subset V_{\mathbf{f}}$, only the first 2 terms are nonzero (over X^ϵ) and we can even take $\partial^{-1}_{(X,V_{\mathbf{f}})} T_{\mathbf{f}} = 0$ (since $T_{\mathbf{f}} \subset V_{\mathbf{f}}$). (Again, we work over X^ϵ because our test forms ω are compactly supported there.) So, with $T^\epsilon_f = T_f \cap X^\epsilon$, $\theta(\tilde{\gamma}^{\epsilon}_{\mathbf{f}}) =$

$$\Big(x;\, \theta(\widetilde{f(x)}, t), 1-f(x), \mathbf{g}(x)\Big)_{x\in X^\epsilon} - \Big(x;\, \underbrace{\theta(\widetilde{f(x)}, t_1), \theta(\widetilde{1-f(x)}, t_2)}_{\Psi_I(\widetilde{f(x)})}, \mathbf{g}(x)\Big)_{x\in T^\epsilon_f}$$

where by abuse of notation $\widetilde{f(x)} = f(x)^+ - f(x)^-$ in the second term alone (and this does *not* apply to $\widetilde{1-f(x)}$.) Now the original idea was to do the initial "pull-to-$\{1\}$" in a different direction, along $z_1 + z_2 = 1$, to $(1, 0, \mathbf{g}(x))$ (instead of $(1, 1-f(x), \mathbf{g}(x))$ as in $\theta_1(\tilde{\gamma}^{\epsilon}_{\mathbf{f}})$). This corresponds to replacing the first term above by

$$\Big(x;\, \theta(t, \widetilde{f(x)}), 1-\theta(t, \widetilde{f(x)}), \mathbf{g}(x)\Big)_{x\in X^\epsilon} =: st(\theta_1(\tilde{\gamma}^{\epsilon}_{\mathbf{f}_0}))$$

and these two alternative "first terms" together constitute two pieces of the boundary of

$$\Big(x;\, \theta(t_2, \tilde{\theta}(t_1, \widetilde{f(x)})), 1-\theta(t_1, \widetilde{f(x)}), \mathbf{g}(x)\Big)_{x\in X^\epsilon} =: \Psi_1(\tilde{\gamma}^{\epsilon}_{\mathbf{f}_0}).$$

Now we set

$$\Psi(\tilde{\gamma}^{\epsilon}_{\mathbf{f}}) := \Psi_1(\tilde{\gamma}^{\epsilon}_{\mathbf{f}_0}) - \Upsilon_1(\tilde{\gamma}^{\epsilon}_{\mathbf{f}_0} \cap \tilde{T}_{z_1}) := \Big(x;\, \theta(t_2, \tilde{\theta}(t_1, \widetilde{f(x)})),\, 1-\theta(t_1, \widetilde{f(x)}), \mathbf{g}(x)\Big)_{x\in X^\epsilon}$$

$$- \left(x;\, \theta(t_2, \tilde{\theta}(t_1, \widetilde{f(x)})),\, \theta(t_3, 1 - \widetilde{\theta(t_1, \widetilde{f(x)})}), \mathbf{g}(x)\right)_{x \in T_f^\epsilon}$$

where again for $x \in T_f^\epsilon$, $\widetilde{f(x)} = f(x)^+ - f(x)^-$; and so the second term is just

$$\big(x;\, \Upsilon(f(x)^+), \mathbf{g}(x)\big)_{x \in T_f^\epsilon} - \big(x;\, \Upsilon(f(x)^-), \mathbf{g}(x)\big)_{x \in T_f^\epsilon}.$$

Modulo $X \times \mathbb{I}^n$ and $N_\epsilon(V) \times \hat{\square}^n$ we then have simply

$$\partial\Psi(\tilde{\gamma}_{\mathbf{f}}^\epsilon) \;=\; \Big\{ - \Big(x;\, \theta(t_1, \widetilde{f(x)}), 1 - \theta(t_1, \widetilde{f(x)}), \mathbf{g}(x)\Big)_{X^\epsilon}$$

$$+ \Big(x;\, \theta(t_2, \widetilde{f(x)}), 1 - f(x), \mathbf{g}(x)\Big)_{X^\epsilon} + \Big(x;\, \Psi_{(IV)}(\widetilde{f(x)}), \mathbf{g}(x)\Big)_{T_f^\epsilon} \Big\}$$

$$- \Big\{ \Big(x;\, \Psi_I(\widetilde{f(x)}), \mathbf{g}(x)\Big)_{T_f^\epsilon} + \Big(x;\, \Psi_{II}(\widetilde{f(x)}), \mathbf{g}(x)\Big)_{T_f^\epsilon} + \Big(x;\, \Psi_{IV}(\widetilde{f(x)}), \mathbf{g}(x)\Big)_{T_f^\epsilon} \Big\}$$

$$= \theta(\tilde{\gamma}_{\mathbf{f}}^\epsilon) \,+\, st(\theta_1(\tilde{\gamma}_{\mathbf{f}_0}^\epsilon)) \,-\, \Big(x;\, \Psi_{II}(\widetilde{f(x)}), \mathbf{g}(x)\Big)_{T_f^\epsilon},$$

so that if ω is a d-closed $(2d - n + 1)$-form compactly supported in X^ϵ,

$$0 \;=\; \int_{\Psi(\tilde{\gamma}_{\mathbf{f}}^\epsilon)} \wedge^n \mathrm{dlog} z_i \wedge \mathrm{d}\pi_X^* \omega \;=\; \int_{\partial\Psi(\tilde{\gamma}_{\mathbf{f}}^\epsilon)} \wedge^n \mathrm{dlog} z_i \wedge \pi^* \omega$$

$$= \int_{\theta(\tilde{\gamma}_{\mathbf{f}}^\epsilon)} \wedge^n \mathrm{dlog} z_i \wedge \pi^* \omega \,+\, \int_{\theta_1(\tilde{\gamma}_{\mathbf{f}_0}^\epsilon)} \underbrace{st^*(\wedge^n \mathrm{dlog} z_i)}_{=0} \wedge \pi^* \omega \,-\, \int_{\Psi_{II}\text{-term}} \wedge^n \mathrm{dlog} z \wedge \pi^* \omega.$$

In order to show the last integral $= 0$ we must examine the "monodromy" of Ψ_{II}: $\Psi_{II}(\widetilde{f(x)}) = \Psi_{II}(f(x)^+) - \Psi_{II}(f(x)^-)$ for $x \in T_f^\epsilon$ as above, and we must consider the difference of the two corresponding integrals. This reduces to computing

$$\int_{\Psi_{II}(\tilde{w})} \mathrm{dlog}\tilde{z}_1 \wedge \mathrm{dlog}\tilde{z}_2 \;=\; - \int_{\theta(t_1, \tilde{w})} \left(\int_{\theta(t_2, 1 - \widetilde{\theta(t_1, \tilde{w})})} \mathrm{dlog}\tilde{z}_2 \right) \mathrm{dlog}\tilde{z}_1$$

$$= \int_{\tilde{z}_1 = 1}^{\tilde{w}} \left(\int_{z_2 = 1}^{\widetilde{1 - z_1}} \mathrm{dlog}\tilde{z}_2 \right) \mathrm{dlog}\tilde{z}_1 \;=\; \int_1^{\tilde{w}} \log(1 - z) \mathrm{dlog} z$$

for $\tilde{w} \;=\; f(x)^+$ and $f(x)^-$. The difference is the integral

$$\int_{\mathcal{O}(f(x))} \log(1 - z) \mathrm{dlog} z$$

where $\mathcal{O}(f(x))$ is a loop around 0 going through 1 and $f(x) \in \mathbb{R}^- \in T_z$, and $\log(1 - z)$ (while having a singularity at 1) has no branch cut on $\mathcal{O}$ (since its cut is $\mathbb{R}^{>1} = T_{1-z}$). As it stands the integral is convergent; so, moving $\mathcal{O}$ off 1 (to the left) and letting it approach 1 from there, we may view the above as a *limit* of (nonsingular) integrals. These are zero by the Cauchy residue formula since the given branch of $\log(1-z)$ is exactly zero at $z=0$. The vanishing of AJ for the first Steinberg is proved.

Now notice that by Stokes theorem, if we extend the Υ-construction to all of $\mathbb{D}[\backslash\mathbb{R}^{>1}]$,

$$-\int_{\Psi_{II}(\tilde{w})}\frac{dz_1}{z_1}\wedge\frac{dz_2}{z_2}=\int_{\Psi_I(\tilde{w})}\frac{dz_1}{z_1}\wedge\frac{dz_2}{z_2}+\int_{\Psi_{(IV)}(\tilde{w})}\frac{dz_1}{z_1}\wedge\frac{dz_2}{z_2}$$

where

$$\int_{\Psi(\tilde{w})}\frac{dz_1}{z_1}\wedge\frac{dz_2}{z_2}\;=\;\int_{1-\theta(t_1,\tilde{w})}\left(\int_{\theta(t_2,\tilde{\theta}(t_1,\tilde{w}))}\mathrm{dlog}\tilde{z}_1\right)\mathrm{dlog}\tilde{z}_2$$

$$=\int_0^{1-w}\left(\int_1^{1-z_2}\mathrm{dlog}z_1\right)\mathrm{dlog}z_2\;=\;\int_0^{1-w}\log(1-z)\mathrm{dlog}z\;=:\;-\mathrm{Li}_2(1-w)$$

gives the dilogarithm[4] so that (with our "standard" branches of log)

$$\int_1^w\log(1-z)\mathrm{dlog}z\;=\;\log(1-w)\log w\;+\;\mathrm{Li}_2(1-w).$$

Since the argument above basically amounted to saying that the l.h.s. is continuous at $\mathbb{R}^-=T_z$ (or has no monodromy around 0), this equation says that the jumps there in the two right-hand terms must cancel. Since $\log(1-w)\log w$ changes by $2\pi i\log(1-w)$ at T_z (in the $+i$ -direction), $\mathrm{Li}_2(1-w)$ must change by $-2\pi i\log(1-w)$. This is the "branch cut" in $\mathrm{Li}_2(1-w)$ corresponding to our cuts in log, and in $\mathrm{Li}_2(w)$ it is $2\pi i\log(w)$ at T_{1-z} (in the $+i$-direction). One may check this simply by integrating $-\log(1-z)\mathrm{dlog}z$ from "$y+0i$" to "$y-0i$" ($y\in T_{1-z}$) around T_{1-z} to obtain $2\pi i\log y$.

2.1.3. Log branch-change and switching tensor factors. Now we take care of the Steinbergs corresponding to "multilinearity": $\mathbf{f}=f_1f_2\otimes\mathbf{g}-f_1\otimes\mathbf{g}-f_2\otimes\mathbf{g}$, for $\mathbf{g}\in\otimes^{n-1}\mathbb{Z}[\mathbb{P}^1_{\mathbb{C}(X)}\setminus\{0,\infty\}]$. The philosophy is then to rely on the remaining "antisymmetry" Steinbergs to transport f (and $f\otimes(1-f)$ in the above) to all positions in the tensor product via permutations. However in this section we shall only do $n=2,3$ and leave the general case to the next section where it is easier.

The crucial point in computing the AJ map $\int_X R'_{f_1f_2\otimes\mathbf{g}}\wedge(\cdot)-\int_X R'_{f_1\otimes\mathbf{g}}\wedge(\cdot)-\int_X R'_{f_2\otimes\mathbf{g}}\wedge(\cdot)$ is that $\log f_1+\log f_2\equiv\log f_1f_2$ only up to $\mathbb{Z}(1)=(2\pi\sqrt{-1})\mathbb{Z}$; and so the difference, multiplied by say $\bigwedge^{n-1}\mathrm{dlog}\mathbf{g}\wedge\omega$, is an *a priori* nontrivial creature. Setting $f_1f_2=:f$, $\log f_1f_2=\log f$ gives the standard branch while $\log f_1+\log f_2={}'\log f$ defines a different branch with cut ${}'T_f=T_{f_1}+T_{f_2}$. Since T_f and ${}'T_f$ (which each yield classes $\in H_{2d-1}(X,V_{\mathbf{f}})$) have the same boundary $=(f)\subset V_{\mathbf{f}}$, $T_f-{}'T_f$ gives a class $\in H_{2d-1}(X)$. We would like to show this class is zero.

The exact sequence for (relative) singular homology

[4](for the basic information on Li_2 see **[Za]** or **[Ha]**.)

$$0 \to H_{2d-1}(X,\mathbb{Z}) \hookrightarrow H_{2d-1}(X,V;\mathbb{Z}) \to H_{2d-2}(V,\mathbb{Z}) \to \dots$$

tells us that it is sufficient to show $[T_f -' T_f] = 0$ in $H_{2d-1}(X,V)$. Over $\mathbb{Q}$ there is a perfect pairing

$$H_{2d-1}(X,V;\mathbb{Q}) \otimes H_1(X-V,\mathbb{Q}) \to \mathbb{Q}$$

so nontriviality of this class, or equivalently the difference between $[T_f]$ and $['T_f]$, should be picked up by intersection with a loop $\gamma \in H_1(X-V)$. Supposing then that $\gamma \cdot T_f = m$ but $\gamma \cdot' T_f = 0$ for such a path γ, we remark that $2\pi\sqrt{-1}(\gamma \cdot T_f)$ tells how much $\log f_1 f_2$ changes around γ if *analytically* continued; likewise, $2\pi\sqrt{-1}(\gamma \cdot' T_f)$ gives the monodromy of $\log f_1 + \log f_2$. But, analytically continued, these are the same function (if they atart at the same value) and so this is impossible.

Therefore on X there exists a $2d$-chain Δ_f with $\partial\Delta_f = T_f -' T_f$. For $n=2$, triviality of the AJ map then goes as follows (where again ω is d-closed and c.s. on $X \setminus N_\epsilon(V)$):

$$\int_X (R'_{f_1 f_2 \otimes g} - R'_{f_1 \otimes g} - R_{f_2 \otimes g}) \wedge \omega \;=\; \int_X (\log f -' \log f)\mathrm{dlog} g \wedge \omega$$

$$-(2\pi\sqrt{-1})\int_{T_f - 'T_f} (\log g)\omega - 4\pi^2 \int_{\partial^{-1}(T_f \cap T_g) - \partial^{-1}('T_f \cap T_g)} \omega.$$

Now writing Δ_f also for the integrable function which is 1 on Δ_f and 0 otherwise, $\log f -' \log f = (2\pi\sqrt{-1})\Delta_f$, while $T_f -' T_f = \partial\Delta_f$ and we may choose $\partial^{-1}(T_f \cap T_g) - \partial^{-1}('T_f \cap T_g) = \partial^{-1}((T_f -' T_f) \cap T_g) = \Delta_f \cap T_g$. So the above

$$= 2\pi\sqrt{-1}\left\{\int_{\Delta_f \setminus \Delta_f \cap T_g} \mathrm{dlog} g \wedge \omega \;-\; \int_{\partial\Delta_f \setminus \partial\Delta_f \cap T_g} (\log g)\omega \;+\; 2\pi\sqrt{-1}\int_{\Delta_f \cap T_g} \omega\right\}.$$

Since ω is d-closed, $\mathrm{d}\log g \wedge \omega = \mathrm{d}((\log g)\omega)$ on $\Delta_f \setminus \Delta_F \cap T_g$, and $\partial(\Delta_f \setminus \Delta_f \cap T_g) = \partial\Delta_f \setminus \partial\Delta_f \cap T_g + (T_g^+ - T_g^-) \cap \Delta_f \implies$ the a bove is zero by Stokes theorem.

For $n=3$ the computation reduces to

$$2\pi\sqrt{-1}\int_{\Delta_f} \mathrm{dlog} g \wedge \mathrm{dlog} h \wedge \omega \;+\; 2\pi\sqrt{-1}\int_{\partial\Delta_f} \log g \, \mathrm{dlog} h \wedge \omega$$

$$-4\pi^2 \int_{\partial\Delta_f} (\log h)\omega \pm \int_{\Delta_f \cap T_g \cap T_h} \omega \;=\; \underbrace{2\pi\sqrt{-1}\Big\{\textstyle\int_{\Delta_f \setminus \Delta_f \cap T_g} \mathrm{d}\,(\log g \, \mathrm{dlog} h \wedge \omega)}_{-\int_{\partial\Delta_f} \log g \, \mathrm{dlog} h \wedge \omega + 2\pi\sqrt{-1}\int_{\Delta_f \cap T_g} \mathrm{dlog} h \wedge \omega}$$

$$\int_{\partial\Delta_f \setminus \partial\Delta_f \cap T_g} \log g \, \mathrm{dlog} h \wedge \omega \;+\; 2\pi\sqrt{-1}\int_{\partial\Delta_f \cap T_g \setminus \partial\Delta_f \cap T_g \cap T_h} (\log h)\omega$$

$$\pm 4\pi^2 \int_{\Delta_f \cap T_f \cap T_g} \omega \Bigg\} \quad = \quad 2\pi\sqrt{-1}\left\{ 2\pi\sqrt{-1} \int_{(\Delta_f \cap T_g)\setminus T_h} \partial\left((\log h)\omega\right)\right.$$

$$\left. + 2\pi\sqrt{-1}\int_{(\partial\Delta_f \cap T_g)\setminus T_h} (\log h)\omega \,\pm\, 4\pi^2 \int_{\Delta_f \cap T_g \cap T_h} \omega \right\} \;=\; 0.$$

We shall do two examples of the Steinbergs inducing "alternation": $\mathbf{f} = f\otimes g + g\otimes f$ for $n=2$ and $f\otimes g\otimes h + f\otimes h\otimes g$ for $n=3$, leaving the general case again for §2.2. Since $T_f\cap T_g = -T_g\cap T_f$, we may choose $\partial^{-1}_{(X,V)}(T_{\mathbf{f}}) = \partial^{-1}(0) = 0$, and so

$$\int_X \left(R'_{f\otimes g} + R'_{g\otimes f}\right)\wedge\omega \;=\; \int_X \left(\log f\,\mathrm{dlog}g + \log g\,\mathrm{dlog}f\right)\wedge\omega$$

$$-2\pi\sqrt{-1}\int_{T_f}(\log g)\omega - 2\pi\sqrt{-1}\int_{T_g}(\log f)\omega,$$

which is 0 since $\log f\,\mathrm{dlog}g + \log g\,\mathrm{dlog}f \;=\; \mathrm{d}\left(\log f\;\log g\right)$ on $X\setminus T_f\cup T_g$ and by Stokes theorem

$$\int_{X\setminus T_f\cap T_g} \mathrm{d}\left((\log f\;\log g)\omega\right) \;=\; 2\pi\sqrt{-1}\int_{T_g}(\log f)\omega \,+\, 2\pi\sqrt{-1}\int_{T_f}(\log g)\omega.$$

For $n=3$, again $T_f\cap T_g\cap T_h = -T_f\cap T_h\cap T_g$ and

$$\int_X \left(R'_{f\otimes g\otimes h} + R_{f\otimes h\otimes g}\right)\wedge\omega \;=\; \int \log f\;\Big(\overbrace{\mathrm{dlog}g\wedge\mathrm{dlog}h + \mathrm{dlog}h\wedge\mathrm{dlog}g}^{=0}\Big)\wedge\omega$$

$$+\,2\pi\sqrt{-1}\int_{T_f}\left(\log g\,\mathrm{dlog}h + \log h\,\mathrm{dlog}g\right)\wedge\omega \,-\, 4\pi^2\left\{\int_{T_f\cap T_g}(\log h)\omega \,+\, \int_{T_f\cap T_h}(\log g)\omega\right\}$$

which reduces to the same computation as above (on T_f rather than X).

2.2. Milnor-Regulator Currents

2.2.1. "Pushing" the Abel-Jacobi map down to X. The proofs in the last section can be vastly simplified by thinking of $\int_X R'_{\mathbf{f}} \wedge \omega$ as integration against something like a differential form, and working with these "outside the integral." So we introduce the language of currents, intuitively differential forms with distributuion coefficients, and reduce $AJ(\gamma_{\mathbf{f}})$ to such an object using formula (1.4.1). Formally, an m-current is a section of a sheaf

$$'\mathcal{D}_X^m = \mathcal{D}(\Omega_{X^\infty}^{2d-m})$$

of distributions on C^∞-forms. Given a suitable ℓ-form η on a real-codimension-k analytic subset $Y \subset X$, with associated "delta function" δ_Y, $\eta \cdot \delta_Y$ defines a current $\in \Gamma('\mathcal{D}_X^{k+\ell})$ by the formula

$$\int_X (\eta \cdot \delta_Y) \wedge \omega \;:=\; \int_Y \eta \wedge \iota_Y^* \omega \,, \qquad\qquad \omega \in \Gamma(\Omega_{X^\infty}^{2d-k-\ell});$$

of course if $\eta = \iota_Y^* \tilde{\eta}$ for $\tilde{\eta}$ on X then "$\tilde{\eta} \cdot \delta_Y$" is the same current. Like C^∞-forms, currents form a complex of sheaves with hypercohomology computing $H^*_{DR}(X, \mathbb{C})$; since they are (like the sheaves of C^∞-forms) acyclic, we have $H^*(\Gamma(X, '\mathcal{D}_X^\bullet)) \cong H^*(X, \mathbb{C})$. Accrding to **[GH]**, the differential of an m-current is just given by the "adjointness" property

$$\int_X \mathrm{d}[S] \wedge \omega \;:=\; (-1)^{m+1} \int_X S \wedge \mathrm{d}\omega$$

where the r.h.s. may have to be computed as a (finite) limit. The obvious example on X is $\mathrm{d}[\mathrm{dlog} f] = (2\pi\sqrt{-1})\, \delta_{(f)}$.

But a differnt kind of example concerns us in this section, where we are not interested in "residues" of this sort, and want to work "away" from $V_{\mathbf{f}}$. Recall that $\iota_! \Omega^{2d-m}_{(X \setminus V_{\mathbf{f}})^\infty}$ is the sheaf of C^∞-forms compactly supported away from $V_{\mathbf{f}}$, and let $'\mathcal{D}^m_{V^\infty} \subset {}'\mathcal{D}_X^m$ be the subsheaf of currents annihilating these forms. They are called the "currents [on X] supported on V" and we get currents on $(X - V)$ by ignoring them:[5]

$$'\mathcal{D}^m_{(X\setminus V)^\infty} := {'\mathcal{D}^m_X}/{'\mathcal{D}^m_{V^\infty}} = \mathcal{D}\left(\iota_! \Omega^{2d-m}_{(X\setminus V)^\infty}\right).$$

These form a complex, and $\int_X S \wedge \omega$ induces a perfect pairing between

$$H^m(\Gamma(X, {'\mathcal{D}^\bullet_{(X\setminus V)^\infty}})) \cong H^m(X - V, \mathbb{C})$$

and

$$H^{2d-m}(\Gamma(X, \iota_! \Omega^\bullet_{(X\setminus V)^\infty})) \cong H^{2d-m}_c(X - V, \mathbb{C}).$$

The prototypical example of d in this complex is, considering $\log f$ (with the $2\pi\sqrt{-1}$ jump at T_f) as a 0-current, $\mathrm{d}[\log f] = \mathrm{dlog} f - (2\pi\sqrt{-1})\delta_{T_f}$. Here T_f is oriented so that $\partial T_f = (f) = (f)_0 - (f)_\infty$ as usual. With the convention that the $\delta_{T_{f_i}}$'s anti-commute with $\mathrm{dlog} f_i$'s and the like, we may differentiate combinations of these exactly like forms (with regard to signs).[6] Formally, one has

$$\int_X \mathrm{d}[\log f] \wedge \omega = -\int_X (\log f)\mathrm{d}\omega = -\int_{X\setminus T_f} \mathrm{d}\left((\log f)\omega\right) + \int_X \mathrm{dlog} f \wedge \omega$$

$$= -2\pi\sqrt{-1}\int_{T_f} \omega + \int_X \mathrm{dlog} f \wedge \omega.$$

---

[5]There are actually a few different constructions of currents "on $X - V$", arising as quotients by the following sheaves of currents supported on V. Writing $\jmath : X - V \hookrightarrow X$, $\iota : V \hookrightarrow X$, we have (partailly after **[Ki]**)

$$\left({'\mathcal{D}^{m-2}_V} \cdot \delta_V =\right) \iota_* {'\mathcal{D}^{m-2}_V} \subset {'\mathcal{D}^m_X}(\mathrm{on} V) \subset {'\mathcal{D}^m_{V^\infty}} \subset {'\mathcal{D}^m_X},$$

which may be defined respectively in terms of what they annihilate:

$$\Omega^{2d-m}_{X^\infty} < \mathrm{null} V > \supset \Omega^{2d-m}_{X^\infty}(\mathrm{null} V) \supset \jmath_! \Omega^{2d-m}_{(X\setminus V)^\infty} \supset 0,$$

where $\Omega^{2d-m}_{X^\infty} < \mathrm{null}V >$ and $\Omega^{2d-m}_X < \mathrm{null}V >$ are C^∞ and holomorphic forms (respectively) *pulling back* to 0 via ι^* (e.g. any holomorphic d-form, $d = \dim X$), and $\Omega^\bullet_{X^\infty}(\mathrm{null}V) := \Omega^\bullet_{X^\infty} \wedge \Omega^\bullet_X < \mathrm{null}V >$. (the difference is that $\Omega^\bullet_{X^\infty}(\mathrm{null}V)$ has a dz *or* z in a neighborhood of a component $\{z = 0\}$ of V, as opposed to e.g. $d\bar{z}$ or $\bar{z}$.) So one has the $(X - V)$-currents

$${'\mathcal{D}^m_X} < V > := {'\mathcal{D}^m_X}/\iota_* {'\mathcal{D}^{m-2}_V} \cong \mathcal{D}\left(\Omega^{2d-m}_{X^\infty} < \mathrm{null}V >\right),$$

$${'\mathcal{D}^m_X}(\log V) := \oplus_{p+q=n} {'\mathcal{D}^p_X} \wedge \Omega^q_X(\log V) \underset{\text{(King)}}{\cong} {'\mathcal{D}^m_X}/{'\mathcal{D}^m_X}(\mathrm{on}V) \cong \mathcal{D}\left(\Omega^{2d-m}_{X^\infty}(\mathrm{null}V)\right),$$

$${'\mathcal{D}^m_{(X\setminus V)^\infty}} := {'\mathcal{D}^m_X}/{'\mathcal{D}^m_{V^\infty}} = \mathcal{D}\left(\jmath_! \Omega^{2d-m}_{(X\setminus V)^\infty}\right).$$

We have chosen the latter in order that $R'_{\mathbf{f}}$ can be d-closed "on $(X - V)$" for $n > d$ even for "bad" $\mathbf{f}$ (when $R'_{\mathbf{f}}$ has worse than log simgularities). Formally, to check in general in which complex of sheaves of $(X - V)$-currents R is closed, let ω be an arbitrary section of one of the three sheaves of forms "vanishing" on V and check if, as $\epsilon \to 0$, $\int_{N_\epsilon(V)} R \wedge \mathrm{d}\omega - \int_{\partial N_\epsilon(V)} R \wedge \omega \to 0$.

[6]with the caveat that a δ-function *in front* corresponds to integration so signs differ between the currents and (1.4.1) for n odd.

We now define a map

$$R : \otimes^n \mathbb{Z}[\mathbb{P}^1_{\mathbb{C}(X)} \setminus \{0, \infty\}] \to \Gamma({}'\mathcal{D}^{n-1}_X)$$

sending

$$f_1 \otimes \ldots \otimes f_n = \mathbf{f} \longmapsto$$

$$R_{\mathbf{f}} := \sum_{i=1}^{n} (\pm 2\pi\sqrt{-1})^{i-1} \log f_i \, \mathrm{dlog} f_{i+1} \wedge \ldots \wedge \mathrm{dlog} f_n \cdot \delta_{T_{f_1} \cap \ldots \cap T_{f_{i-1}}}$$

$$= \log f_1 \, \mathrm{dlog} f_2 \wedge \ldots \wedge \mathrm{dlog} f_n \, + \, (\pm 2\pi\sqrt{-1}) \log f_2 \, \mathrm{dlog} f_3 \wedge \ldots \wedge \mathrm{dlog} f_n \cdot \delta_{T_{f_1}}$$

$$+ \ldots + \, (2\pi\sqrt{-1})^{n-1} \log f_n \cdot \delta_{T_{f_1} \cap \ldots \cap T_{f_{n-1}}},$$

where $\pm = (-1)^{n-1}$. We note that the singularities are integrable even for $\mathbf{f}$ "bad" so this makes sense as a current on all of X and not just $X - V$. But it is closed on $X - V$, as applying d yields *modulo* ${}'\mathcal{D}^n_{V^\infty}$ a collapsing sum:

$$\mathrm{d}[R_{\mathbf{f}}] \, = \, \sum_{i=1}^{n} (\pm 2\pi\sqrt{-1})^{i-1} \mathrm{dlog} f_i \wedge \ldots \wedge \mathrm{dlog} f_n \cdot \delta_{T_{f_1} \cap \ldots \cap T_{f_{i-1}}}$$

$$- \sum_{i=1}^{n} (\pm 2\pi\sqrt{-1})^{i} \mathrm{dlog} f_{i+1} \wedge \ldots \wedge \mathrm{dlog} f_n \cdot \delta_{T_{f_1} \cap \ldots \cap T_{f_i}}$$

$$= \, \mathrm{dlog} f_1 \wedge \ldots \wedge \mathrm{dlog} f_n \, - \, (2\pi\sqrt{-1})^n \, \delta_{T_{f_1} \cap \ldots \cap T_{f_n}} \, = \, \Omega_{\mathbf{f}} \, - \, (2\pi\sqrt{-1})^n T_{\mathbf{f}}.$$

Assuming $n > d$, $\Omega_{\mathbf{f}} = 0$ and $\partial^{-1}_{(X,V)} T_{\mathbf{f}}$ exists, so that the $(n-1)$-current

$$R'_{\mathbf{f}} = R_{\mathbf{f}} + (2\pi\sqrt{-1})^n \, \delta_{\partial^{-1} T_{\mathbf{f}}} \in \Gamma({}'\mathcal{D}^{n-1}_X)$$

now has $\mathrm{d}[R'_{\mathbf{f}}] \in \Gamma({}'\mathcal{D}^n_{V^\infty})$ and so is d-closed in the complex $\Gamma({}'\mathcal{D}^\bullet_{(X \setminus V)^\infty})$. Therefore we have a class $\in H^{n-1}(X - V, \mathbb{C})$, well-defined up to classes generated by currents $\left\{ (2\pi\sqrt{-1})^n \, \delta_{\mathcal{C}} \middle| \, \mathcal{C} \in H_{n-1}(X, V; \mathbb{Z}) \right\}$.

PROPOSITION 2.2.1. *(a) For $n > d$, sending $\mathbf{f} \mapsto R'_{\mathbf{f}}$ gives a well-defined map*

$$R' : \otimes^n \mathbb{Z}[\mathbb{P}^1_{\mathbb{C}(X)} \setminus \{0, \infty\}] \to H^{n-1}(\eta_X, \mathbb{C}/\mathbb{Z}(n)) \cong \frac{H^{2d-n+1}(X_{rel}, \mathbb{C})^\vee}{im\,\{H_{n-1}(X_{rel}, \mathbb{Z}(n))\}}.$$

(b) This map factors through $K^M_n(\mathbb{C}(X))$.

Most of the remainder of the section is devoted to showing (b), by writing $R'_{\mathbf{f}}$ (for $\mathbf{f}$ a Steinberg) as a coboundary $\mathrm{d}[S]$ in the quotient complex $\Gamma({}'\mathcal{D}^\bullet_{(X \setminus V)^\infty})$. Note that in $\Gamma({}'\mathcal{D}^\bullet_X)$, the "residues" $\mathrm{d}[R'_{\mathbf{f}}]$ and $\mathrm{d}[S] - R'_{\mathbf{f}}$ are in general *not* zero, and are considered in the next section; knowing residues of

S in particular is useful for our discussion of relative regulators in Chapter 5. In any case, for ω d- closed with c.s. on X^ϵ,

$$R'_{\mathbf{f}} = \mathrm{d}[S] \implies \int_X R'_{\mathbf{f}} \wedge \omega = \int_X \mathrm{d}[S] \wedge \omega := \pm \int_X S \wedge \mathrm{d}\omega = 0.$$

2.2.2. Explicit coboundary currents.

CASE 1. $\mathbf{f} = f \otimes (1-f) \otimes \mathbf{g}$.

As in §2.1, $T_f \cap T_{1-f} \subset V_{\mathbf{f}}$ so (we may choose)

$$R'_{\mathbf{f}} = \log f \,\mathrm{dlog}(1-f) \wedge \mathrm{dlog} g_1 \wedge \ldots \wedge \mathrm{dlog} g_{n-2}$$
$$+(-1)^{n-1} 2\pi\sqrt{-1}\log(1-f)\mathrm{dlog} g_1 \wedge \ldots \wedge \mathrm{dlog} g_{n-2} \cdot \delta_{T_f}.$$

As a 0-current[7], $\mathrm{Li}_2(x)$ may be differentiated (using FTC and the discussion in §2.1) $\mathrm{d}[\mathrm{Li}_2(x)] =$

$$\mathrm{d}\left[-\int_0^x \log(1-z)\mathrm{dlog} z\right] = -\log(1-x)\mathrm{dlog} x + 2\pi\sqrt{-1}\log x \cdot \delta_{T_{1-z}}.$$

Setting

$$S = -\mathrm{Li}_2(1-f) \bigwedge^{n-2} \mathrm{dlog}\mathbf{g},$$

we have (on $X \setminus V$) $\mathrm{d}[S] =$

$$\log f \,\mathrm{dlog}(1-f) \wedge \bigwedge^{n-2} \mathrm{dlog}\mathbf{g} - 2\pi\sqrt{-1}\log(1-f) \cdot \delta_{T_f} \wedge \bigwedge^{n-2} \mathrm{dlog}\mathbf{g} = R'_{\mathbf{f}}.$$

CASE 2. $\mathbf{f} = \mathbf{g} \otimes f_1 f_2 - \mathbf{g} \otimes f_1 - \mathbf{g} \otimes f_2$.

Again we must show the regulator-current's cohomology class is invariant with respect to branch change in $f = f_1 f_2$. The $2d$-chain Δ_f of §2.1 translates to a 0-current with

$$\mathrm{d}[\Delta_f] = \delta_{T_f} - \delta_{{}'T_f} = \delta_{T_{f_1 f_2}} - \delta_{T_{f_1}} - \delta_{T_{f_2}}.$$

Notice that most of the terms in $R_{\mathbf{f}} = R_{\mathbf{g}\otimes f_1 f_2} - R_{\mathbf{g}\otimes f_1} - R_{\mathbf{g}\otimes f_2} =$

$$\log g_1 \,\mathrm{dlog} g_2 \wedge \ldots \wedge \mathrm{dlog} g_{n-1} \wedge (\mathrm{dlog} f_1 f_2 - \mathrm{dlog} f_1 - \mathrm{dlog} f_2) + \ldots$$
$$+(\pm 2\pi\sqrt{-1})^{n-1}(\log f_1 f_2 - \log f_1 - \log f_2) \cdot \delta_{T_{g_1} \cap \ldots \cap T_{g_{n-1}}}$$

[7]Incidentally, one can view all the polylogarithms as single-valued functions with a branch cut at $T_{1-z} = [1, \infty] \in \mathbb{R}^+$, and so as 0-currents on $\mathbb{P}^1$. We write $\mathrm{Li}_1(x) = -\log(1-x)$ in our preferred branch ($-\pi \leq \Im\{\mathrm{Li}_1(x)\} \leq \pi$), and thereby iteratively obtain preferred branches of all

$$\mathrm{Li}_n(x) = \int_0^x \mathrm{Li}_{n-1}(t)\mathrm{dlog} t,$$

by integrating away from T_{1-z}. The general differentiation rule (using the monodromy result in **[Ha]**) is then

$$d[\mathrm{Li}_n(x)] = \mathrm{Li}_{n-1}(x)\mathrm{dlog} x + \frac{2\pi\sqrt{-1}}{(n-1)!}\log^{n-1} x \cdot \delta_{T_{1-z}}$$

(where $\log^{n-1} x \in \mathbb{R}$ along T_{1-z}).

$$= \pm(2\pi\sqrt{-1})^n \Delta_f \cdot \delta_{T_{g_1}\cap\ldots\cap T_{g_{n-1}}}$$

cancel since $\mathrm{dlog} f_1 f_2 = \mathrm{dlog} f_1 + \mathrm{dlog} f_2$ exactly; by choosing

$$\partial^{-1}_{(X,V)}\left(T_{g_1}\cap\ldots\cap T_{g_{n-1}}\cap(T_f - {}'T_f)\right) = T_{g_1}\cap\ldots\cap T_{g_{n-1}}\cap\Delta_f$$

we may write

$$R'_{\mathbf{f}} = (2\pi\sqrt{-1})^n \Delta_f \cdot \delta_{T_{g_1}\cap\ldots\cap T_{g_{n-1}}} - (2\pi\sqrt{-1})^n \delta_{T_{g_1}\cap\ldots\cap T_{g_{n-1}}\cap\Delta_f} = 0.$$

Of course for the branch-change occurring in another factor of the $\otimes$-product, we can define an S using Δ_f (or just Case 3 below).

Case 3. Alternation is more involved. We do increasing levels of difficulty.

$\mathbf{n=2}$: $\mathbf{f} = f\otimes g + g\otimes f$. On $X - V$

$$R'_{\mathbf{f}} = \log f\, \mathrm{dlog} g - (2\pi\sqrt{-1})\log g\cdot\delta_{T_f} + \log g\,\mathrm{dlog} f - (2\pi\sqrt{-1})\log f\cdot\delta_{T_g}$$

$$= \mathrm{d}\left[\log f\ \log g\right]$$

$\mathbf{n=3}$: $\mathbf{f} = f\otimes g\otimes h + g\otimes f\otimes h \implies R'_{\mathbf{f}} = \mathrm{d}\left[\log f\ \log g\,\mathrm{dlog} h\right]$

$$\mathbf{f} = f\otimes g\otimes h + f\otimes h\otimes g \implies R'_{\mathbf{f}} = \mathrm{d}\left[2\pi\sqrt{-1}\ \log g\ \log h\cdot\delta_{T_f}\right]$$

$$\mathbf{f} = f\otimes g\otimes h + h\otimes g\otimes f \implies$$

$$R'_{\mathbf{f}} = \log f\,\mathrm{dlog} g\wedge\mathrm{dlog} h + (2\pi\sqrt{-1})\log g\,\mathrm{dlog} h\cdot\delta_{T_f} - 4\pi^2\log h\cdot\delta_{T_f\cap T_g}$$

$$+\log h\,\mathrm{dlog} g\wedge\mathrm{dlog} f + (2\pi\sqrt{-1})\log g\,\mathrm{dlog} f\cdot\delta_{T_h} - 4\pi^2\log f\cdot\delta_{T_h\cap T_g}$$

$$= \mathrm{d}\left[-\log f\ \log h\,\mathrm{dlog} g + 2\pi\sqrt{-1}\log f\ \log g\cdot\delta_{T_h} + 2\pi\sqrt{-1}\log h\log g\cdot\delta_{T_f}\right].$$

$\mathbf{n>3}$: $\mathbf{f} = f_1\otimes\ldots\otimes f_i\otimes\ldots\otimes f_j\otimes\ldots\otimes f_n + f_1\otimes\ldots\otimes f_j\otimes\ldots\otimes f_i\otimes\ldots\otimes f_n$.
Then on $X\setminus V_{\mathbf{f}}$, $R'_{\mathbf{f}} = \mathrm{d}[S]$ (choosing $\partial^{-1}_{(X,V)}T_{\mathbf{f}} = \partial^{-1}0 = 0$) where

$$S_{ij} = (2\pi\sqrt{-1})^{i-1}\log f_i\ \log f_j\,\mathrm{dlog} f_{i+1}\wedge\ldots\wedge\mathrm{dlog} f_{j-1}\wedge\mathrm{dlog} f_{j+1}\wedge\ldots\wedge\mathrm{dlog} f_n\cdot\delta_{T_{f_1}\cap\ldots\cap T_{f_{i-1}}}$$

$$+(2\pi\sqrt{-1})^i\Big[\log f_i\ \log f_{i+1}\,\mathrm{dlog} f_{i+2}\wedge\ldots\wedge\mathrm{dlog} f_{j-1}\wedge\mathrm{dlog} f_{j+1}\wedge\ldots\wedge\mathrm{dlog} f_n\cdot\delta_{T_{f_1}\cap\ldots\cap T_{f_{i-1}}\cap T_{f_j}}$$

$$+\log f_j\ \log f_{i+1}\,\mathrm{dlog} f_{i+2}\wedge\ldots\wedge\mathrm{dlog} f_{j-1}\wedge\mathrm{dlog} f_{j+1}\wedge\ldots\wedge\mathrm{dlog} f_n\cdot\delta_{T_{f_1}\cap\ldots\cap T_{f_{i-1}}\cap T_{f_i}}\Big]$$

$$+\ \cdot\ \cdot\ \cdot\ +$$

$$(2\pi\sqrt{-1})^{j-2}\Big[\log f_i\ \log f_{j-1}\,\mathrm{dlog} f_{j+1}\wedge\ldots\wedge\mathrm{dlog} f_n\cdot\delta_{T_{f_1}\cap\ldots\cap T_{f_{i-1}}\cap T_{f_j}\cap T_{f_{i+1}}\cap\ldots\cap T_{f_{j-2}}}$$

$$\log f_j\ \log f_{j-1}\,\mathrm{dlog} f_{j+1}\wedge\ldots\wedge\mathrm{dlog} f_n\cdot\delta_{T_{f_1}\cap\ldots\cap T_{f_{i-1}}\cap T_{f_i}\cap T_{f_{i+1}}\cap\ldots\cap T_{f_{j-2}}}\Big]$$

We mention one consequence of CASE 3 before proceeding. If

$$Alt_n(\mathbf{f}) := \sum_{\sigma\in\mathcal{S}_n} \mathrm{sgn}(\sigma)\cdot f_{\sigma(1)}\otimes\ldots\otimes f_{\sigma(n)}$$

then we may alternate the regulator current for free – that is, there must exist a current S so that on $X\setminus V_{\mathbf{f}}$ (for appropriate choice of ∂^{-1})

$$n!\,R'_{\mathbf{f}} - R'_{Alt_n(\mathbf{f})} = d[S].$$

This is easier said than done, even for $n=3$ and $\mathbf{f}=f\otimes g\otimes h$. The reader is warmly invited to check that

$$S = 6\pi\sqrt{-1}\log g\,\log h\cdot\delta_{T_f} + 2\log f\,\log g\,\mathrm{dlog}h - 2\log f\,\log h\,\mathrm{dlog}g$$

$$+2\pi\sqrt{-1}\log f\,\log g\cdot\delta_{T_h} - 2\pi\sqrt{-1}\log f\,\log h\cdot\delta_{T_g}$$

does the job. It is instructive to look at the residues of S; bearing in mind that on X we have $\mathrm{d}[\delta_{T_f}]=\delta_{(f)}$ and $\mathrm{d}[\mathrm{dlog}f]=2\pi\sqrt{-1}\,\delta_{(f)}$,

$$\mathrm{d}[S] - R'_{\{n!\mathbf{f}-Alt_n\mathbf{f}\}} =$$

$$6\pi\sqrt{-1}\left\{\log g\,\log h\cdot\delta_{(f)} + \log f\,\log g\cdot\delta_{(h)} - \log f\,\log h\cdot\delta_{(g)}\right\}.$$

The symmetry here is strongly connected to the way we shall use the "Levine lemma" (Proposition 1.2.1) in §3.1.

So we have our map

$$R':\ K_n^M(\mathbb{C}(X))\to H^{n-1}(\eta_X,\mathbb{C}/\mathbb{Z}(n))$$

in the event that $n>d$ so that $R'_{\mathbf{f}}$ is indeed closed. In the event that $n\leq d$, we have first

$$\Omega:\ K_n^M(\mathbb{C}(X))\to H^n(\eta_X,\mathbb{Z}(n))\cap F^nH^n(\eta_X,\mathbb{C})$$

$$\mathbf{f}\longmapsto\Omega_{\mathbf{f}} := \bigwedge^n \mathrm{dlog}f$$

which may be thought of as a "holomorphic n-current" on X (closed on $X\setminus V$). Then R' is defined on $\ker(\Omega)$ and altogether we get a map

$$(\Omega,R'):\ K_n^M(\mathbb{C}(X))\to H^n_{\mathcal{D}}(\eta_X,\mathbb{Z}(n)).$$

The overall picture is simplified if we consider the currents as functionals on $H_{n-1}(X-V,\mathbb{Z})$ via integration over singular cycles $\mathcal{C}$. Assume once again $n>\dim X$. If $[R'_{\mathbf{f}}]$ is "trivial", i.e. $\in im\left\{H_{n-1}(X,V;\mathbb{Z}(n))\hookrightarrow H^{n-1}(X-V,\mathbb{C})\right\}$, then while $\int_X R'_{\mathbf{f}}\wedge\omega$ (for an individual ω) can be anything, $\int_{\mathcal{C}} R'_{\mathbf{f}}$ is necessarily in $\mathbb{Z}(n)$; so we may think of $H^{n-1}(\eta_X,\mathbb{C}/\mathbb{Z}(n))$ as $hom\left(H_{n-1}(\eta_X,\mathbb{Z}),\mathbb{C}/\mathbb{Z}(n)\right)$. In fact, in integration we may altogether drop the last term of $R'_{\mathbf{f}}$, as

$$"\int_{\mathcal{C}}(2\pi\sqrt{-1})^n\partial^{-1}_{(X,V)}T_{\mathbf{f}}"$$

is just an intersection number $\in\mathbb{Z}(n)$. So $[R_{\mathbf{f}}]$ alone gives the desired cohomology class, from this perspective.

For $n \leq \dim X$, $\mathrm{d}[R_{\mathbf{f}}] = \Omega_{\mathbf{f}} - (2\pi\sqrt{-1})^n T_{\mathbf{f}}$; therefore we may think of $\int_{\mathcal{C}} R_{\mathbf{f}}$ as giving a $\mathbb{C}/\mathbb{Z}(n)$-valued holomorphic differential character. This is a functional χ on cycles (not cycle-classes) which obeys the rule

$$\mathcal{C} = \partial\zeta \text{ on } X - V \implies \chi(\mathcal{C}) \equiv \int_{\zeta} \Omega \mod (2\pi\sqrt{-1})^n\mathbb{Z},$$

for some fixed $\Omega \in \Omega^n(X \setminus V)$. Such objects give classes in $H^n_{\mathcal{D}}(\eta_X, \mathbb{Z}(n))$, see **[Ga]** or (essentially) §2.4.

Notice also that cycles $\mathcal{C}$ may be expressed as limits of (Poincaré dual) closed forms ω with c.s. on X^{ϵ}, and so whatever we prove for such forms also holds for cycles.

2.3. Residues in the Local-Global Setting

2.3.1. Res^1 and Tame^1 in the context of currents. We first show how to compute codimension-1 residues of $R'_{\mathbf{f}}$, using the language of §2.2, then introduce the relevant local-global spectral sequences for computing higher residues. We initially work with a particular (convenient) choice of $R'_{\mathbf{f}}$, ignoring the $\mathbb{Z}(n)$-ambiguities in the regulator and its residues, then show how to reinstate them in a way compatible with the local-global setting.[8]

For any $\mathbf{f}$ we may write $\mathbf{f} = \mathbf{f}_0 + \mathbf{g}$, $\mathbf{g}$ a Steinberg and $\mathbf{f}_0$ "good" in the sense that $\overline{\gamma_{\mathbf{f}_0}} \in Z^n(X, n)$. Trivially $R_{\mathbf{f}} = R_{\mathbf{f}_0} + R_{\mathbf{g}}$ on $X \setminus V_{\mathbf{f}}$, enlarging $V_{\mathbf{f}}$ if necessary to include $V_{\mathbf{f}_0}$ and $V_{\mathbf{g}}$. One may then choose simply $\partial^{-1}_{(X,V_{\mathbf{f}})} T_{\mathbf{f}} = \partial^{-1}_{(X,V_{\mathbf{f}_0})} T_{\mathbf{f}_0} + \partial^{-1}_{(X,V_{\mathbf{g}})} T_{\mathbf{g}}$ so that $R'_{\mathbf{f}} = R'_{\mathbf{f}_0} + R'_{\mathbf{g}}$, so that $[R'_{\mathbf{f}}] \equiv [R'_{\mathbf{f}_0}] + [R'_{\mathbf{g}}] \in H^{n-1}(X - V_{\mathbf{f}}, \mathbb{C})$ (where e.g. $[R'_{\mathbf{f}_0}]$ is the image of the class under $H^{n-1}(X - V_{\mathbf{f}_0}) \to H^{n-1}(X - V_{\mathbf{f}})$). Now we know that the class $[R'_{\mathbf{g}}]$ is zero (for the right choice of ∂^{-1}), and so $[R'_{\mathbf{f}}] \equiv [R'_{\mathbf{f}_0}]$ and the classes of their residues *must* be equal. Formally, if $R'_{\mathbf{g}} = \mathrm{d}[S]$ on $X - V_{\mathbf{g}}$, we take (in the spirit of §1.4) a d-closed form β on $V_{\mathbf{f}}$ compactly supported away from $W_{\mathbf{f}}$; we may then arrange an extension $\tilde{\beta}$ to $N_{2\epsilon_1}(V_{\mathbf{f}})$ so that $\mathrm{d}\tilde{\beta}$ is supported away from $N_{\epsilon}(V_{\mathbf{f}}) \supseteq N_{\epsilon}(V_{\mathbf{g}})$, and compute

$$\pm \int_X \mathrm{d}[R'_{\mathbf{g}}] \wedge \tilde{\beta} := \int_X R'_{\mathbf{g}} \wedge \mathrm{d}\tilde{\beta} = \int_{X \setminus N_{\epsilon}(V_{\mathbf{g}})} R'_{\mathbf{g}} \wedge \mathrm{d}\tilde{\beta} = \int_{X \setminus N_{\epsilon}(V_{\mathbf{g}})} \mathrm{d}[S] \wedge \mathrm{d}\tilde{\beta}$$

$$= \int_{N_{\epsilon}(V_{\mathbf{g}})} S \wedge \mathrm{d}\tilde{\beta} = 0.$$

This does *not* prove that the residues of $R'_{\mathbf{g}}$ are "physically zero", but that they are cohomologically so (and one could do this for all Res^i below, so that $\mathrm{d}[R'_{\mathbf{g}}]$ is cohomologous to zero on $V_{\mathbf{f}}$, not just $V_{\mathbf{f}} \setminus W_{\mathbf{f}}$). The main point is

[8]Note one slight shift to simplify notation: here V^i shall mean what $'V^i$ did in §1.4. Namely, $V^2_{\mathbf{f}} (= W)$ consists of *all* codimension-1 intersections (and self-intersections) of all components of $V^1_{\mathbf{f}} (= V)$, then $V^3_{\mathbf{f}} (= Y)$ is derived from $V^2_{\mathbf{f}}$ in the same way, and so on.

now that the residues of $R'_{\mathbf{f}_0}$ (which has only log singularities) are "directly" computable unlike those of $R'_{\mathbf{f}}$.

EXAMPLE 2.3.1. On a curve, let $\mathbf{f} = f \otimes f$. Then

$$R_{\mathbf{f}} = \log f \mathrm{dlog} f \; - \; 2\pi\sqrt{-1}(\log^{[+]} f) \cdot \delta_{T_f}$$

is not convenient to work with. Thus we write

$$\mathbf{f} = \; f \otimes (-1) \; + \; [f \otimes f - f \otimes (-1)] \; =: \mathbf{f}_0 + \mathbf{g},$$

so that (with standard branches of log $\implies$ arg $\in (-\pi i, \pi i]$

$$R_{\mathbf{f}_0} = \; \log f \mathrm{dlog}(-1) \; - \; 2\pi\sqrt{-1} \log(-1) \cdot \delta_{T_f} \; = \; 2\pi^2 \delta_{T_f}$$

and

$$\mathrm{d}[R'_{\mathbf{f}_0}] = 2\pi\sqrt{-1} \left(-\pi\sqrt{-1} \cdot \delta_{(f)} \right).$$

As for

$$R_{\mathbf{g}} \; = \; \log f \, \mathrm{dlog} f \; - \; 2\pi\sqrt{-1} \log f \cdot \delta_{T_f} \; - \; 2\pi^2 \cdot \delta_{T_f},$$

across T_f the change in $\log^2 f$ is

$$(\log f - 2\pi\sqrt{-1})(\log f - 2\pi\sqrt{-1}) - \log^2 f \; = \; -4\pi\sqrt{-1} \log f \; - 4\pi^2$$

so that

$$\mathrm{d}\left[\frac{1}{2}\log^2 f\right] = R_{\mathbf{g}}$$

on $X \setminus |(f)|$.

Recall that in §1.4 we computed residues of AJ on a "good" graph like $\gamma_{\mathbf{f}_0}$; that result gives the second equality in

$$\pm \int_X \mathrm{d}[R'_{\mathbf{f}_0}] \wedge \tilde{\beta} \; := \; \int_{N_{2\epsilon_1}(V)} R'_{\mathbf{f}_0} \wedge \mathrm{d}\tilde{\beta} \; = \; \int_V R'_{\partial \mathbf{f}_0} \wedge \beta.$$

However, in the present setting we can see equality of the first and last terms more immediately. To start with, suppose on a surface $\mathbf{f}_0 = f \otimes g \otimes h$ (adding more terms changes nothing) so that all components of $|(f)|$, $|(g)|$, $|(h)|$ are distinct, and compute on X (not $X - V$)

$$\mathrm{d}[R_{\mathbf{f}_0}] \; = \; \mathrm{d}\left[\log f \, \mathrm{dlog} g \wedge \mathrm{dlog} h \; + \; 2\pi\sqrt{-1} \log g \, \mathrm{dlog} h \cdot \delta_{T_f} \; - \; 4\pi^2 \log h \cdot \delta_{T_f \cap T_g}\right]$$

$$= \; 2\pi\sqrt{-1} \left\{ \left(\log f \mathrm{dlog} g \cdot \delta_{(h)} - \log f \mathrm{dlog} h \cdot \delta_{(g)} \right) + \left(\log g \mathrm{dlog} h \cdot \delta_{(f)} - 2\pi\sqrt{-1} \log g \cdot \delta_{T_f} \cdot \delta_{(h)} \right) \right.$$

$$\left. + \left(2\pi\sqrt{-1} \log h \cdot \delta_{T_f} \cdot \delta_{(g)} - 2\pi\sqrt{-1} \log h \cdot \delta_{T_g} \cdot \delta_{(f)} \right) \right\} \; + \; 8\pi^3\sqrt{-1} \, \delta_{T_f \cap T_g \cap T_h}$$

$$= 2\pi\sqrt{-1} \times \left\{ \left(\log f \mathrm{dlog} g - 2\pi\sqrt{-1} \, \log g \cdot \delta_{T_f} \right) \cdot \delta_{(h)} - \left(\log f \mathrm{dlog} h - 2\pi\sqrt{-1} \log h \cdot \delta_{T_f} \right) \cdot \delta_{(g)} \right.$$

$$\left. \left(\log g \mathrm{dlog} h - 2\pi\sqrt{-1} \log h \cdot \delta_{T_g} \right) \cdot \delta_{(f)} \right\} \; + \; 8\pi^3\sqrt{-1} \, T_{\mathbf{f}_0}.$$

More generally, for $\mathbf{f}_0 = f_1 \otimes \ldots \otimes f_n$, on X

$$\mathrm{d}[R_{\mathbf{f}_0}] = \mathrm{d}\left[\sum_{i=1}^{n}(\pm 2\pi\sqrt{-1})^{i-1}\log f_i \mathrm{dlog} f_{i+1}\wedge\ldots\wedge\mathrm{dlog} f_n\cdot\delta_{T_{f_1}\cap\ldots\cap T_{f_{i-1}}}\right]$$

$$= \sum_{\ell=1}^{n}\left[\sum_{i=1}^{\ell-1}(\pm 2\pi\sqrt{-1})^{i-1}2\pi i(-1)^{\ell-i-1}\log f_i\mathrm{dlog} f_{i+1}\wedge\ldots\wedge\widehat{\mathrm{dlog} f_\ell}\wedge\ldots\wedge\mathrm{dlog} f_n\cdot\delta_{T_{f_1}\cap\ldots\cap T_{f_{i-1}}}\right.$$

$$\left.+\sum_{i=\ell+1}^{n}(2\pi\sqrt{-1})^{i-1}(-1)^{n+\ell-i-1}\log f_i\mathrm{dlog} f_{i+1}\wedge\ldots\wedge\mathrm{dlog} f_n\cdot\delta_{T_{f_1}\cap\ldots\cap\widehat{T_{f_\ell}}\cap\ldots\cap T_{f_{i-1}}}\right]\cdot\delta_{(f_\ell)}$$

$$\pm(2\pi\sqrt{-1})^n T_{\mathbf{f}_0} \quad = \quad 2\pi\sqrt{-1}\sum_{\ell=1}^{n}(-1)^\ell R_{f_1\otimes\ldots\otimes\widehat{f_\ell}\otimes\ldots\otimes f_n}\cdot\delta_{(f_\ell)} \pm (2\pi\sqrt{-1})^n T_{\mathbf{f}_0}.$$

Now (see §1.4 for the picture)

$$\partial_X\left(\partial^{-1}_{(X,V)}T_{\mathbf{f}_0}\right) = T_{\mathbf{f}_0} - \sum_\ell(-1)^\ell\partial^{-1}_{(V_\ell,W)}T_{\partial_\ell\mathbf{f}_0}$$

so that reinstating $(2\pi\sqrt{-1})^n\delta_{\partial^{-1}T_{\mathbf{f}_0}}$ in $R_{\mathbf{f}_0}$

$$\mathrm{d}[R'_{\mathbf{f}_0}] = 2\pi\sqrt{-1}\sum_{\ell=1}^{n}(-1)^\ell R'_{\partial_\ell\mathbf{f}_0}\cdot\delta_{(f_\ell)} \qquad (*)$$

gives classes $\in H^{n-2}(V\backslash W,\mathbb{C}) = \oplus_\ell H^{n-2}(V_\ell\backslash W,\mathbb{C})$ which are cohomologous to the residues of $R'_{\mathbf{f}}$. In fact, since the map $\mathbf{f}_0 \mapsto \partial_i\mathbf{f}_0$ on good $\mathbf{f}_0$ generates $\mathrm{Tame} : K_n^M(\mathbb{C}(X)) \to \coprod_{x\in X^1} K^M_{n-1}(\mathbb{C}(x))$, one may write

$$\frac{1}{2\pi\sqrt{-1}}\mathrm{d}[R'_{\mathbf{f}}] \equiv R'_{\mathrm{Tame}\mathbf{f}}\cdot\delta_V \in H^{n-2}(V\setminus W,\mathbb{C}/\mathbb{Z}(n-1))$$

where Tame$\mathbf{f}$ means any representative (e.g., $T(\mathbf{f})$ in the §1.2 footnote).[9]

PROPOSITION 2.3.2. *We therefore have a commutative diagram ($n > d$)*

$$\begin{tikzcd}
K_n^M(\mathbb{C}(X)) \arrow[r, "R'"] \arrow[d, "T"] & H^{n-1}(\eta_X,\mathbb{C}/\mathbb{Z}(n)) \arrow[d, "\frac{Res^1}{2\pi\sqrt{-1}}"] \\
\coprod_{x\in X^1} K^M_{n-1}(\mathbb{C}(x)) \arrow[r, "\coprod R'"] & \coprod_{x\in X^1} H^{n-2}(\eta_x,\mathbb{C}/\mathbb{Z}(n-1))
\end{tikzcd}$$

[9]However for computing higher Res^i this is *not* sufficiently sensitive – we must use $\mathbf{f}_0$ and (*) above. (That is, while the Res^i are invariant with respect to choice of $\mathbf{g}$ on $X\setminus V$, we are *not* free to modify by Steinbergs on $V\setminus W$ unless we are interested only in Res^1 as is the case here.)

2.3.2. Higher residues of currents [dual to section (1.4.4)]. Following a suggestion in [**F**], we now construct a local-global picture which will turn out to be dual to that of §1.4. We add one more inclusion to those employed there:

$$\jmath^{(i)} : X - V_{\mathbf{f}}^{i} \hookrightarrow X\,, \quad \jmath^{(i,i+1)} : V_{\mathbf{f}}^{i} \setminus V_{\mathbf{f}}^{i+1} \hookrightarrow V_{\mathbf{f}}^{i},$$

$$\iota_{(i)} : V_{\mathbf{f}}^{i} \hookrightarrow X, \quad \iota_{(i+1,i)} : V_{\mathbf{f}}^{i+1} \hookrightarrow V_{\mathbf{f}}^{i}.$$

Pushing forward by ι we have the exact sequence of sheaves

$$\begin{array}{ccccccc} 0 \to \iota_*^{(i+1)}\mathbb{C} \to \iota_*^{(i)}\mathbb{C} \to & \iota_*^{(i)}\mathbb{C}/\iota_*^{(i+1)}\mathbb{C} & \to 0 \\ & \iota_*^{(i)} \uparrow\cong & \\ & \jmath_*^{(i,i+1)}\mathbb{C} & \end{array}$$

in which we may resolve terms by

$$\begin{array}{cc} \iota_*^{(i)}\mathbb{C} \to {}'\mathcal{D}^\bullet_{(V_{\mathbf{f}}^{i})^\infty}\,, \quad \iota_*^{(i)}\mathbb{C}/\iota_*^{(i+1)}\mathbb{C} \to & {}'\mathcal{D}^\bullet_{(V_{\mathbf{f}}^{i})^\infty}/{}'\mathcal{D}^\bullet_{(V_{\mathbf{f}}^{i+1})^\infty} \\ & \iota_*^{(i)} \uparrow\simeq \\ & {}'\mathcal{D}^\bullet_{(V_{\mathbf{f}}^{i}\setminus V_{\mathbf{f}}^{i+1})^\infty}[-2i] \end{array}$$

where by ${}'\mathcal{D}^\bullet_{(V_{\mathbf{f}}^{i}\setminus V_{\mathbf{f}}^{i+1})^\infty} := {}'\mathcal{D}^\bullet_{V_{\mathbf{f}}^{i}} \Big/ \Big\{ {}'\mathcal{D}^\bullet_{(V_{\mathbf{f}}^{i+1})^\infty} \subset {}'\mathcal{D}^\bullet_{V_{\mathbf{f}}^{i}} \Big\}$ we actually mean (a quotient of) currents "on $V_{\mathbf{f}}^{i}$" and not "on X supported on $V_{\mathbf{f}}^{i}$",[10] which is what it is mapping to quasi-isomorphically (via $\iota_*^{(i)}$, which is essentially mutiplication by $\delta_{V_{\mathbf{f}}^{i}}$). The short exact sequence reflects an ascending filtration $(F_{p-1}^{N} \subset F_{p}^{N})$ by niveau

$${}'F_p^N\mathbb{C} = \iota_*^{(d-p)}\mathbb{C}, \quad {}'Gr_p^N\mathbb{C} = \frac{{}'F_p^N}{{}'F_{p-1}^N} \cong \jmath_*^{(d-p,d-p+1)}\mathbb{C}$$

(compare $Gr_p^N\mathbb{C} \cong \jmath_!^{(d-p,d-p+1)}\mathbb{C}$), and from the resolutions we get the initial exact triangles for the corresponding spectral sequence

$$\begin{array}{cc} \ldots \to H^*_{V_{\mathbf{f}}^{i+1}}(X,\mathbb{C}) \to H^*_{V_{\mathbf{f}}^{i}}(X,\mathbb{C}) \to & H^*_{V_{\mathbf{f}}^{i}\setminus V_{\mathbf{f}}^{i+1}}(X,\mathbb{C}) \to \ \ldots \\ & \| \\ & H^{*-2i}(V_{\mathbf{f}}^{i} - V_{\mathbf{f}}^{i+1},\mathbb{C}) \end{array}$$

The E_1-terms of the two spectral sequences are (computed *a priori* using the quotient sheaves)

$$E_1^{p,q} := H^{p+q}\left\{\Gamma\left(X,\, \jmath_!\Omega^\bullet_{(X\setminus V_{\mathbf{f}}^{d-p+1})^\infty} \Big/ \jmath_!\Omega^\bullet_{(X\setminus V_{\mathbf{f}}^{d-p})^\infty}\right)\right\} \underset{\jmath^*_{(d-p)}}{\overset{\cong}{\longrightarrow}}$$

$$H^{p+q}\left\{\Gamma\left(V_{\mathbf{f}}^{d-p},\, \jmath_!\Omega^\bullet_{(V_{\mathbf{f}}^{d-p}\setminus V_{\mathbf{f}}^{d-p+1})^\infty}\right)\right\} \cong H_c^{p+q}(V_{\mathbf{f}}^{d-p} - V_{\mathbf{f}}^{d-p+1},\mathbb{C})$$

[10] it helps to note that there are 2 different ${}'\mathcal{D}^\bullet_{(V_{\mathbf{f}}^{i+1})^\infty}$'s at work here, one in ${}'\mathcal{D}^\bullet_{(V_{\mathbf{f}}^{i})^\infty} \subset {}'\mathcal{D}^\bullet_{X}$, the other in ${}'\mathcal{D}^\bullet_{V_{\mathbf{f}}^{i}}$.

$$\cong H^{p+q}(V_{\mathbf{f}}^{d-p}, V_{\mathbf{f}}^{d-p+1}; \mathbb{C}) \implies Gr_p^N H^{p+q}(X, \mathbb{C}) = E_\infty^{p,q}$$

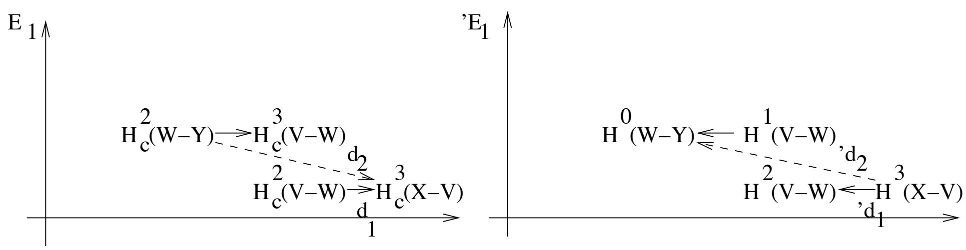
from §1.4 and from directly above (and somewhat unconventionally)

$$'E_1^{p,q} := H^{2d-(p+q)} \left\{ \Gamma \left(X, \, '\mathcal{D}^\bullet_{(V_{\mathbf{f}}^{d-p})\infty} \Big/ '\mathcal{D}^\bullet_{(V_{\mathbf{f}}^{d-p+1})\infty} \right) \right\} \underset{\iota_*^{(d-p)}}{\overset{\cong}{\longleftarrow}}$$

$$H^{p-q} \left\{ \Gamma \left(V_{\mathbf{f}}^i, \, '\mathcal{D}^\bullet_{(V_{\mathbf{f}}^{d-p} \setminus V_{\mathbf{f}}^{d-p+1})\infty} \right) \right\} \cong H^{p-q}(V_{\mathbf{f}}^{d-p} - V_{\mathbf{f}}^{d-p+1}, \mathbb{C})$$

$$\implies \; 'Gr_p^N H^{2d-(p+q)}(X, \mathbb{C}) = \, 'E_\infty^{p,q}$$

and none of this requires normal crossings. Clearly $'E_1^{p,q}$ and $E_1^{p,q}$ are (Lefschetz-)dually isomorphic by the usual pairing $\int_X (\cdot) \wedge (\cdot)$, and moreover (as we shall demonstrate for $i = 1, 2$) the $(')d_i$ of the spectral sequences are adjoint (as are $\iota^*_{(i)}$ and $\iota_*^{(i)}$) in this pairing. For $n = 4$, $d = 3$ one should have in mind the picture



where of course $Y \subset W \subset V \subset X$ are analytic subsets of increasing codimension (and Y is a collection of points).

We describe the $'d_i = \mathrm{Res}^i$ on currents for $n = 4$, X a 3-fold. $[R'_{\mathbf{f}}] \in H^3(X - V_{\mathbf{f}})$ lifts to a closed section $R'_{\mathbf{f}} \in \Gamma(X, \, '\mathcal{D}^3_X / '\mathcal{D}^3_{V\infty})$, thence to a non-closed section $\in \Gamma(X, \, '\mathcal{D}^3_X)$, so that

$$\mathrm{d}[R'_{\mathbf{f}}] \in \Gamma(X, \, '\mathcal{D}^4_{V\infty}) \to \Gamma(X, \, '\mathcal{D}^4_{V\infty} / '\mathcal{D}^4_{W\infty}).$$

Modulo "exact" currents in this last complex, the image (of $\mathrm{d}[R'_{\mathbf{f}}]$) lifts to

$$\widetilde{\mathrm{d}[R'_{\mathbf{f}}]} \in \Gamma(V, \, '\mathcal{D}^2_{(V \setminus W)\infty} = \, '\mathcal{D}^2_V / '\mathcal{D}^2_{W\infty})$$

under $\iota_*^{(1)}$; then $\mathrm{Res}^1[R'_{\mathbf{f}}]$ is the class of either the image or the lift, in $H^4_{V \setminus W}(X, \mathbb{C}) \cong H^2(V - W, \mathbb{C})$. Now

$$\tilde{\beta} \in \Gamma(X, \, \jmath_! \Omega^2_{(X \setminus W)\infty} / \jmath_! \Omega^2_{(X \setminus V)\infty})$$

was a lift of the (d-closed) form $\beta \in \Gamma(V, \, \jmath_! \Omega^2_{(V \setminus W)\infty})$ under $\iota^*_{(1)}$, which then got differentiated as a section $\Gamma(X, \, \jmath_! \Omega^2_{(X \setminus W)\infty})$ to give $\mathrm{d}\tilde{\beta} \in \Gamma(X, \, \jmath_! \Omega^3_{(X \setminus V)\infty})$

and a corresponding class in $H^3_c(X-V,\mathbb{C})$. The procedures are therefore exactly dual (or "adjoint") and

$$\pm\int_X R'_{\mathbf{f}}\wedge \mathrm{d}\tilde{\beta} \;=\; \int_X \mathrm{d}[R'_{\mathbf{f}}]\wedge\tilde{\beta} \;=\; \int_V \widetilde{\mathrm{d}[R'_{\mathbf{f}}]}\wedge\beta.$$

This is why we took the "lift"; although localized at V the second integral is still on X.

Remark 2.3.3. If the class of $\mathrm{d}[R'_{\mathbf{f}}]\in H^4\left\{\Gamma\left(X,{}'\mathcal{D}^\bullet_{V\infty}/{}'\mathcal{D}^\bullet_{W\infty}\right)\right\}$ is zero (i.e. Res^1 is trivial), then one does *not* "lift" it to V until arriving at Res^2 in codimension 2 – the sensitive "sub-cohomological" information must be retained along the way. The same goes for all Res^i; this is very similar to our use of Bloch's moving lemma in the local-global setting in §1.2.

For aid in computing ${}'d_2=\mathrm{Res}^2$ we may once again draw a diagram of "exact triangles" (where $[R'_{\mathbf{f}}]\in H^3(X-V)$),

$$\begin{tikzcd}
 & {} \arrow[r] & H^3(X-V) \arrow[r, "\Delta"] & H^4_V(X) \arrow[r] \arrow[d, equal, "{'d_1}"] & H^4(X) \arrow[r] & {} \\
{} \arrow[r] & H^1(V-W) \arrow[r, "\Delta"] & H^4_W(X) \arrow[r] \arrow[d, equal, "{'d_1}"', "{'d_2}"] & H^4_V(X) \arrow[r] & H^2(V-W) \arrow[r, "\Delta"] & {} \\
 & {} \arrow[r] & H^4_W(X) \arrow[r] & H^0(W-Y) \arrow[r, "\Delta"] & {} &
\end{tikzcd}$$

in which we have already traced through the top ${}'d_1$. If it is zero, i.e. $\mathrm{d}[R'_{\mathbf{f}}]$ is exact in $\Gamma(X,{}'\mathcal{D}^\bullet_{V\infty}/{}'\mathcal{D}^\bullet_{W\infty})$, then there exists $\mathrm{S}^1_V\in\Gamma(X,{}'\mathcal{D}^3_{V\infty})$ such that

$$\mathrm{d}[R'_{\mathbf{f}}]=\mathrm{d}[S^1_V]\qquad \mathrm{mod}\ {}'\mathcal{D}^4_{W\infty},$$

and we "take residues"

$$\mathrm{d}[R'_{\mathbf{f}}]-\mathrm{d}[S^1_V]\in\Gamma(X,{}'\mathcal{D}^4_{W\infty})\to\Gamma(X,{}'\mathcal{D}^4_{W\infty}/{}'\mathcal{D}^4_{Y\infty}).$$

Now it's o.k. to move the image by a coboundary and lift via $\iota_*^{(2)}$ to

$$\mathrm{d}[\widetilde{R'_{\mathbf{f}}-S^1_V}]\in\Gamma(W,{}'\mathcal{D}^0_W\,/\{{}'\mathcal{D}^0_{Y\infty}\subset{}'\mathcal{D}^0_W\})=\Gamma(W,{}'\mathcal{D}^0_{(W\setminus Y)\infty}).$$

We have to mod out by the image of the bottom ${}'d_1$ here: take a d-closed current in

$$R^1_V\in\Gamma(V,{}'\mathcal{D}^1_{(V\setminus W)\infty})\ \underset{\iota_*^{(1)}}{\longrightarrow}\ \Gamma(X,{}'\mathcal{D}^3_{V\infty}/{}'\mathcal{D}^3_{W\infty})$$

so that

$$\mathrm{d}[(\iota_*^{(1)})R^1_V]\in\Gamma(X,{}'\mathcal{D}^4_{W\infty})\to\Gamma(X,{}'\mathcal{D}^4_{W\infty}/{}'\mathcal{D}^4_{Y\infty}),$$

and we may move/lift this to $\widetilde{\mathrm{d}[R^1_V]} \in \Gamma(W, {}'\mathcal{D}^0_{(W\setminus Y)^\infty})$ with class in $H^0(W-Y)$. The difference was that in $V-W$, $\mathrm{d}[R^1_V]=0$ while $\mathrm{d}[S^1_V]=\mathrm{d}[R'_{\mathbf{f}}](\neq 0)$. So in brief one should (abusing notation slightly)[11] think of Res^2 as $\mathrm{Res}^1_W \circ \mathrm{d}^{-1} \circ \mathrm{Res}^1_V$, modulo the image of Res^1_W on "closed" currents on $V-W$. To show the desired "adjointness" of the $(')d_i$, we recall the explicit version of $d_2\eta$ from §1.4 and write (ignoring signs)

$$\int_X R'_{\mathbf{f}} \wedge d_2\eta \;=:\; \int_X R'_{\mathbf{f}} \wedge \mathrm{d}(\tilde{\eta}-\tilde{\beta}) \;=:\; \int_X \mathrm{d}[R'_{\mathbf{f}}] \wedge (\tilde{\eta}-\tilde{\beta})$$

$$= \int_X \mathrm{d}[R'_{\mathbf{f}} - S^1_V] \wedge (\tilde{\eta}-\tilde{\beta}) \;+\; \int_X \mathrm{d}[S^1_V] \wedge (\tilde{\eta}-\tilde{\beta}).$$

Since $\tilde{\beta}$ is supported away from W, this

$$= \int_X \mathrm{d}[R'_{\mathbf{f}} - S^1_V] \wedge \tilde{\eta} \;+\; \int_X S^1_V \wedge \mathrm{d}(\tilde{\eta}-\tilde{\beta})$$

$$= \int_X \widetilde{\mathrm{d}[R'_{\mathbf{f}} - S^1_V]} \wedge \eta \;=\; \int_W \mathrm{Res}^2[R'_{\mathbf{f}}] \wedge \eta.$$

Here the second term vanished because $S^1_V \in \Gamma({}'\mathcal{D}^2_{V^\infty})$, while $\iota^*_V\tilde{\eta} - \beta$ is d-closed on V so that $\mathrm{d}(\tilde{\eta}-\tilde{\beta}) \in \Gamma(\Omega^4_{(X\setminus V)^\infty})$. More generally for $\xi \in$ an appropriate subquotient of $H^{2d-n}_c(V^i_{\mathbf{f}} - V^{i+1}_{\mathbf{f}})$,

$$\int_X R'_{\mathbf{f}} \wedge d_i\xi \;=\; \int_{V^i_{\mathbf{f}}} \mathrm{Res}^i[R'_{\mathbf{f}}] \wedge \xi$$

where $\mathrm{Res}^i[R'_{\mathbf{f}}]$ lands in a subquotient of $H^{n-2i}(V^i_{\mathbf{f}} - V^{i+1}_{\mathbf{f}})$.

REMARK 2.3.4. One possibility is that $\{\mathbf{f}\} \in \ker(\mathrm{Tame})$ and $\mathrm{d}[R'_{\mathbf{f}}]$ is (along some components of V) trivialized on $V-W$ by an S^1_V of the sort encountered in §2.2 (CASE 1), so that Res^2 is dilogarithmic. More on this later; Res^i in general should be i-logarithmic in nature.

2.3.3. Lifting $[\mathbf{R}'_{\mathbf{f}}]$ to X. Reinstating the $\mathbb{Z}(\mathbf{n})$-ambiguities. Now replacing the right-hand column ${}'E_1^{d,q} \cong H^{d-q}(X-V,\mathbb{C})$ by zeroes we arrive at a spectral sequence converging to

$${}'E_\infty^{p,q}(V) \cong {}'Gr^N_p H_V^{2d-(p+q)}(X,\mathbb{C})$$

which is dually isomorphic to $Gr^N_p H^{p+q}(V)$. (Perhaps the notation $H^{*-2}(\hat{V}) := H^*_V(X)$, so that Poincaré duality $H^m(\hat{V}) = H^{m+2}_V(X) \cong H^{2d-2-m}(V)$ involves the dimension of V rather than X, makes more sense.) A simple

[11] This approach is essentially valid in the case of normal crossings and good $\mathbf{f}$, so one may use log-currents and build a double complex. In that case the "Res_W" at a component of W typically involves the difference of two residues from two components V_i and V_j. The computation then looks something like $\mathrm{Res}_{W_{ij}}\{\mathrm{d}^{-1}(\mathrm{Res}_{V_i}[R'_{\mathbf{f}}])\} - \mathrm{Res}_{W_{ij}}\{\mathrm{d}^{-1}(\mathrm{Res}_{V_j}[R'_{\mathbf{f}}])\}$, noting that while these two terms are not necessarily closed currents on W_{ij}, their difference imust be.

algebraic argument then gives the long exact sequence

$$\to H^*_V(X) \underset{\iota_*^{(1)}}{\overset{Gy}{\longrightarrow}} H^*(X) \underset{\jmath^*_{(1)}}{\longrightarrow} H^*(X-V) \underset{\text{Res}}{\longrightarrow} H^{*+1}_V(X) \to$$

in which the graded pieces of Res are exactly the Res^i. So if successive $\text{Res}^i[R'_{\mathbf{f}}]$ are all trivial, $[R'_{\mathbf{f}}]$ comes from $H^*(X)$, that is, gives a well-defined functional on

$$\text{im}\{H^*_c(X-V) \underset{\jmath_*^{(1)}}{\to} H^*(X)\} \cong H^*_c(X-V)/\ker(\jmath_*^{(1)}).$$

Actually lifting $R'_{\mathbf{f}}$ to $H^*(X)$ means going through the whole process of finding S^1_V as above, then S^2_W, and so on (there are different possible choices, too, and they produce ambiguities). Then $(R'_{\mathbf{f}} - S^1_V - S^2_W - \dots)$ gives a closed current on X, and the class is ambiguous by $\text{im}(Gy)$ just as completing a graph cycle (for $\mathbf{f}$ good) involves ambiguities. (Also as with completions, this may be very difficult to carry out, which is why we have done as much as possible over $X - V$.)

The trouble is now that one wants $R'_{\mathbf{f}}$ to come from $H^*(X)$ (mod $\mathbb{Z}(n)$) if the $\frac{1}{(2\pi\sqrt{-1})^i}\text{Res}^i$ are only trivial $\mod \mathbb{Z}(n-i)$; so we have to modify the above argument. There is an easy homology spectral sequence converging to $E^\infty_{p,q} \cong Gr^N_p H_{p+q}(X, \mathbb{Z})$ with

$$E^1_{p,q} = H_{p+q}(V^{d-p}, V^{d-p+1}; \mathbb{Z}),$$

given by filtering singular chains on X by dimension of support (in terms of the system of analytic subsets $X \supset V^1[= V] \supset V^2[= W] \supset \dots$); omitting the (first i) r.h. column(s) gives graded pieces of singular homology $H_{p+q}(V^{(i)}, \mathbb{Z})$.

Now relative singular chains include

$$H_{p+q}(V^{d-p}, V^{d-p+1}; \mathbb{Z}(n)) \cong (2\pi\sqrt{-1})^n E^1_{p,q} \hookrightarrow H^{p-q}(V^{d-p} - V^{d-p+1}, \mathbb{C}) \cong {}'E_1^{p,q}$$

both as currents and as functionals on

$$E_1^{p,q} \cong H_c^{p+q}(V^{d-p} - V^{d-p+1}, \mathbb{C}) \cong H^{p+q}(V^{d-p}, V^{d-p+1}; \mathbb{C}).$$

The terms of the relevant exact triangles (=long-exact sequences) also include, producing (by an easy diagram chase) a $\mathbb{C}/\mathbb{Z}(n)$ exact triangle; so one has a quotient spectral sequence with differentials $\frac{1}{(2\pi\sqrt{-1})^i}\text{Res}^i$ into subquotients of $H^{n-2i}(V^i_{\mathbf{f}} - V^{i+1}_{\mathbf{f}}, \mathbb{C}/\mathbb{Z}(n-i))$. If these vanish on $[R'_{\mathbf{f}}] \in H^{n-1}(X - V, \mathbb{C}/\mathbb{Z}(n))$, then $[R'_{\mathbf{f}}]$ lies in $\text{im}\left\{H^{n-1}(X, \mathbb{C}/\mathbb{Z}(n))\right\}$; to get $R'_{\mathbf{f}}$ then as a restriction of a closed current on X, we may have to change our choice of $R'_{\mathbf{f}}$ by an integral cycle on (X, V). Abstractly, the idea is that in the diagram

$$
\begin{tikzcd}
\longrightarrow H^{n-1}(X,\mathbb{C}/\mathbb{Z}(n)) \arrow[r] & H^{n-1}(X-V,\mathbb{C}/\mathbb{Z}(n)) \arrow[r, "\overline{\mathrm{Res}}"] & H^n_V(X,\mathbb{C}/\mathbb{Z}(n)) \longrightarrow & \\
\longrightarrow H^{n-1}(X,\mathbb{C}) \arrow[r] \arrow[u] & H^{n-1}(X-V,\mathbb{C}) \arrow[r, "\mathrm{Res}"] \arrow[u] & H^n_V(X,\mathbb{C}) \arrow[r] \arrow[u] & H^n(X,\mathbb{C}) \longrightarrow \\
 & \longrightarrow H_{2d-n+1}(X,V;\mathbb{Z}(n)) \arrow[r, "\partial"] \arrow[u, hook, "\text{P.D.}"'] & H_{2d-n}(V,\mathbb{Z}(n)) \arrow[r, "\iota_*"] \arrow[u, hook, "\text{P.D.}"'] & H_{2d-n}(X,\mathbb{Z}(n)) \longrightarrow \arrow[u, hook, "\text{P.D.}"']
\end{tikzcd}
$$

we suppose that an initial "choice" of $R'_{\mathbf{f}}$ gives a class in $H^{n-1}(X-V,\mathbb{C})$ with trivial $\overline{\mathrm{Res}}$. That is, $\mathrm{Res}[R'_{\mathbf{f}}] \in \mathrm{im}\,\{H_{2d-n}(V,\mathbb{Z}(n)) \to H^n_V(X,\mathbb{C})\}$ is [co]homologous (on V) to [the Poincaré dual of] a topological cycle $\mathcal{K}$ supported on V, $[\mathcal{K}] \in H_{2d-n}(V,\mathbb{Z}(n))$. By the diagram $[\mathcal{K}] \in \ker(\iota_*) = \mathrm{im}(\partial)$; modifying the original $R'_{\mathbf{f}}$ by the resulting $\partial^{-1}\mathcal{K}$ yields $R''_{\mathbf{f}} = R'_{\mathbf{f}} - \partial_X^{-1}\mathcal{K}$ with $\mathrm{Res}[R''_{\mathbf{f}}] = 0$, and so

$$[R''_{\mathbf{f}}] \in \mathrm{im}\left\{H^{n-1}(X,\mathbb{C}) \to H^{n-1}(X-V,\mathbb{C})\right\}.$$

In §2.4.2 we will find that for[12] $\{\mathbf{f}\} \in K_n^M(X)$, $[R'_{\mathbf{f}}] \in \ker(\overline{\mathrm{Res}})$; therefore we will have a map $K_n^M(X) \to \mathrm{im}\left\{H^{n-1}(X,\mathbb{C}/\mathbb{Z}(n)) \to H^{n-1}(\eta_X,\mathbb{C}/\mathbb{Z}(n))\right\}$ which will be referred to at times as the "holomorphic" Milnor regulator.

Explicitly, if $\overline{\mathrm{Res}}^1$ vanishes then (writing T and B for $(2\pi\sqrt{-1})^n\times$singular chains)

$$\mathrm{d}[R'_{\mathbf{f}}] = \mathrm{d}[S^1_V] + T^1_V \quad \text{on } X-W,$$

so that $[T^1_V]$ is trivial under $H_{2d-*}(X,W;\mathbb{Z}(n)) \to H^*(X-W,\mathbb{C})$ and injectivity of this map

$$\implies \quad T^1_V = \partial B^1_X \quad \mathrm{mod}\ W$$

(for some B^1_X). If $\overline{\mathrm{Res}}^2$ is trivial then

$$\mathrm{d}[(R'_{\mathbf{f}} - B^1_X) - S^1_V] = \mathrm{d}[S^2_W] + T^2_W \;\; \text{on } X-Y;$$

again $T^2_W = \partial B^2_X$ and so on, until

$$\mathrm{d}[(R'_{\mathbf{f}} - B^1_X - B^2_X - \dots) - (S^1_V + S^2_W + \dots)] = 0 \;\; \text{on } X;$$

and indeed the restriction of this to $X-V$ is $R'_{\mathbf{f}} - (B^1_X + B^2_X + \dots)$, a new "choice" of $R'_{\mathbf{f}}$.

[12]that is, for $[\gamma_{\mathbf{f}}]$ ∈the kernel of all Res^i on $CH^n(\eta_X,n)$, $[R'_{\mathbf{f}}]$ ∈kernel of all the $\overline{\mathrm{Res}}^i$ on currents. To see this we need the commutative diagram in §2.4.2, which is a byproduct of the maps $\mathcal{R}$ defined in §2.4.1.

2.3.4. Higher Tube maps on cycles and a simplified picture. In order to explain the original motivation for this section, we introduce one more spectral sequence, with

$$'E^1_{p,q} = H_{p-q}(V^{d-p} - V^{d-p+1}, \mathbb{Z}) \implies {}'E^{\infty}_{p,q} = {}'Gr^N_p H_{2d-(p+q)}(X, \mathbb{Z}),$$

to complete the local-global duality picture. Removing the d^{th} column computes not $'Gr^N_p H_{2(d-1)-(p+q)}(V)$ but graded pieces of something we will call $H_*(\hat{V})$. At least if $V_{\mathbf{f}}$ has normal crossings, it seems possible to produce $E^r_{p,q}$ and $'E^r_{p,q}$ via double complexes of singular chains, viz.

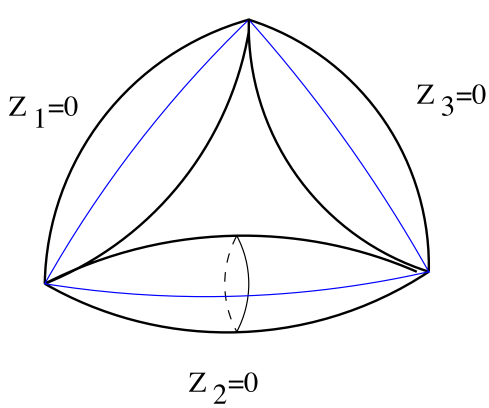
$$E^0_{p,q} := \mathcal{C}_{p+q}(V^{d-p}) \Big/ \mathcal{C}_{p+q}(V^{d-p+1}) \, , \quad 'E^0_{p,q} := \mathcal{C}_{p+q}(V^{d-p} - V^{d-p+1})$$

where the chains on the right have to be supported away from V^{d-p+1}, and the horizontal differentials $'E^0_{p,q} \to {}'E^0_{p+1,q}$ are Tube maps set up in such a way that Tube $\circ$ Tube $= 0$. $(')E^{\infty}$ then computes the homologies of the associated simple complexes, and the results are dual:

$$H_{2(d-1)-m}(\hat{V}) \cong H_m(V).$$

This is the right generalization of Poincaré duality to this kind of singular space (a union of divisors), and there is no need for intersection homology.

EXAMPLE 2.3.5. V =the "triangle" $\bigcup_{i=0}^2 \{z_i = 0\} \subset \mathbb{P}^2$, W =its vertices. Then we expect a duality between $H_1(\hat{V})$ and $H_1(V)$, given by $\cap$ on the components. The *actual triangle* given by the union of the three relative cycles $\subset (V_i, W)$ connecting $\{0\}$ and $\{\infty\}$ ought to be a singular cycle on V; and indeed this does give a class in $H_1(V)$ via the above picture. On the other hand a nontrivial element of $H_1(\hat{V})$ is given by a cycle on $(V_2 - W)$ going around $z_1 = \{0\}$. They intersect in one point (on V_2); here's a picture:



Still assuming normal crossings, we set up a double complex

$$'E_0^{p,q} = \Gamma\left(V^{d-p}, {}'\mathcal{D}^{p-q}_{V^{d-p}}(\log V^{d-p+1})\right)$$

computing $H^*(X)$ (and $H^*_V(X)$, omitting the d^{th} column) and record below for reference $'E^0_{*,*}$ and $'E_0^{*,*}$ for the case of $X =$ a 3-fold.

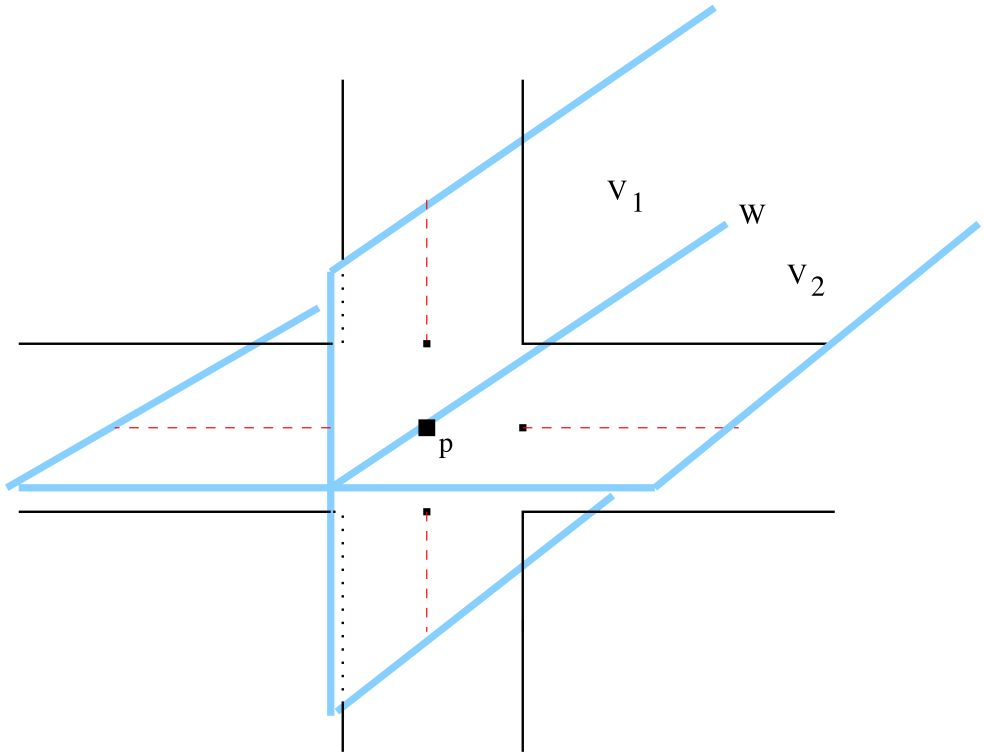
$$
\begin{tikzcd}
 & & & \mathcal{C}_0(X-V) \\
 & & \mathcal{C}_0(V-W) \arrow[r] & \mathcal{C}_1(X-V) \arrow[u] \\
 & \mathcal{C}_0(W-Y) \arrow[r] & \mathcal{C}_1(V-W) \arrow[r] \arrow[u] & \mathcal{C}_2(X-V) \arrow[u] \\
\mathcal{C}_0(Y) \arrow[r] & \mathcal{C}_1(W-Y) \arrow[r] \arrow[u] & \mathcal{C}_2(V-W) \arrow[r, "\text{Tube}"] \arrow[u] & \mathcal{C}_3(X-V) \arrow[u, "\partial"'] \\
 & \mathcal{C}_2(W-Y) \arrow[r] \arrow[u] & \mathcal{C}_3(V-W) \arrow[r] \arrow[u] & \mathcal{C}_4(X-V) \arrow[u, "\partial"'] \\
 & & \mathcal{C}_4(V-W) \arrow[r] \arrow[u] & \mathcal{C}_5(X-V) \arrow[u] \\
 & & & \mathcal{C}_6(X-V) \arrow[u]
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & & \Gamma('\mathcal{D}^0_X(\log V)) \arrow[d] \\
 & & \Gamma('\mathcal{D}^0_V(\log W)) \arrow[d] & \Gamma('\mathcal{D}^1_X(\log V)) \arrow[l] \arrow[d] \\
 & \Gamma('\mathcal{D}^0_W(\log Y)) \arrow[d] & \Gamma('\mathcal{D}^1_V(\log W)) \arrow[l] \arrow[d] & \Gamma('\mathcal{D}^2_X(\log V)) \arrow[l] \arrow[d, "\mathrm{d}"'] \\
\Gamma('\mathcal{D}^0_Y) & \Gamma('\mathcal{D}^1_W(\log Y)) \arrow[l] \arrow[d] & \Gamma('\mathcal{D}^2_V(\log W)) \arrow[l] \arrow[d] & \Gamma('\mathcal{D}^3_X(\log V)) \arrow[l, "\mathrm{Res}"'] \arrow[d, "\mathrm{d}"'] \\
 & \Gamma('\mathcal{D}^2_W(\log Y)) & \Gamma('\mathcal{D}^3_V(\log W)) \arrow[l] \arrow[d] & \Gamma('\mathcal{D}^4_X(\log V)) \arrow[l] \arrow[d] \\
 & & \Gamma('\mathcal{D}^4_V(\log W)) & \Gamma('\mathcal{D}^5_X(\log V)) \arrow[l] \arrow[d] \\
 & & & \Gamma('\mathcal{D}^6_X(\log V))
\end{tikzcd}
$$

Here $\mathcal{C}_0(Y)$ and $\Gamma('\mathcal{D}^0_Y)$, respectively, are in the $(0,0)$ positions. We refer to the d_i in $'E^i_{*,*}$ as Tube^i; Tube^1 is the usual tube map from $H_*(V-W) \to H_{*+1}(X-V)$, $\text{Tube}^2 = \text{Tube} \circ \partial^{-1} \circ \text{Tube}$, etc., and together they give the graded pieces of the tube map in the exact sequence

$$
\to H_{*-1}(\hat{V}, \mathbb{Z}) \xrightarrow{\text{Tube}} H_*(X-V, \mathbb{Z}) \longrightarrow H_*(X, \mathbb{Z}) \xrightarrow{\cap} H_{*-2}(\hat{V}, \mathbb{Z}) \to .
$$

In the case corresponding to the dotted arrow above the local picture of a Tube^2 of a point $p \in W - Y$ is, in a picture $/\mathbb{R}$ (i.e., we draw S^0's where there should be S^1's):



The four points (one of which is hidden) are Tube(p), and the light dotted lines are $\partial^{-1} \circ \mathrm{Tube}(p)$. In fact for one point p, in the global picture this may not exist – one must choose a 0-cycle $\sum n_i p_i$ (on $W - Y$) in ker(Tube^1), to construct Tube^2. A simple computation shows

$$\int_{\mathrm{Tube}^2(\sum n_i p_i)} R_{\mathbf{f}} \equiv \sum_i n_i [\mathrm{Res}^2 R'_{\mathbf{f}}](p_i) \quad \mod \mathbb{Z}(4),$$

and the previous remarks on the dilogarithmic nature of $\mathrm{Res}^2 R_{\mathbf{f}}$ stand (also see the next section). In general

$$\int_{\mathrm{Tube}^i \mathcal{C}} R_{\mathbf{f}} \equiv \int_{\mathcal{C}} \mathrm{Res}^i R'_{\mathbf{f}} \quad \mod \mathbb{Z}(n),$$

where

$$\mathrm{Tube}^i : H_{n-2i}(V^i \setminus V^{i+1}, \mathbb{Z}) \supseteq \ker(\mathrm{Tube}^{i-1}) \longrightarrow H_{n-1}(X - V, \mathbb{Z}) \Big/ \bigcup_{j<i} \mathrm{im}(\mathrm{Tube}^j) \ .$$

Remark 2.3.6. At least in the case of V having normal crossings and $\mathbf{f}$ good, one can get results similar to those in this section by working with $\Omega^\bullet_X(\mathrm{null} V)$ forms (which one is more likely to be able to write down) instead of $\Omega^\bullet_{(X \setminus V)^\infty}$ forms. (Dually one must use $\log V$ currents, which accounts for the more stringent conditions.) The spectral sequence would be obtained by, e.g., resolving $\jmath^{(i)}_! \mathbb{C} \to \Omega^\bullet_X(\mathrm{null} V^i)$ instead of $\jmath_! \Omega^\bullet_{(X \setminus V)^\infty}$.

2.4. Abel-Jacobi for Higher Chow Groups

2.4.1. The triple ($\mathbf{T}_{\mathcal{Z}}$,$\Omega_{\mathcal{Z}}$,$\mathbf{R}_{\mathcal{Z}}$). We begin by constructing explicitly an Abel-Jacobi map

$$\mathcal{R}: CH^p(X,n) \to H_{\mathcal{D}}^{2p-n}(X,\mathbb{Z}(p))$$

for X projective $/\mathbb{C}$. Then we extend it to some other cases and give applications. In particular, on η_X (for $n=p$) it coincides with the Milnor regulator.

On $\square^n$, define[13]

$$T_{\square}^n := T_{z_1} \cap \ldots \cap T_{z_n} \in \mathcal{C}_n(\square^n),$$

$$\Omega_{\square}^n := \Omega(z_1,\ldots,z_n) = \bigwedge^n \mathrm{dlog} z_i \in F^n{}'\mathcal{D}^n(\square^n),$$

$$R_{\square}^n := R(z_1,\ldots,z_n) =$$

$$\log z_1 \mathrm{dlog} z_2 \wedge \ldots \wedge \mathrm{dlog} z_n \pm (2\pi\sqrt{-1}) \log z_2 \mathrm{dlog} z_3 \wedge \ldots \wedge \mathrm{dlog} z_n \cdot \delta_{T_{z_1}}$$

$$- \ldots + (\pm 2\pi\sqrt{-1})^{n-1} \log z_n \cdot \delta_{T_{z_1}\cap\ldots\cap T_{z_{n-1}}} \quad \in {}'\mathcal{D}^{n-1}(\square^n).$$

Put

$$\sum_{i,e} = \sum_{i=1}^n \sum_{e=0,\infty} (-1)^{i+s(e)}, \quad \text{where } s(e) = \begin{cases} 1 & e=0 \\ 0 & e=\infty \end{cases}$$

so that we may conveniently write

$$\partial T_{\square}^n = \sum_{i,e} \rho_{i\,*}^e T_{\square}^{n-1} = \sum_{i,e} \partial_i^e T_{\square}^n,$$

$$\mathrm{d}[\Omega_{\square}^n] = 2\pi\sqrt{-1}\sum_{i,e} \rho_{i\,*}^e \Omega_{\square}^{n-1} = 2\pi\sqrt{-1}\sum_i (-1)^i \Omega(z_1,\ldots,\widehat{z_i},\ldots,z_n)\cdot\delta_{(z_i)},$$

$$\mathrm{d}[R_{\square}^n] = -2\pi\sqrt{-1}\sum_{i,e} \rho_{i\,*}^e R_{\square}^{n-1} + \Omega_{\square}^n - (2\pi\sqrt{-1})^n T_{\square}^n$$

$$= -2\pi\sqrt{-1}\sum_i (-1)^i R(z_1,\ldots,\widehat{z_i},\ldots,z_n)\cdot\delta_{(z_i)} + \Omega_{\square}^n - (2\pi\sqrt{-1})^n T_{\square}^n.$$

Consider a cycle $\mathcal{Z} \in Z^p(X \times \square^n)$:

[13] (using the more concise notation $\mathcal{A}(X) = \Gamma(X,\mathcal{A})$ for sections of sheaves)

$$\begin{tikzcd} & & \mathcal{Z} \arrow[dll, "\pi_{\square}^{\mathcal{Z}}"'] \arrow[dl] \arrow[ddl, "\pi_X^{\mathcal{Z}}"] \\ \square^n & X \times \square^n \arrow[l, "\pi_{\square}"] \arrow[d, "\pi_X"] & \\ & X & \end{tikzcd}$$

Set $\partial_i^e \mathcal{Z} := \rho_i^{e^*} \mathcal{Z} \; = \mathcal{Z} \cap \left(X \times \rho^e_{i\,*} \square^{n-1} \right)$, so that $\partial_{\mathcal{B}} \mathcal{Z} = \sum_{i,e} \partial_i^e \mathcal{Z}$, and we have the diagram of projections and inclusions:

$$\begin{tikzcd} \square^{n-1} \arrow[d, hook, "\rho_i^e"'] & \partial_i^e \mathcal{Z} \arrow[l, "\pi_{\square}^{(\partial_i^e \mathcal{Z})}"'] \arrow[r, "\pi_X^{(\partial_i^e \mathcal{Z})}"] \arrow[d, hook, "\iota_i^e"] & X \arrow[d, equal] \\ \square^n & \mathcal{Z} \arrow[l, "\pi_{\square}^{\mathcal{Z}}"'] \arrow[r, "\pi_X^{\mathcal{Z}}"] & X \end{tikzcd}$$

Now we define the central objects of the section, noting that the $\pi^{\mathcal{Z}}_{X\,*}$'s (in what follows) can both involve integration over the *compact* fibers of $\mathcal{Z}$ (e.g. generically over X if $n > p$) and multiplication by δ_{V^i}-functions for components of $\mathcal{Z}$ supported over V^i. Put

$$R_{\mathcal{Z}} := \pi^{\mathcal{Z}}_{X\,*} \pi_{\square}^{\mathcal{Z}^*} R^n_{\square} \in {}'\mathcal{D}^{2p-n-1}(X),$$

$$\Omega_{\mathcal{Z}} := \pi^{\mathcal{Z}}_{X\,*} \pi_{\square}^{\mathcal{Z}^*} \Omega^n_{\square} \in F^p{}'\mathcal{D}^{2p-n}(X),$$

$$T_{\mathcal{Z}} := \pi_X \left((X \times T^n_{\square}) \cap \mathcal{Z} \right) \in \mathcal{C}_{2d-2p+n}(X, \mathbb{Z});$$

applying differentials gives

$$\partial T_{\mathcal{Z}} = \pi_X \left((X \times \partial T^n_{\square}) \cap \mathcal{Z} \right) = \pi_X \left(\sum_{i,e} (X \times \rho^e_{i\,*} T^{n-1}_{\square}) \cap \mathcal{Z} \right)$$

$$= \pi_X \left(\sum_{i,e} (X \times T^{n-1}_{\square}) \cap \rho_i^{e^*} \mathcal{Z} \right) = \pi_X \left((X \times T^{n-1}_{\square}) \cap \partial_{\mathcal{B}} \mathcal{Z} \right) = T_{\partial_{\mathcal{B}} \mathcal{Z}},$$

$$\mathrm{d}[\Omega_{\mathcal{Z}}] = \; \pi^{\mathcal{Z}}_{X\,*} \pi_{\square}^{\mathcal{Z}^*} \mathrm{d}[\Omega^n_{\square}] \; = \; 2\pi\sqrt{-1} \sum_{i,e} \pi^{\mathcal{Z}}_{X\,*} \pi_{\square}^{\mathcal{Z}^*} \rho^e_{i\,*} \Omega^{n-1}_{\square}$$

$$= 2\pi\sqrt{-1} \sum_{i,e} \pi^{\mathcal{Z}}_{X\,*} \iota^e_{i\,*} \pi_{\square}^{(\partial_i^e \mathcal{Z})*} \Omega^{n-1}_{\square} \; = \; 2\pi\sqrt{-1} \sum_{i,e} \pi_X^{(\partial_i^e \mathcal{Z})}{}_{*} \pi_{\square}^{(\partial_i^e \mathcal{Z})*} \Omega^{n-1}_{\square}$$

$$= 2\pi\sqrt{-1}\, \Omega_{\partial_{\mathcal{B}} \mathcal{Z}}, \text{ and similarly}$$

$$\mathrm{d}[R_{\mathcal{Z}}] \;=\; -2\pi\sqrt{-1}\,R_{\partial_{\mathcal{B}}\mathcal{Z}} \;+\; \Omega_{\mathcal{Z}} \;-\; (2\pi\sqrt{-1})^n T_{\mathcal{Z}}.$$

We are already familiar with this last formula in the case $n = p$, $\mathcal{Z} = \overline{\gamma_{\mathbf{f}}}$ for $\mathbf{f}$ good.

Define a complex of cochains for the Deligne homology[14]

$$\begin{aligned}
&\mathcal{C}_{\mathcal{D}}^{\bullet-2d}(X, \mathbb{Z}(p-d)) := \\
&\mathrm{Cone}\left\{\mathcal{C}_{2d-\bullet}(X, \mathbb{Z}(p)) \oplus F^p{}'\mathcal{D}_X^\bullet(X) \to {}'\mathcal{D}_X^\bullet(X)\right\}[-1](-d) \\
&= \left\{\mathcal{C}_{2d-\bullet}(X, \mathbb{Z}(p)) \oplus F^p{}'\mathcal{D}_X^\bullet(X) \oplus {}'\mathcal{D}_X^{\bullet-1}(X)\right\}(-d)
\end{aligned}$$

with differential ∂ taking

$$(a,\, b,\, c) \mapsto (-\partial a,\, -\mathrm{d}[b],\, \mathrm{d}[c] - b + a).$$

Then the following is a map of (cohomological) complexes:

$$\mathcal{R}_X :\, Z^p(X, -\bullet) \longrightarrow \mathcal{C}_{\mathcal{D}}^{2p-2d+\bullet}(X, \mathbb{Z}(p-d))$$

induced (for $-\bullet = n$) by

$$\mathcal{Z} \longmapsto (-2\pi\sqrt{-1})^{p-n} \times \left((2\pi\sqrt{-1})^n\, T_{\mathcal{Z}},\, \Omega_{\mathcal{Z}},\, R_{\mathcal{Z}}\right) =: \mathcal{R}_X(\mathcal{Z}).$$

(Technically one should also throw in a $(2\pi\sqrt{-1})^{-d}$ here; we'll sometimes ignore the $(-d)$ like this). The only thing worth writing out here is to check that powers of $(2\pi\sqrt{-1})$ work out in the third term of the cone complex:

$$\begin{aligned}
\mathrm{d}[c] - b + a \;=\; &(-2\pi\sqrt{-1})^{p-n}\left[-2\pi\sqrt{-1}R_{\partial_{\mathcal{B}}\mathcal{Z}} + \Omega_{\mathcal{Z}} - (2\pi\sqrt{-1})^n T_{\mathcal{Z}}\right] \\
-(-2\pi\sqrt{-1})^{p-n}\Omega_{\mathcal{Z}} + (-1)^{p-n}(2\pi\sqrt{-1})^p T_{\mathcal{Z}} \;&=\; (-2\pi\sqrt{-1})^{p-(n-1)} R_{\partial_{\mathcal{B}}\mathcal{Z}},
\end{aligned}$$

and so

$$\partial\mathcal{R}_X(\mathcal{Z}) = (-2\pi\sqrt{-1})^{p-(n-1)} \times \left((2\pi\sqrt{-1})^{n-1} T_{\partial_{\mathcal{B}}\mathcal{Z}}\,,\, \Omega_{\partial_{\mathcal{B}}\mathcal{Z}},\, R_{\partial_{\mathcal{B}}\mathcal{Z}}\right) = \mathcal{R}_X(\partial_{\mathcal{B}}\mathcal{Z}).$$

Therefore if $\mathcal{Z}$ is a higher Chow cycle ($\partial_{\mathcal{B}}\mathcal{Z} = 0$), we have that $\partial\mathcal{R}_X(\mathcal{Z}) = 0$. Using the formula for $\mathrm{d}[R_{\mathcal{Z}}]$,

$$\mathrm{d}[R_{\mathcal{Z}}] = \Omega_{\mathcal{Z}} - (2\pi\sqrt{-1})^n T_{\mathcal{Z}} \quad\Longrightarrow\quad [\Omega_{\mathcal{Z}}] \sim (2\pi\sqrt{-1})^n [T_{\mathcal{Z}}]$$

gives a class in

$$F^p H^{2p-n}(X, \mathbb{C}) \cap H^{2p-n}(X, \mathbb{Z}(p)).$$

In the event that this vanishes there are primitives $\mathrm{d}_X^{-1}\Omega_{\mathcal{Z}}$ and $\partial_X^{-1}T_{\mathcal{Z}}$; we may use them to modify $R_{\mathcal{Z}}$ to get something d-closed. The choice of $\mathrm{d}^{-1}\Omega_{\mathcal{Z}}$ is ambiguous by $F^p H^{2p-n-1}(X, \mathbb{C})$ and that of $(2\pi\sqrt{-1})^n \partial_X^{-1} T_{\mathcal{Z}}$ by $H^{2p-n-1}(X, \mathbb{Z}(p))$, so we get a well-defined class

$$[R'_{\mathcal{Z}}] \in \frac{H^{2p-n-1}(X, \mathbb{C})}{F^p H^{2p-n-1}(X, \mathbb{C}) + H^{2p-n-1}(X, \mathbb{Z}(p))}.$$

---

[14] we try to agree here with the conventions used by Jannsen and Lewis (**[Ja]**, **[L2]**), and forewarn the reader that this renders the indexing slightly ridiculous (due to the use of *co*chains to compute homology).

This just goes to say that Deligne homology and cohomology are the same by a kind of "Poincaré duality":

$$H^{-n}\left(\mathcal{C}_{\mathcal{D}}^{2p-2d+\bullet}(X,\mathbb{Z}(p-d)\right) =: H^{\mathcal{D}}_{2d-2p+n}(X,\mathbb{Z}(d-p)) \underset{\text{P.D.}}{\cong} H^{2p-n}_{\mathcal{D}}(X,\mathbb{Z}(p)),$$

and we have the promised map for X projective.[15]

We show how to modify this picture for $X \setminus V$ in case V has normal crossings, so that $'\mathcal{D}^\bullet_X(\log V)$ resolves $\jmath_*^{(1)}\mathbb{C}_{X-V}$ and cohomology of

$$\mathcal{C}_{\mathcal{D}}^{\bullet-2d}(X-V,\mathbb{Z}(p-d)) :=$$

$$\text{Cone}\left\{\mathcal{C}_{2d-\bullet}(X,V;\mathbb{Z}(p))\oplus\Gamma(F^{p\prime}\mathcal{D}^\bullet_X(\log V))\to\Gamma('\mathcal{D}^\bullet_X(\log V))\right\}[-1](-d)$$

computes Deligne (co)homology of $X \setminus V$. In fact we may view this as $Gr^0_N\mathcal{C}^{\bullet-2d}_{\mathcal{D}}$ by defining a coniveau filtration

$$N^i\mathcal{C}_{\mathcal{D}}^{\bullet-2d}(X,\mathbb{Z}(p-d)) :=$$

$$\text{Cone}\left\{\mathcal{C}_{2d-\bullet}(V^i,\mathbb{Z}(p))\oplus\Gamma(F^{p\prime}\mathcal{D}^\bullet_X(\text{on}V^i))\to\Gamma('\mathcal{D}^\bullet_X(\text{on}V^i)\right\}[-1](-d).$$

It is clear that $\mathcal{R}_X$ respects coniveau, restricting to a map

$$N^iZ^p(X,-\bullet) \overset{\mathcal{R}_{V^i}}{\longrightarrow} N^i\mathcal{C}_{\mathcal{D}}^{2p-2d+\bullet}(X,\mathbb{Z}(p-d))$$

and therefore

$$Gr^0_NZ^p(X,-\bullet) \overset{\mathcal{R}_{X\setminus V}}{\longrightarrow} \mathcal{C}_{\mathcal{D}}^{2p-2d+\bullet}(X\setminus V,\mathbb{Z}(p-d)),$$

which is to say that $\mathcal{R}_{X\setminus V}$ only operates on cycles $\mathcal{Z}$ on $X \setminus V$ with good closure on X. Bloch's moving lemma then implies that $\mathcal{R}_{X\setminus V}$ descends to a map

$$\mathcal{R}_{X\setminus V} : CH^p(X\setminus V,n)\to H^{2p-n}_{\mathcal{D}}(X\setminus V,\mathbb{Z}(p)).$$

For $p = n$, $\mathcal{R}_{X\setminus V}(\gamma_{\mathbf{f}})$ is just the map we constructed in §2.2-3 using $\Omega_{\mathbf{f}}$ and $R'_{\mathbf{f}}$, since the definitions $\implies \Omega_{\gamma_{\mathbf{f}}} = \Omega_{\mathbf{f}}$, $T_{\gamma_{\mathbf{f}}} = T_{\mathbf{f}}$, $R_{\gamma_{\mathbf{f}}} = R_{\mathbf{f}}$. However, we got around the requirements that $\mathbf{f}$ be "good" and V have normal crossings there. (This was the point of using $'\mathcal{D}^\bullet_{X\setminus V}$ in §2.2, also see the discussion preceding lemma 1.3.7.) The same is possible for $\mathcal{R}_{X\setminus V}$ if $n > p$, because (for purposes of computing $H^{2p-n}_{\mathcal{D}}$ for $n > p$) we can alter the terms of $\mathcal{C}_{\mathcal{D}}^{2p-2d-n}(X\setminus V,\mathbb{Z}(p-d))$ up to $n = p$ by altogether omitting the second entry (as $F^{p\prime}\mathcal{D}^{2p-n}_X = 0$ for $n > p$). This gives us the freedom to replace $'\mathcal{D}^{2p-n-1}_X(\log V)$ by $'\mathcal{D}^{2p-n-1}_{(X\setminus V)^\infty}$, so that we may drop the normal crossings

[15] Note that the version of this map given by Goncharov **[Go1]** was not correct. One really needs the Deligne homology cochains here to get a map of complexes inducing AJ: there is no getting around the triple (T, Ω, R) for general n, p, d. The major problem, however, was that he uses real $(n-1)$-currents r_n instead of a complex current like the $R^n_\square$ employed here.

assumption on V, and even the assumption that $\mathcal{Z}$ has good closure. Taking the limit, we get

$$\mathcal{R}_{\eta_X} : CH^p(\mathbb{C}(X), n) \to H_{\mathcal{D}}^{2p-n}(\eta_X, \mathbb{Z}(p)) = H^{2p-n-1}(\eta_X, \mathbb{C}/\mathbb{Z}(p)),$$

induced by

$$\mathcal{Z} \mapsto (-2\pi\sqrt{-1})^{p-n} \times \left((2\pi\sqrt{-1})^n T_{\mathcal{Z}},\, R_{\mathcal{Z}}\right),$$

for $n > p$. For the remaining n: set it equal to our previous construction for $n = p$, and for $n < p$ define it to be 0 since all the cycles in $Z^p(X, n)$ are then contained in higher coniveau.

REMARK 2.4.1. So the cases $n \geq p$ are *arithmetically* interesting; they amount to something already on fields, and they are interesting for projective X if the coefficients of the defining equations are in $\bar{\mathbb{Q}}$ (or perhaps extensions thereof, of small transcendence degree). If these coefficients are general (say, enough independent transcendentals) they are not interesting in most cases, $n = p = 2$ and X an elliptic curve being a notable exception (see Chapter 4). On the other hand, the cases $n < p$ are *geometrically* interesting, and do tend to have interesting images for "general" projective X.

2.4.2. Why residues of $R'_{\mathfrak{f}}$ are polylogarithmic. Again assuming normal crossings, the maps $\mathcal{R}_{V^i}$ on coniveau induce a map of spectral sequences, so that the $d_i = \mathrm{Res}^i$ and $\mathcal{R}$ (on the appropriate subquotients of graded pieces) automatically commute; the central step is that the square

$$\begin{array}{ccc}
H^*(Gr^i_N Z^p(X, -\bullet)) & \xrightarrow{\Delta} & H^{*+1}(N^{i+1} Z^p(X, -\bullet)) \\
\Big\downarrow \mathcal{R} & & \Big\downarrow \mathcal{R} \\
H^*(Gr^i_N \mathcal{C}_{\mathcal{D}}^{2p-2d+\bullet}) & \xrightarrow{\Delta} & H^{*+1}(N^{i+1} \mathcal{C}_{\mathcal{D}}^{2p-2d+\bullet})
\end{array}$$

commutes. Cohomology of

$$Gr^i_N \mathcal{C}_{\mathcal{D}}^{2p-2d+\bullet} \underset{\iota_*^{(i)}}{\overset{\simeq}{\longrightarrow}} \mathrm{Cone}\left\{\mathcal{C}_{2d-2p-\bullet}(V^i, V^{i+1}; \mathbb{Z}(p-i))\right.$$

$$\left.\oplus\ \Gamma(F^{p-i\prime}\mathcal{D}_{V^i}^{2(p-i)+\bullet}(\log V^{i+1}) \longrightarrow \Gamma({}'\mathcal{D}_{V^i}^{2(p-i)+\bullet}(\log V^{i+1})\right\}[-1](-(d-i))$$

at $\bullet = -n$ computes $H_{\mathcal{D}}^{2(p-i)-n}\left(V^i \setminus V^{i+1}, \mathbb{C}/\mathbb{Z}(p-i)\right)$, and so one has a commutative square

$$\begin{array}{ccc} CH^p(X\setminus V,n)\supseteq \ker(\mathrm{Res}^{i-1}) & \xrightarrow{\mathrm{Res}^i} & CH^{p-i}(V^i\setminus V^{i+1},n-1)\Big/\bigcup_{j<i}\mathrm{im}(\mathrm{Res}^j) \\ \downarrow{\mathcal{R}} & & \downarrow{\mathcal{R}} \\ H^{2p-n}_{\mathcal{D}}(X\setminus V,\mathbb{Z}(p))\supseteq \ker(\mathrm{Res}^{i-1}) & \xrightarrow{\mathrm{Res}^i[/(2\pi i)^i]} & H^{2(p-i)-n+1}_{\mathcal{D}}(V^i\setminus V^{i+1},\mathbb{Z}(p-i))\Big/\bigcup_{j<i}\mathrm{im}(\mathrm{Res}^j) \end{array}$$

Now suppose $p > d = \dim X$. For all n, $F^p{}'\mathcal{D}^{2p-n}_X = 0$ (so that $H^*_{\mathcal{D}}(X,\mathbb{Z}(p)) = H^{*-1}(X,\mathbb{C}/\mathbb{Z}(p))$) and we may write instead

$$N^i\mathcal{C}^{\bullet-2d}_{\mathcal{D}}(X,\mathbb{Z}(p-d)) = \mathrm{Cone}\left\{\mathcal{C}_{2d-\bullet}(V^i,\mathbb{Z}(p)) \to {}'\mathcal{D}^{\bullet}_{(V^i)^\infty}(X)\right\}[-1](-d)$$

so that

$$Gr^i_N\mathcal{C}^{2p-2d+\bullet}_{\mathcal{D}}(X,\mathbb{Z}(p-d)) \underset{\iota_*^{(i)}}{\overset{\simeq}{\longleftarrow}}$$

$$\mathrm{Cone}\left\{\mathcal{C}_{2d-2p-\bullet}(V^i,V^{i+1};\mathbb{Z}(p-i)) \to {}'\mathcal{D}^{2(p-i)+\bullet}_{(V^i\setminus V^{i+1})^\infty}(X)\right\}[-1](-d+i)$$

computes

$$H^{2(p-i)-n}_{\mathcal{D}}(V^i\setminus V^{i+1},\mathbb{Z}(p)) \;=\; H^{2(p-i)-n-1}(V^i\setminus V^{i+1},\mathbb{C}/\mathbb{Z}(n))$$

without the assumption of normal crossings.

Let $\gamma_{\mathbf{f}} \in CH^n(X\setminus V_{\mathbf{f}},\, n)$ be any graph (we are frequently interested in the case $d = n-1$). Then exactly as above d_i and $\mathcal{R}$ commute and so[16]

$$\mathrm{Res}^i\left((2\pi\sqrt{-1})^n T_{\mathbf{f}},\, R_{\mathbf{f}}\right) =$$

$$\mathrm{Res}^i(\mathcal{R}(\gamma_{\mathbf{f}})) \;=\; \mathcal{R}(\mathrm{Res}^i(\gamma_{\mathbf{f}})) = \;(2\pi\sqrt{-1})^i \times \left((2\pi\sqrt{-1})^{n-i}T_{\mathrm{Res}^i(\gamma_{\mathbf{f}})},\, R_{\mathrm{Res}^i(\gamma_{\mathbf{f}})}\right)$$

$$\in \text{[a subquotient of]}\; H^{n-2i}(V^i\setminus V^{i+1},\mathbb{C}/\mathbb{Z}(p-i))(i),$$

or more concisely

$$\frac{1}{(2\pi\sqrt{-1})^i}\mathrm{Res}^i R'_{\mathbf{f}} \;=\; R'_{\mathrm{Res}^i(\gamma_{\mathbf{f}})}.$$

We could not accomplish this before because $\mathrm{Res}^i(\gamma_{\mathbf{f}})$ is not a graph for $i \geq 2$ (see Chapter 1): over each point in $V^i\setminus V^{i+1}$ it has fiber dimension $i-1$ and so some integration is required (along these fibers) in pushing $\pi^* R^n_{\square}$ down to get $R_{\mathrm{Res}^i(\gamma_{\mathbf{f}})}$.

Provided the conjectural picture at the end of §1.2 can be completely realized, we would have ($i \geq 2$)

$$\mathrm{Res}^i(\gamma_{\mathbf{f}}) \;=\; \mathrm{Res}^i(\bar{\rho}_n(n)\mathbf{f}) \;=\; \bar{\rho}_{n-i}(n-1)(\mathrm{Tame}^i\mathbf{f}) \;=:\; \bar{\rho}(\mathrm{Tame}^i\mathbf{f})$$

[16]Of course one can take a limit here and replace $X\setminus V_{\mathbf{f}}$ by η_X, $V^i\setminus V^{i+1}$ by [generic points of] codimension i points in X.

where $\mathrm{Tame}^i\mathbf{f}$ lives in a subquotient of

$$\coprod_{x\in X^i} \mathcal{B}_i(\mathbb{C}(x)) \otimes \bigwedge^{n-2i} \mathbb{C}(x)^*$$

and so

$$\frac{1}{(2\pi\sqrt{-1})^i}\mathrm{Res}^i R'_{\mathbf{f}} \ = \ R'_{\bar{\rho}(\mathrm{Tame}^i\mathbf{f})},$$

which makes more concrete the expected "i-logarithmic" behavior of $\mathrm{Res}^i R'_{\mathbf{f}}$. (Recall that the relations on $\mathcal{B}_i$ correspond to functional equations for the i^{th} polylogarithm; for more evidence, particularly in the case $n = 2i$, see below.)

Recall that

$$K_n^M(X) \subseteq K_n^M(\mathbb{C}(X)) \ \underset{\gamma}{\overset{\cong}{\longrightarrow}} \ CH^n(\eta_X, n)$$

is defined to be $\gamma^{-1}(\cap \ker(\mathrm{Res}^i))$. Since $\mathcal{R}$ commutes with Res^i, it maps $K_n^M(X)$ to Deligne cohomology of η_X (i.e. to currents with trivial Res^i). So we get a well-defined map

$$\mathcal{R}: \ K_n^M(X) \longrightarrow \mathrm{im}\,\{H^n_{\mathcal{D}}(X,\mathbb{Z}(n)) \to H^n_{\mathcal{D}}(\eta_X,\mathbb{Z}(n))\} \ \cong \ H^n_{\mathcal{D}}(X,\mathbb{Z}(n))\,/\mathrm{im}(Gy)$$

(see diagram below). This is the "holomorphic" part of the Milnor regulator, the stuff that doesn't come from residues and presumably has more to do with the arithmetic and geometry of X. To put this in a context we write down the (commuting) map of "exact triangles" (=localization sequences) mentioned above, for $i = 1$. We write $H^{*-2}_{\mathcal{D}}(\hat{V},\mathbb{Z}(n-1))(1)$ (where $(1) = 2\pi\sqrt{-1}$) for

$$H^*_{\mathcal{D},V}(X,\mathbb{Z}(n)) := H^*(N^1\mathcal{C}^{\bullet-2d}_{\mathcal{D}}(X,\mathbb{Z}(n-d))),$$

note that both Res maps break up into Res^i, and refer the reader to Totaro [**T**] for a definition of the norm map $\mathcal{N}$ (*a priori* defined into $K_n^M(\mathbb{C}(X))$).

$$
\begin{tikzcd}
 & & \varinjlim CH^{n-1}(Z,n) \arrow[r, "\mathcal{R}_V"] \arrow[d, "\iota_*^{(1)}"] & \varinjlim H_{\mathcal{D}}^{n-2}(Z,\mathbb{Z}(n-1))(1) \arrow[d, "Gy\,|\,\iota_*^{(1)}"] \\
 & & CH^n(X,n) \arrow[r, "\mathcal{R}_X"] \arrow[d, "\jmath^*_{(1)}"] \arrow[lld, "\mathcal{N}"', "(norm)"] & H_{\mathcal{D}}^n(X,\mathbb{Z}(n)) \arrow[d, "\jmath^*_{(1)}"] \\
K_n^M(X) \arrow[r, hook] & K_n^M(\mathbb{C}(X)) \arrow[r, "\gamma(graph)", "\cong"'] \arrow[d, "\mathrm{Tame}^1"] & CH^n(\eta_X,n) \arrow[r, "\mathcal{R}_{\eta_X}"] \arrow[dd, "\mathrm{Res}"] & H_{\mathcal{D}}^n(\eta_X,\mathbb{Z}(n)) \arrow[dd, "\mathrm{Res}"] \\
 & \coprod_{x\in X^1} K_{n-1}^M(\mathbb{C}(x)) \arrow[d, "\cong\,\gamma"] & & \\
 & \coprod_{x\in X^1} CH^{n-1}(\eta_X,n-1) & \varinjlim CH^{n-1}(Z,n-1) \arrow[l, "\jmath^*_{(1,2)}"'] \arrow[r, "\mathcal{R}_V"] & \varinjlim H_{\mathcal{D}}^{n-1}(Z,\mathbb{Z}(n-1))(1)
\end{tikzcd}
$$

Of course $\gamma(K_n^M(X)) \subset \ker(\mathrm{Res})$, not just $\ker(\mathrm{Res}^1)$; and Gy can be described as simply

$$H_V^{n-1}(X,\mathbb{C}/\mathbb{Z}(n)) \to H^{n-1}(X,\mathbb{C}/\mathbb{Z}(n))$$

since we are assuming $n > d$.

2.4.3. Some special cases of the construction. We now run through the basic examples, for $p = 2$ $(0 \leq n \leq 3)$, for the most part recovering familiar formulas (whose analogues for $p \geq 3$ are perhaps less familiar). We assume in each case that the "cycle-class" part of the AJ-map is zero, so that there exist $\mathrm{d}^{-1}\Omega_{\mathcal{Z}} \in F^{p\prime}\mathcal{D}^{2p-n-1}_{X \text{ or } (\eta_X)^\infty}$ and $\partial^{-1}_{X \text{ or } X_{rel}} T_{\mathcal{Z}}$ (on X or η_X, depending on the example).

Example 2.4.2. We actually consider $CH^n(\eta_X,n) \cong K_n^M(\mathbb{C}(X))$ for $n = 2,3$. The above assumption always $(\forall d) \implies \Omega_{\mathcal{Z}} = 0$ for $p = n$ over η_X (holomorphic log forms inject into cohomology); we may also ignore $\partial^{-1}T_{\mathcal{Z}}$ by defining our AJ-map as a functional on integral cycles.

If $d = 1$ (X a curve), $n = 2$, $\mathcal{Z} = \sum m_i \gamma_{f_i \otimes g_i} \longleftrightarrow \prod\{f_i, g_i\}^{m_i}$, then $R_{\mathcal{Z}} = \sum m_i \left(\log f_i \,\mathrm{dlog} g_i \,-\, \log g_i \cdot \delta_{T_{f_i}}\right)$, and the "regulator" $\to H^1(\eta_X, \mathbb{C}/\mathbb{Z}(2))$ viewed as a functional on 1-cycles $\mathcal{C}$ in $X \setminus V_{\mathbf{f}}$

$$\int_{\mathcal{C}} R_{\mathcal{Z}} \underset{\mathrm{mod}\ 4\pi^2\mathbb{Z}}{\equiv} \sum m_i \int_{\mathcal{C}} \left(\log^* f_i \,\mathrm{dlog} g_i \,-\, \log g_i(p_0)\,\mathrm{dlog} f_i\right).$$

Here we have picked a "base point" $p_0 \in |\mathcal{C}|$ and continued $\log f_i$ to get $\log^* f_i$ (only defined on [a cover of] $|\mathcal{C}|$, with cut only at [a lift of] p_0). This is the formula used in [**GG1**], [**C1**], etc. If $\mathcal{Z} \longleftrightarrow \{\mathbf{f}\} \in \ker(\mathrm{Tame}^{(1)})$ this functional lifts to a well-defined class in $H^1(X, \mathbb{C}/\mathbb{Z}(2))$ because $\mathrm{im}(Gy) = 0$ in the above diagram $(H^0_{\mathcal{D}}(\mathrm{pt}, \mathbb{Z}(1)) = 0)$.

If $d = 2$ (X a complex surface), $n = 3$, $\mathcal{Z} \longleftrightarrow \{\mathbf{f}\} = \prod \{f_i, g_i, h_i\}^{m_i}$, then the regulator is a functional on 2-cycles $\mathcal{C}$ in $X \setminus V$ given by $\int_{\mathcal{C}} R_{\mathcal{Z}} =$

$$\sum m_i \left\{ \int_{\mathcal{C}} \log f_i \, \mathrm{dlog} g_i \wedge \mathrm{dlog} h_i \; + 2\pi\sqrt{-1} \int_{\mathcal{C}\cap T_{f_i}} \log g_i \, \mathrm{dlog} h_i \; - 4\pi^2 \sum_{\mathcal{C}\cap T_{f_i} \cap T_{g_i}} \log h_i \right\}.$$

Though $K_3^M(X)$ is still $\ker(\mathrm{Tame}^{(1)})$, $\mathrm{im}(Gy) \neq 0$ so one only has

$$\mathcal{R}_X : K_3^M(X) \longrightarrow H^2(X, \mathbb{C}/\mathbb{Z}(3)) \, / \mathrm{im}(H^2_V(X, \mathbb{C})) \; .$$

This is because one doesn't get all of $H_2(X, \mathbb{Z})$ by considering cycles $\mathcal{C}$ avoiding V, although for X sufficiently general (see Proposition 4.5.5) such $\mathcal{C}$ do give everything but the hyperplane class.

EXAMPLE 2.4.3. $[\mathcal{Z}] \in CH^2(X, 1)$ for X a (regular) smooth projective surface; $\mathcal{R}$ maps to

$$H^3_{\mathcal{D}}(X, \mathbb{Z}(2)) \cong H^2(X, \mathbb{C}) \, / \, F^2 H^2(X, \mathbb{C}) + H^2(X, \mathbb{Z}(2)) \; .$$

$\mathcal{Z}$ consists of functions f_i on divisors V_i such that $\sum \iota_*^{V_i}(f_i) = 0$ (the V_i may be singular). We get data

$$R_{\mathcal{Z}} = \sum \log f_i \cdot \delta_{V_i}, \qquad \Omega_{\mathcal{Z}} = \sum \mathrm{dlog} f_i \cdot \delta_{V_i}, \qquad T_{\mathcal{Z}} = \cup \iota^{V_i}(T_{f_i})$$

and note that we may choose $\partial^{-1}\Omega_{\mathcal{Z}} \in \Gamma(F^{2\prime}\mathcal{D}^2_X)$ (no assumption is needed for this if X is regular). Therefore we may ignore it in $R'_{\mathcal{Z}} = R_{\mathcal{Z}} - \partial^{-1}\Omega_{\mathcal{Z}} + 2\pi\sqrt{-1}\, \delta_{\partial_X^{-1} T_{\mathcal{Z}}}$ for purposes of integrating against closed forms $[\omega] \in F^1 H^2(X, \mathbb{C})$:

$$\int_X R'_{\mathcal{Z}} \wedge \omega \; = \; \sum_i \left\{ \int_{V_i} \log f_i \cdot \omega \; + \; 2\pi\sqrt{-1} \int_{\partial_X^{-1} T_{\mathcal{Z}}} \omega \right\}$$

is then the desired formula, as used in [**?**], [**C1**], [**GL**], etc.

EXAMPLE 2.4.4. $[\mathcal{Z}] \in CH^2(X, 0) = CH^2(X)$ for X any smooth projective variety (we recover the classical AJ map from our construction). $R_{\mathcal{Z}} = 0$, $\Omega_{\mathcal{Z}} = T_{\mathcal{Z}} = \mathcal{Z}$ (or $\delta_{\mathcal{Z}}$) $\in H^4(X, \mathbb{Z}(2)) \cap F^2 H^2(X, \mathbb{C})$. We assume this class is trivial, and take two primitives:

$$\frac{1}{2\pi\sqrt{-1}} \beta \, [= \mathrm{d}^{-1}\Omega] \in F^{2\prime}\mathcal{D}^3_X(X), \quad \Gamma \, [= \partial^{-1} T] \in \mathcal{C}_{2d-3}(X)$$

so that $\mathrm{d}[\beta] = 2\pi\sqrt{-1} \cdot \delta_{\mathcal{Z}}$, $\partial\Gamma = \mathcal{Z}$. Pick a d-closed $\omega \in F^{d-1}\Omega^{2d-3}_{X^\infty}(X)$, so that $\beta \wedge \omega = 0$. Then our map "$\int_X R_{\mathcal{Z}} \wedge \omega$" reduces to

$$\int_\Gamma \omega,$$

and if $\omega = \mathrm{d}\alpha$ (where it suffices to choose $\alpha \in F^{d-1}\Omega^{2d-4}_{X^\infty}(X)$),

$$\int_\Gamma \mathrm{d}\alpha \;=\; \int_{\partial\Gamma} \alpha \;=\; \int_{\mathcal{Z}} (\iota^*_{\mathcal{Z}})\alpha \;=\; 0$$

by type. So integration over Γ gives a functional on $F^{d-1}H^{2d-3}(X,\mathbb{C})$, i.e. by duality an element of $H^3(X,\mathbb{C})/F^2H^3$.

Now suppose $\mathcal{Z} \underset{rat}{\equiv} 0$, so that there exist divisors $V_i \subset X$ and functions f_i on V_i with $\sum \iota^{V_i}_*(f_i) = \mathcal{Z}$. Then one may choose

$$\Gamma = \sum \iota^{V_i}(T_{f_i}), \quad \beta = \sum \mathrm{dlog} f_i \cdot \delta_{V_i};$$

clearly $\beta - (2\pi\sqrt{-1})\Gamma \;=\; \mathrm{d}[\mathrm{r}]$, where $\mathrm{r} = \sum \log f_i \cdot \delta_{V_i}$. Notice that

$$(-2\pi\sqrt{-1})^{p-1} \times (2\pi\sqrt{-1}\,\Gamma,\, \beta,\, \mathrm{r}) \longmapsto (-2\pi\sqrt{-1})^{p} \times (\mathcal{Z},\, \delta_{\mathcal{Z}},\, 0)$$

in the Deligne complex. Therefore it's no surprise that we can write

$$\int_\Gamma \omega \;=\; \frac{-1}{2\pi\sqrt{-1}} \int_X \mathrm{d}[\mathrm{r}] \wedge \omega \;-\; \frac{1}{2\pi\sqrt{-1}} \int_X \beta \wedge \omega \;=\; 0,$$

where the first term is zero because $\mathrm{d}\omega = 0$ (and X is compact).

So $\int_\Gamma = 0 \in H^3/F^2H^3$ for $\mathcal{Z}$ rationally equivalent to zero, for the above choice of Γ. This result is ambiguous by a period because one may modify $\Gamma = \partial^{-1}\mathcal{Z}$ by a $\mathbb{Z}(2)$-cycle.

EXAMPLE 2.4.5. $[\mathcal{Z}] \in CH^2(\eta_X, 3)$, so that fibers of $\mathcal{Z} \overset{\pi}{\to} X$ are (compact) curves. Since $R^3_\square$ is a 2-current, $R_{\mathcal{Z}}$ is a $\mathbb{C}/\mathbb{Z}(2)$-valued 0-current given by integrating along fibers[17] $\pi^{-1}(p) = (\pi^{\mathcal{Z}}_X)^{-1}(p)$:

$$R_{\mathcal{Z}}(p) \;=\; \frac{1}{2\pi\sqrt{-1}} \int_{\pi^{-1}(p)} \log z_1 \mathrm{dlog} z_2 \wedge \mathrm{dlog} z_3$$

$$+ \int_{\pi^{-1}(p)\cap T_{z_1}} \log z_2 \mathrm{dlog} z_3 \;+\; 2\pi\sqrt{-1} \sum_{\pi^{-1}(p)\cap T_{z_1}\cap T_{z_2}} \log z_3.$$

This gives a class in $H^0(\eta_X, \mathbb{C}/\mathbb{Z}(2))$ and is therefore a "constant" with $\mathbb{Z}(2)$ jumps. $\Omega_{\mathcal{Z}} = 0$, while the $(2d{-}1)$-cycle $T_{\mathcal{Z}}$ with support on $\left\{p \in X \setminus V_{\mathcal{Z}} \,\middle|\, \pi^{-1}(p) \cap T^3_\square\right\}$ has a primitive. This is just a $2d$-dimensional "patch" which annihilates $R_{\mathcal{Z}}$'s jumps: so that $R_{\mathcal{Z}} - 4\pi^2 T_{\mathcal{Z}} = R'_{\mathcal{Z}}$ really *is* a constant (this is unimportant, though). This formula already seems new.

[17]Here one should think of $p \in X \setminus V_{\mathcal{Z}}$, where $V_{\mathcal{Z}}$ is those points over which (a component of) the fiber lies entirely in a face of $\square^3$.

2.4.4. An attractive application to polylogarithms. Recall the maps $\bar{\rho}_n(\ell)$ of §1.2, which in general were only conjecturally well-defined on the terms of the Goncharov complex $G^n(\mathbb{C}(X), -\bullet)$ (where essentially $-\bullet = \ell$, and $n \leq \ell \leq 2n-1$ is where the nonzero entries lie). If these are well-defined and Beilinson-Soulé holds then we get a diagram of maps of complexes

$$\begin{tikzcd}
 & Z^n(\mathbb{C}(X), -\bullet) \arrow[r, "\mathcal{R}"] & \mathcal{C}_{\mathcal{D}}^{2n-2d+\bullet}(\eta_X, \mathbb{Z}(n-d)) \\
 & C^n(\mathbb{C}(X), -\bullet) \arrow[u, hook, "\simeq"'] \arrow[d, "\simeq"] & \\
G^n(\mathbb{C}(X), -\bullet) \arrow[r, "\bar{\rho}_n(-\bullet)"] & A^n(\mathbb{C}(X), -\bullet) &
\end{tikzcd}$$

descending to a composite

$$\begin{array}{ccccc}
H^*\left(G^n(\mathbb{C}(X), -\bullet)\right) & \overset{\bar{\rho}}{\longrightarrow} & CH^n(\mathbb{C}(X), -*) & \overset{\mathcal{R}}{\longrightarrow} & H_{\mathcal{D}}^{2n+*}(\eta_X, \mathbb{Z}(i)) \\
 & & & & \downarrow \pi_{\mathbb{R}} \\
 & & & & H_{\mathcal{D}}^{2n+*}(\eta_X, \mathbb{R}(i)).
\end{array}$$

It seems consonant with Goncharov's program[18] that this should agree (on the cohomological level) with his maps defined explicitly on the level of complexes[19]

$$r_n(2n+\bullet) : G^n(\mathbb{C}(X), -\bullet) \to \Gamma(\Omega^{2n+\bullet-1}_{(\eta_X)^\infty, \mathbb{R}})$$

defining a map

$$H^*\left(G^n(\mathbb{C}(X), -\bullet)\right) \to H^{2n+*-1}(\eta_X, \mathbb{R}(n-1)).$$

In particular, $r_n(1)$ is given explicitly on $G^n(\mathbb{C}(X), 2n-1) = \mathcal{B}_n(\mathbb{C}(X))$ by $\{f\}_n \mapsto \mathcal{L}_n(f)$ (where $\mathcal{L}_n$ is the generalized Bloch-Wigner "real single-valued n-logarithm"),[20] and the homology of the Goncharov complex at that term

[18] namely, the comparison of the Lie-motivic and Grassmanian "polylogarithms", as discussed in **[GZ]**. These are presented as cochains in the double (triple) complex computing Deligne cohomology of the (bi)Grassmanian *complex*, and these should differ by a coboundary. Our composition corresponds roughly to a pullback of the Grassmanian polylogarithm via the section $\bar{\rho}(\cdot)$; the $r_n(*)$-type maps correspond to the Lie-motivic side of things.

[19] to avoid confusion: these are *not* the maps with which we compare in §3.1.

[20] e.g., for $n = 2, 3$ these are, putting $\mathrm{Li}_1(z) = -\log(1-z)$,

$$\mathcal{L}_2(z) = \Im\{\mathrm{Li}_2(z)\} - \log|z| \cdot \Im\{\mathrm{Li}_1(z)\} \qquad \text{and}$$

$$\mathcal{L}_3(z) = \Re\{\mathrm{Li}_3(z)\} - \log|z| \cdot \Re\{\mathrm{Li}_2(z)\} + \frac{1}{3}\log^2|z| \cdot \Re\{\mathrm{Li}_1(z)\}.$$

is just $\ker(\delta)$ (see §1.2). So we can make a very concrete conjecture here, namely that the composition

$$\begin{array}{ccccc} \mathcal{B}_n(\mathbb{C}(X)) \supseteq \ker\delta & \overset{\bar{\rho}_n(2n-1)}{\longrightarrow} & CH^n(\mathbb{C}(X),\, 2n-1) & \overset{\mathcal{R}}{\longrightarrow} & H^1_{\mathcal{D}}(\eta_X, \mathbb{Z}(n)) \\ & & & & \| \\ & & & & H^0(\eta_X, \mathbb{C}/\mathbb{Z}(n)) \\ & & & & \downarrow \pi_{\mathbb{R}} \\ & & & & \mathbb{R}(n-1) \end{array}$$

should coincide with (and so give explicit $\mathbb{C}/\mathbb{Z}(n)$-valued "lifts" of[21]) the $\mathcal{L}_n$ on $\ker\delta$. (Here $\pi_{\mathbb{R}}$ is $\Im$ for n even, $\Re$ for n odd.) We will actually prove this for $n = 2$ in §3.1.2. Notice that (on $\ker\delta$) the answer is always a constant; this reflects the "rigidity" of the kernel, which is (at least conjecturally) already generated over $\bar{\mathbb{Q}}$ (at least this is well-known for $n = 2$). So for purposes of computation we lose nothing by setting $X =$a point. Now while we can write down the maps here, e.g. for $x \in \mathbb{C}$

$$\rho_n(2n-1)(x) \;=\; \pm Alt_{2n-1}\left(1-z_1, 1-\frac{z_2}{z_1}, \ldots, 1-\frac{z_{n-1}}{z_{n-2}}, 1-\frac{x}{z_{n-1}}, z_1, \ldots, z_{n-1}\right)$$

$$\in A^n(\mathbb{C}, 2n-1) \;=\; \frac{C^n(\mathbb{C}, 2n-1)}{S^n(\mathbb{C}, 2n-1)},$$

which induces $\bar{\rho}_n := \bar{\rho}_n(2n-1)$ on $\xi = \sum m_i\{x_i\}_n \in \mathcal{B}_n(\mathbb{C})$, there is the problem of "lifting" $\bar{\rho}(\xi)$ from a $\bar{\partial}_{\mathcal{B}}$-closed element of A^n to a $\partial_{\mathcal{B}}$-closed element of C^n, and this relies on Beilinson-Soulé. This conjecture is known for only $n = 2$, although $\bar{\rho}_n(2n-1)$ is known to be well-defined into $A^n(\mathbb{C}, 2n-1)$ for both $n = 2$ **[GM]** and $n = 3$ **[Zh]**. So one could still try computations for $n = 3$.

Before proving our conjecture for $n = 2$, we had "checked" it on the two elements (writing now $i = \sqrt{-1}$)

$$4\{i\}_2 \quad \text{and} \quad 2\left\{\frac{1+\sqrt{-7}}{2}\right\}_2 + \left\{\frac{-1+\sqrt{-7}}{4}\right\}_2 \qquad \in \ker(\delta = st),$$

and we now run through just the first one, to indicate what sort of computation we have in mind.

First, $st(4\{i\}_2) =$

$$4\cdot(1-i)\wedge i \;=\; (1-i)\wedge i^4 \;=\; (1-i)\wedge 1 \;=\; 1 \quad\Longrightarrow\quad 4\{i\}_2 \in \ker(st) \subseteq \mathcal{B}_2(\mathbb{C})$$

More generally, **[Go1]** has

$$\mathcal{L}_n(z) := \left\{\begin{array}{c} \Re\ (n \text{ odd}) \\ \Im\ (n \text{ even}) \end{array}\right\} \sum_{k=0}^{n-1}\left(\frac{2^k\beta_k}{k!}\right)\log^k|z|\cdot \mathrm{Li}_{n-k}(z)$$

where β_k are Bernoulli numbers (see footnote in §2.2.2 for the Li).

[21] This seems relevant (for the $n = 2$ case) to the approach to the Rogers function in **[Ha]** or Bloch's other book **[B2]**.

as asserted. The associated rationally parametrized (or "fractional linear") cycle is

$$\bar{\rho}_2(3)\,(4\{i\}_2) \;=\; 4\,Alt_3\left(1-z,\,1-\frac{i}{z},\,z\right)\;\in A^2(\mathbb{C},3)$$

which is $\bar{\partial}_{\mathcal{B}}$-closed; that is,

$$\partial_{\mathcal{B}}\left(\bar{\rho}_2(3)(4\{i\}_2)\right) \;=\; 4\,Alt_2\,(1-i,\,i)\;\in\; S^2(\mathbb{C},2)\;=\;C^1(\mathbb{C},1)\bigwedge\partial_{\mathcal{B}}C^1(\mathbb{C},2).$$

Since $S^2(\mathbb{C},*)$ is acyclic, one may write this as $\partial_{\mathcal{B}}$ of something $[\in S^2(\mathbb{C},3) = C^1(\mathbb{C},1)\bigwedge C^1(\mathbb{C},2)]$; in fact,

$$4\,(1-i,\,i)\;=\;\partial_{\mathcal{B}}\left(\frac{(z-i)^4}{(z-1)^4},\,1-i,\,z\right),$$

and so we can modify the original A-cycle to get a ($\partial_{\mathcal{B}}$-closed) C-cycle

$$Alt_3\left\{4\left(1-z,\,1-\frac{i}{z},\,z\right)-\left(\frac{(z-i)^4}{(z-1)^4},\,1-i,\,z\right)\right\}\in C^2(\mathbb{C},3).$$

Now we must integrate $R^3_\square$ over this; we first try to calm it down a bit (get rid of the alternation if possible). Indeed, we get lucky here because

$$\mathcal{Z}\;=\;-4\left(1-\frac{i}{z},\,1-z,\,z\right)\;+\;\left(1-i,\,\frac{(z-i)^4}{(z-1)^4},\,z\right)$$

is also $\partial_{\mathcal{B}}$-closed and so (by proposition 1.2.1) differs from our C-cycle by $\mathrm{im}\partial_{\mathcal{B}}$, on which $\mathcal{R}$ is trivial (in this case that means it would change the $\int$ by only $\mathbb{Z}(2)$). Now the integral $\int_{\mathcal{Z}} R^3_\square$ pulls back to

$$\begin{aligned}\int_{\mathbb{P}^1}\iota^*_{\mathcal{Z}}R^3_\square \;&=\; -4\left\{\frac{1}{2\pi i}\int_{\mathbb{P}^1}\log(1-\frac{i}{z})\mathrm{dlog}(1-z)\wedge\mathrm{dlog}z\right.\\ &\qquad\left. +\int_{T_{(1-\frac{i}{z})}}\log(1-z)\mathrm{dlog}z\;+\;2\pi i\int_{T_{(1-\frac{i}{z})}\cap T_{1-z}}\log z\right\}\end{aligned}$$

plus zero for the second term of $\mathcal{Z}$, since $T_{1-i}=\emptyset$ and $\mathrm{dlog}\frac{(z-i)^4}{(z-1)^4}\wedge\mathrm{dlog}z\;=0$ on $\mathbb{P}^1$ by type; the same goes for the first term of the above. Moreover, $T_{(1-\frac{i}{z})}$ is a vertical segment from 0 to i while $T_{1-z}=\mathbb{R}^{\geq 1}\cup\{\infty\}$, and so they never intersect. so we are left with the middle term

$$-4\int_0^i\log(1-z)\mathrm{dlog}z\;=\;4\mathrm{Li}_2(i),$$

in the "standard" branch. Now all we have to do is check that its imaginary part agrees with

$$r_2(1)(4\{i\}_2)\;=\;4\mathcal{L}_2(i)\;=\;4\left\{\Im\mathrm{Li}_2(i)\;+\;\log|i|\arg(1-i)\right\},$$

but since $\log|i|=1$ this is trivial. In general the terms involving combinations of log (and lower-order polylogarithms, if $i>2$) won't be zero like this.

So we cannot get the "lift" of the $\mathcal{L}_n$ just by taking Li_n on $\ker\delta$ (and so this "application" would solve a nontrivial problem).

REMARK 2.4.6. Already for $n = 2$ I have no idea whether (a) $\ker\delta$ is dense in $\mathbb{C}$ or (b) the $\mathbb{C}/\mathbb{Z}(2)$-valued function obtained by this procedure is continuous. Clearly the imaginary part $(= \mathcal{L}_2)$ is, but my hunch says the real part is "pathological".

CHAPTER 3

Real and Relative Regulators

3.1. Real Regulators and Goncharov's Construction

In the late 1970s Bloch [**B1**] defined a *real* regulator on K_2 of an elliptic curve E, via the Abel-Jacobi map

$$K_2^M(\mathbb{C}(E)) \supseteq \ker(\mathrm{Tame}) \underset{\cong}{\longleftarrow} CH^2(E,2) \longrightarrow H^2_{\mathcal{D}}(E,\mathbb{R}(2)) \cong H^1(E,\mathbb{R})$$

$$\mathbf{f} \longmapsto \underset{\text{completing } \gamma_{\mathbf{f}}}{\Gamma} \longmapsto \left\{ \omega \mapsto \int_{\partial^{-1}\Gamma} \Im\left(\frac{\mathrm{d}z_1}{z_1} \wedge \frac{\mathrm{d}z_2}{z_2}\right) \wedge \pi_E^*\omega \right\}$$

which is a special case of (the imaginary part of) the consruction in §1.3. He computed its image by way of the "elliptic dilogarithm"; his major discovery was that, for E having complex multiplication *and* defined over a number field, this was related to the value of the Hasse zeta function of E at $s = 2$. Five years later in his seminal paper [**Be**], Beilinson introduced real regulators[1] on "motivic cohomology"

$$H^i_{\mathcal{M}}(X,\mathbb{Q}(p)) \underset{r_{\mathrm{Be}}}{\longrightarrow} H^i_{\mathcal{D}}(X,\mathbb{R}(p))$$

along with conjectures predicting that for X smooth projective over a number field (and $2p - i \geq 2$), $\mathrm{im}(r_{\mathrm{Be}})$ should be a lattice with covolume equal to a rational multiple of the special value $L^{(i-1)}(X,p)$.

Now the motivic cohomology groups, say as used by Beilinson, were originally definied in terms of Quillen K-theory and its associated "γ-filtration" by Adams operations [**So**]:

$$H^i_{\mathcal{M}}(X,\mathbb{Q}(p)) \cong Gr_p^{\gamma} K_{2p-i}(X).$$

The higher Chow groups were constructed by Bloch to offer a more geometric, alternative picture for these groups; in particular, he proved the

[1]these in a sense generalize the Dirichlet regulator on K_1 of a number field F, which is just the multiplicative group of units u in the ring of integers (of F). If $\sigma_1, \ldots, \sigma_n$ ($n = [F:\mathbb{Q}]$) is a list of the distinct embeddings $F \hookrightarrow \mathbb{C}$, then

$$r(u) := (\log|\sigma_1(u)|, \ldots, \log|\sigma_n(u)|) \in \mathbb{R}^n\ ;$$

and roughly speaking, it is the covolume of the $\mathbb{Q}$-lattice in $\mathbb{R}^n$ (generated by all $u \in K_1(F)$ in this fashion) that Dirichlet's theorem then relates to the behavior of $\zeta_F(s)$ at $s = 0$. See [**Ra**] for a more complete description.

generalization (mod torsion)

$$K_j(X) \cong \oplus_p CH^p(X,j)$$

of Grothendieck-Riemann-Roch (the case $j=0$), and in fact that

$$H^i_{\mathcal{M}}(X,\mathbb{Q}(p)) \cong CH^p(X,2p-i)_{\mathbb{Q}}.$$

So it would be perfectly natural for someone to try and realize the maps r_{Be} explicitly on $CH^p(X,n[=2p-i])$, and this is essentailly what Goncharov did in [**Go1**]. But we should be more precise here. His (real) regulator formula r is compatible with the product structure on higher Chow and Deligne cohomology over the generic points of all subvarieties of X (of any codimension), as is r_{Be}. So it follows from a spectral sequence argument that r_{Be} and r coincide as maps $Gr^i_N CH^p(X,n)_{\mathbb{Q}} \to Gr^i_N H^{2p-n}_{\mathcal{D}}(X,\mathbb{Q}(p))$. As far as I know this is all that is known at the present stage.

3.1.1. Comparison of §2.4.1 **with Goncharov.** Equally natural is the question (resolved affirmatively in this subsection) as to whether our Abel-Jacobi maps provide $\mathbb{C}/\mathbb{Z}(p)$-"lifts" of r, or (more precisely) whether the compositions

$$CH^p(X,n) \underset{(\S 2.4.1)}{\overset{AJ}{\longrightarrow}} H^{2p-n}_{\mathcal{D}}(X,\mathbb{Q}(p)) \overset{\pi^p_{\mathbb{R}}}{\longrightarrow} H^{2p-n}_{\mathcal{D}}(X,\mathbb{R}(p))$$

are the same as Goncharov's maps (defined below).

In order to prove this we need to slightly modify the map of complexes

$$\mathcal{R}_X : Z^p(X,-\bullet) \longmapsto \text{Cone}\left\{C_{2d-2p-\bullet}(X,\mathbb{Z}(p)) \oplus F^p{}'\mathcal{D}^{2p+\bullet}_{X,\mathbb{C}}(X) \to {}'\mathcal{D}^{2p+\bullet}_{X,\mathbb{C}}(X)\right\}[-1]$$

given at $-\bullet = n$ by

$$\mathcal{Z} \longmapsto \mathcal{R}_X(\mathcal{Z}) := (-2\pi i)^{p-n}\left((2\pi i)^n T_{\mathcal{Z}}, \Omega_{\mathcal{Z}}, R_{\mathcal{Z}}\right)$$

and inducing AJ. recall from §1.2.1 that alternation over $S_n \ltimes (S_2)^{\oplus n}$

$$Alt_n : Z^p(X,-\bullet) \to Z^p(X,-\bullet)$$

is also a map of complexes, which descends to the identity on $CH^p(X,n)$ (by Corollary 1.2.2a); clearly the composition $\mathcal{R}_X \circ Alt_n$ must also induce AJ. Moreover by our branch choices[2] for $\log z_i$ (as 0-currents) we have that $Alt_n T^n_{\square} = T^n_{\square}$, $Alt_n \Omega^n_{\square} = \Omega^n_{\square}$, and $Alt_n R^n_{\square}$ reduces to alternation over S_n; this leads to $T_{Alt_n \mathcal{Z}} = T_{\mathcal{Z}}$, $\Omega_{Alt_n\mathcal{Z}} = \Omega_{\mathcal{Z}}$ and[3]

$$R_{Alt_n\mathcal{Z}} = \pi_X^{Alt\mathcal{Z}}{}_* \pi_{\square}^{Alt\mathcal{Z}*} R^n_{\square} = \pi^{\mathcal{Z}}_{X*}\pi^{\mathcal{Z}*}_{\square}\left(Alt_n R^n_{\square}\right) =: (Alt_n R)_{\mathcal{Z}}.$$

Which is to say, sending

$$\mathcal{Z} \longmapsto (-2\pi i)^{p-n}\left((2\pi i)^n T_{\mathcal{Z}},\, \Omega_{\mathcal{Z}},\, (Alt_n R)_{\mathcal{Z}}\right)$$

---

[2]which are completely symmetric with respect to $S_n \ltimes (S_2)^{\oplus n}$ (the cuts are at $\arg z_I = \pm\pi$, for all i). For AJ to be well-defined we must at least have S_n-symmetry (AJ on all of X is a good deal more sensitive to branch chages than the Milnor regulator [over η_X]).

[3]noting that $Alt_n R^n_{\square} = \frac{1}{n!}\sum_{\sigma\in S_n} R\left(z_{\sigma(1)},\ldots,z_{\sigma(n)}\right)$

also induces AJ.

To compose this with the "real" projection, first set

$$\pi_{\mathbb{R}}^{p} : \mathbb{C} = \mathbb{R}(p) \oplus \mathbb{R}(p-1) \longrightarrow \mathbb{R}(p-1)$$

and define

$$\mathcal{C}^{\bullet}_{\mathcal{D},\mathbb{R}}(X,\mathbb{R}(p)) :\cong \text{Cone}\left\{ F^{p\prime}\mathcal{D}^{\bullet}_{X,\mathbb{C}}(X) \underset{\pi_{\mathbb{R}}^{p}}{\rightarrow} {}^{\prime}\mathcal{D}^{\bullet}_{X,\mathbb{R}(p-1)}(X) \right\}[-1],$$

$$H^{2p-n}_{\mathcal{D}}(X,\mathbb{R}(p)) := H^{-n}\left(\mathcal{C}^{2p+\bullet}_{\mathcal{D},\mathbb{R}}(X,\mathbb{R}(p))\right).$$

We can induce

$$\pi_{\mathbb{R}}^{p} :\ H^{2p-n}_{\mathcal{D}}(X,\mathbb{Z}(p)) \to H^{2p-n}_{\mathcal{D}}(X,\mathbb{R}(p))$$

by sending

$$(a,\, b,\, c) \longmapsto (b,\, \pi_{\mathbb{R}}^{p} c),$$

and so $\pi_{\mathbb{R}}^{p} \circ AJ$ may be computed by

$$\mathcal{Z} \longmapsto \Big((-2\pi i)^{p-n}\Omega_{\mathcal{Z}}, \underbrace{\pi_{\mathbb{R}}^{p}\left\{(-2\pi i)^{p-n}(Alt_n R)_{\mathcal{Z}}\right\}}_{= \pi^{\mathcal{Z}}_{X*}\pi^{\mathcal{Z}*}_{\square}\left(\pi_{\mathbb{R}}^{p}\left[(-2\pi i)^{p-n} Alt_n R^n_{\square}\right]\right)} \Big).$$

Goncharov's map

$$\mathsf{r} :\ CH^p(X,n) \to H^{2p-n}_{\mathcal{D}}(X,\mathbb{R}(p))$$

is given by setting[4]

$$Alt_n \sum_{j\geq 0} \binom{n}{2j+1} \log|z_1| \, (\text{dlog}|z_2| \wedge \ldots \wedge \text{dlog}|z_{2j+1}|) \wedge (\text{d}i \arg z_{2j+2} \wedge \ldots \wedge \text{d}i \arg z_n)$$

$$=: r(z_1, \ldots, z_n) \ =: \ r^n_{\square} \quad \in {}^{\prime}\mathcal{D}^{n-1}_{\mathbb{R}(n-1)}(\square^n)$$

and (for $\mathcal{Z} \in Z^p(X,n)$)

$$r_{\mathcal{Z}} := \pi^{\mathcal{Z}}_{X*}\pi^{\mathcal{Z}*}_{\square} r^n_{\square},$$

and simply sending

$$\mathcal{Z} \longmapsto \mathsf{r}(\mathcal{Z}) := (-2\pi i)^{p-n}\left(\Omega_{\mathcal{Z}}, r_{\mathcal{Z}}\right) \in \mathcal{C}^{2p-n}_{\mathcal{D},\mathbb{R}}(X,\mathbb{R}(p)).$$

The objective is now to show that for $\mathcal{Z}$ any $\partial_{\mathcal{B}}$-cycle, the difference

$$\mathsf{r}(\mathcal{Z}) - \pi_{\mathbb{R}}^{p}\left\{\mathcal{R}(Alt_n\mathcal{Z})\right\} \ = \ \Big(0,\ \pi^{\mathcal{Z}}_{X*}\pi^{\mathcal{Z}*}_{\square} \underbrace{\left[(-2\pi i)^{p-n} r^n_{\square} - \pi_{\mathbb{R}}^{p}\left\{(-2\pi i)^{p-n} Alt_n R^n_{\square}\right\}\right]}_{=:\ \Delta^n} \Big)$$

is a coboundary in the cone complex $\mathcal{C}^{2p-n}_{\mathcal{D},\mathbb{R}}(X,\mathbb{R}(p))$; it is enough to show that $\pi^{\mathcal{Z}}_{X*}\pi^{\mathcal{Z}*}_{\square}\Delta^n = \text{d}[\,\cdot\,]$ on X. In fact, suppose there were an alternating $(n-$

[4] this differs from Goncharov's current by a factor of $(2\pi i)^n$.

2)-current $Alt_n S \in {}'\mathcal{D}^{n-2}_{\mathbb{R}(p-1)}(\square^n)$ with $\mathrm{d}[Alt_n S] = \Delta^n + \text{residues on } \partial\square^n$. The residues must be of the form[5]

$$C = 2\pi i \sum_{\ell} (-1)^{\ell} Q(z_1, \ldots, \widehat{z_\ell}, \ldots, z_n) \cdot \delta_{(z_\ell)} = 2\pi i \sum_{\ell,e} \rho^e_{\ell *} Q^{n-1}_{\square}$$

for some current $Q \in {}'\mathcal{D}^{n-3}_{\mathbb{R}(p-2)}(\square^{n-1})$, because Alt_n is idempotent (and commutes with d) $\implies C$ is Alt-invariant. So we find

$$\mathrm{d}\left[\pi^{\mathcal{Z}}_{X*}\pi^{\mathcal{Z}*}_{\square} Alt_n S\right] - \pi^{\mathcal{Z}}_{X*}\pi^{\mathcal{Z}*}_{\square}\Delta^n = 2\pi i\, \pi^{\mathcal{Z}}_{X*}\pi^{\mathcal{Z}*}_{\square} \sum_{i,e} \rho^e_{i*} Q^{n-1}_{\square}$$

$$= 2\pi i\, \pi^{\partial_{\mathcal{B}}\mathcal{Z}}_{X}{}_{*}\pi^{\partial_{\mathcal{B}}\mathcal{Z}*}_{\square} Q^{n-1}_{\square} = 0$$

since $\partial_{\mathcal{B}}\mathcal{Z} = 0$ is assumed. Therefore to equate the two real regulators it is sufficient to exhibit $Alt_n S$ with $\mathrm{d}[Alt_n S] = \Delta^n$ on $\square^n \setminus \partial\square^n$.

Next put $\pm_n := \begin{cases} i^{n-2}, & n \text{ even} \\ i^{n-1}, & n \text{ odd} \end{cases}$ and notice that

$$\pi^p_{\mathbb{R}}\left\{(-2\pi i)^{p-n} Alt_n R^n_{\square}\right\} = (-2\pi i)^{p-1}\Re\left\{\frac{1}{(-2\pi i)^{n-1}} Alt_n R^n_{\square}\right\}$$

$$= \pm_n (-2\pi i)^{p-1}\frac{1}{(-2\pi)^{n-1}}\left\{\begin{array}{l} \Re,\ n \text{ odd} \\ \Im,\ n \text{ even}\end{array}\right\} Alt_n R^n_{\square}$$

$$= \pm_n (-2\pi i)^{p-n}(i^{n-1})\left\{\begin{array}{c} \Re \\ \Im \end{array}\right\} Alt_n R^n_{\square} = (-2\pi i)^{p-n}\left\{\begin{array}{c} \Re \\ i\cdot\Im \end{array}\right\} Alt_n R^n_{\square},$$

while (with $\varepsilon_n := \begin{cases} i, & n \text{ even} \\ 1, & n \text{ odd} \end{cases}$)

$$r^n_{\square} = \varepsilon_n \cdot \pm_n \cdot \left(r^n_{\square}/i^{n-1}\right) = \varepsilon_n \cdot \left\{\begin{array}{c} \pm_n Alt_n \sum_{j\geq 0}(-1)^j \binom{n}{2j+1} \log|z_1| \cdot \\ (\mathrm{dlog}|z_2| \wedge \ldots \wedge \mathrm{dlog}|z_{2j+1}|) \wedge (\mathrm{d}\arg z_{2j+2} \wedge \ldots \wedge \mathrm{d}\arg z_n) \end{array}\right\}$$

$$=: \varepsilon_n \cdot \tilde{r}^n_{\square}.$$

So we've reduced this problem to producing an alternating current S_0 with

$$\mathrm{d}[S_0] = \left\{\begin{array}{l} \Re,\ n \text{ odd} \\ \Im,\ n \text{ even}\end{array}\right\} Alt_n R^n_{\square} - \tilde{r}^n_{\square}$$

(on $\square^n \setminus \partial\square^n$).

Working under the Alt sign[6] now, we will use the following notation for real $(n-1)$-currents (on $\square^n$):

$$\left\langle \left\{\begin{array}{c} L \\ A \end{array}\right\} | \, i \, | \, j \right\rangle_k := \left\{\begin{array}{c} \log|z_1| \\ \arg z_1 \end{array}\right\} (\mathrm{dlog}|z_1| \wedge \ldots \wedge \mathrm{dlog}|z_{i+1}|) \wedge$$

[5](see §2.4.1 for $\sum_{\ell,e}$)

[6]*Alt* is implicitly applied to everything; so these are alternating currents (other wise the lemma would be false!).

$$\left(\mathrm{d}\arg z_{i+2} \wedge \ldots \wedge \mathrm{d}\arg z_{i+j+1}\right) \cdot \delta_{T_{z_{i+j+2}} \cap \ldots \cap T_{z_n}}$$

where $i+j+k+1=n$; we will also use $(n-2)$-currents

$$\langle L\,A \mid i \mid j \rangle_k := \log|z_1| \arg z_2 \left(\mathrm{dlog}|z_3| \wedge \ldots \wedge \mathrm{dlog}|z_{i+2}|\right) \wedge$$

$$\left(\mathrm{d}\arg z_{i+3} \wedge \ldots \wedge \mathrm{d}\arg z_{z_{i+j+2}}\right) \cdot \delta_{T_{z_{i+j+3}} \cap \ldots \cap T_{z_n}}$$

where $i+j+k+2=n$. We have the following basic[7]

LEMMA 3.1.1. $d[\langle L\,A \mid i \mid j \rangle_k] = \langle L \mid i \mid j+1 \rangle_k - \langle A \mid i+1 \mid j \rangle_k - 2\pi \langle L \mid i \mid j \rangle_{k+1}$,

and it is also easy to see that in this notation

$$\pm_n \cdot \left\{ \begin{array}{l} \Re,\ n \text{ odd} \\ \Im,\ n \text{ even} \end{array} \right\} Alt_n R^n_\square =$$

$$\sum_{i=1}^{n} \sum_{j \geq 0} (2\pi)^{i-1} (-1)^j \left[\binom{n-i}{2j} \langle L\,|2j|\,n-2j-1 \rangle_{i-1} + \binom{n-i}{2j+1} \langle A\,|2j+1|\,n-2j-i-1 \rangle_{i-1} \right]$$

while

$$\tilde{r}^n_\square = \pm_n \sum_{j \geq 0} (-1)^j \binom{n}{2j+1} \langle L \mid 2j \mid n-2j-1 \rangle_0 .$$

Recalling the "Pascal's triangle" rule $\binom{n-1}{2j} - \binom{n}{2j+1} = -\binom{n-1}{2j+1}$,
we find that

$$\pm_n \left(\left\{ {\Re \atop \Im} \right\} Alt_n R^n_\square - \tilde{r}^n_\square \right) =$$

$$\sum_{j \geq 0} (-1)^j \binom{n-1}{2j+1} \left[\langle A|2j+1|n-2j-2 \rangle_0 - \langle L|2j|n-2j-1 \rangle_0 \right]$$

$$+ \sum_{i=2}^{n} \sum_{j \geq 0} (2\pi)^{i-1} (-1)^j \left[\binom{n-i}{2j} \langle L|2j|n-2j-1 \rangle_{i-1} + \binom{n-i}{2j+1} \langle A|2j+1|n-2j-i-1 \rangle_{i-1} \right] \tag{3.1.1}$$

By the Lemma,

$$\mathrm{d} \left[\sum_{j \geq 0} (-1)^j \binom{n-1}{2j+1} \langle L\,A \,|\, 2j \,|\, n-2j-2 \rangle_0 \right] =$$

[7] the important thing to notice in verifying this is that swapping $\log|\cdot|$ and $\arg(\cdot)$, i.e. changing $\log|z_1| \arg z_2 \to \arg z_1 \log|z_2|$, generates a (-1), while swapping the differentials, e.g. $\mathrm{dlog}|z_{i+2}| \wedge \mathrm{d}\arg z_{i+3} \to \mathrm{d}\arg z_{i+2} \wedge \mathrm{dlog}|z_{i+3}|$, does not. (The point is always to keep the order of the $\{z_i\}$ intact.)

$$-\sum_{j\geq 0}(-1)^j \left(\begin{array}{c} n-1 \\ 2j+1 \end{array} \right) [\langle A\,|\,2j+1\,|\,n-2j-2\rangle_0 - \langle L\,|\,2j\,|\,n-2j-1\rangle_0]$$

$$-2\pi \sum_{j\geq 0}(-1)^j \left(\begin{array}{c} n-1 \\ 2j+1 \end{array} \right) \langle L\,|\,2j\,|\,n-2j-2\rangle_1 .$$

Adding this coboundary-current to the current of equation (3.1.1) gives

$$\sum_{j\geq 0} 2\pi(-1)^j \left(\begin{array}{c} n-2 \\ 2j+1 \end{array} \right) [\langle A\,|\,2j+1\,|\,n-2j-3\rangle_1 - \langle L\,|\,2j\,|\,n-2j-2\rangle_1]$$

$$+\sum_{i=3}^{n} \sum_{j\geq 0} \left[\left(\begin{array}{c} n-i \\ 2j \end{array} \right) \langle L\,|\,2j\,|\,n-2j-1\rangle_{i-1} + \left(\begin{array}{c} n-i \\ 2j+1 \end{array} \right) \langle A\,|\,2j+1\,|\,n-2j-i-1\rangle_{i-1} \right],$$

and so on. More generally, the S_0 we are looking for is $\pm_n$ times the current in brackets, in the following

PROPOSITION 3.1.2. *On* $\square^n \setminus \partial\square^n$, $\pm_n \left(\left\{ \begin{array}{c} \Re \\ \Im \end{array} \right\} Alt_n R^n_\square - \tilde{r}^n_\square \right) =$

$$d\left[\sum_{i=1}^{n-1} \sum_{j\geq 0} (2\pi)^{i-1}(-1)^j \left(\begin{array}{c} n-i \\ 2j+1 \end{array} \right) \langle L\,A\,|\,2j\,|\,n-2j-i-1\rangle_{i-1} \right],$$

and so r *and* $\pi^p_{\mathbb{R}} \circ AJ$ *coincide as maps:* $CH^p(X,n) \to H^{2p-n}_{\mathcal{D}}(X, \mathbb{R}(p))$.

We give some applications in the subsections that follow (as well as §3.2).

3.1.2. Application to $CH^m(\mathbb{C}, 2m-1)$. Using a result of Goncharov we can now prove the conjecture of §2.4.4 (regarding $\mathbb{C}/\mathbb{Z}(m)$-lifts of the $\mathcal{L}_m$ on $\ker(\delta) \subset \mathcal{B}_m(\mathbb{C})$) in the case $n = 2$. Writing $\rho_m(\alpha)$ for the linear subvariety of $\square^{2m-1}$

$\rho_m(2m-1)(a) :=$

$$(-1)^{m-1} Alt_{2m-1} \left(1-z_1, 1-\frac{z_2}{z_1}, \ldots, 1-\frac{a}{z_{m-1}}, z_1, \ldots, z_{m-1} \right) \in Z^m(\mathbb{C}, 2m-1)$$

parametrized by $\mathbb{P}^{m-1}$, Theorem 3.6 in **[Go1]** translates[8] to

[8] One needs to use the fact that (since $r(f_1, \ldots, f_{2n-1})$ is alternating multilinear in its entries)

$$r(z_1, \ldots\ z_{n-1},\ 1-z_1,\ z_1-z_2, \ldots,\ z_{m-2}-z_{m-1},\ z_{m-1}-a)$$
$$= r(1-z_1,\ 1-\frac{z_2}{z_1}, \ldots,\ 1-\frac{z_{m-1}}{z_{m-2}},\ 1-\frac{a}{z_{m-1}},\ z_1, \ldots,\ z_{m-1})$$

, together with the $(2\pi i)^{2m-1}$ difference between the $r^{2m-1}_\square$ here and the version in [Go]. Incidentally note that using these properties of r one can easily extend his computation to other linear subvarieties, e.g. $\mathcal{L}_m(\alpha_1 \cdot \ldots \cdot \alpha_m) =$

$$\frac{1}{(2\pi i)^{m-1}} \int_{\mathbb{P}^{n-1}} r(1-\alpha_1 z_1, 1-\alpha_2\frac{z_1}{z_2}, \ldots, 1-\alpha_{m-1}\frac{z_{m-1}}{z_{m-2}}, 1-\frac{\alpha_n}{z_{n-1}}, z_1, \ldots, z_{m-1}).$$

LEMMA 3.1.3. [Goncharov] For any $a \in \mathbb{C}$,

$$\varepsilon_m \cdot (-2\pi i)^{m-1} \mathcal{L}_m(a) \;=\; \int_{\rho_m(a)} r_\square^{2m-1} \;\; \Big(=\; r_{\rho_m(a)}\Big).$$

REMARK 3.1.4. Obviously this holds for an "unalternated" version of $\rho_m(a)$ since $r_\square^{2m-1}$ is itself alternated. (In particular we may use any permutation, with the corresponding sign.)

Another way of putting the Lemma is

$$\mathsf{r}(\rho_m(a)) = (0,\, \varepsilon_m \cdot \mathcal{L}_m(a)) \,\in\, \mathcal{C}^1_{\mathcal{D},\mathbb{R}}(Spec\mathbb{C}[=pt.], \mathbb{R}(m)).$$

Since the only coboundary 0-current is 0, if $\rho_m(a)$ were a $\partial_{\mathcal{B}}$-cycle then Proposition 3.1.2 $\implies$

$$\pi_{\mathbb{R}}^m \frac{1}{(-2\pi i)^{m-1}} R_{\rho_m(a)} = \frac{1}{(-2\pi i)^{m-1}} r_{\rho_m(a)} \left(= \varepsilon_m \cdot \mathcal{L}_m(a)\right).$$

But this is not so; even if one replaces $\rho_m(a)$ [given some $\xi = \sum m_j\{a_j\}_m \in \ker(\delta) \subset \mathcal{B}_m(\mathbb{C})$] by $\bar{\rho}_m(\xi) := \sum m_j \rho_m(a_j)$, this still has to be *completed* to a $\partial_{\mathcal{B}}$-cycle by the addition of $\mathcal{W} \in \mathcal{S}^m(\mathbb{C}, 2m-1)$ using Beilinson-Soulé. This is how we obtained the map from $\ker(\delta) \to CH^m(\mathbb{C}, 2m-1)$ in §2.4.4 (conjecturally for $m \geq 3$).

Now specialize to $m = 2$ and recall that

$$\mathcal{S}^2(\mathbb{C}, 3) = C^1(\mathbb{C}, 1) \wedge C^1(\mathbb{C}, 2),$$

so that one of z_1, z_2, z_3 is constant on each component of $\mathcal{W}$. Thus one can use **[Go1]** Theorem 3.3 or the following simple argument to ahow that

$$\int_{\mathcal{W}} r_\square^3 \;=\; \int_{\mathcal{W}} r(z_1, z_2, z_3) \;=\; 0.$$

Since $\mathcal{W}$ is a sum of curves,

$$\mathrm{dlog} z_k \wedge \mathrm{dlog} z_\ell = 0 \quad\implies\quad \mathrm{d}\arg z_k \wedge \mathrm{d}\arg z_\ell = \mathrm{dlog}|z_k| \wedge \mathrm{dlog}|z_\ell|$$

on $\mathcal{W}$, and so

$$\iota_{\mathcal{W}}^* r(z_1, z_2, z_3) \;=\; \iota_{\mathcal{W}}^* Alt_3 \left\{-3\log|z_1| \mathrm{d}\arg z_2 \wedge \mathrm{d}\arg z_3 + \log|z_1| \mathrm{dlog}|z_2| \wedge \mathrm{dlog}|z_3|\right\}$$

$$=\; -2\, i_{\mathcal{W}}^* Alt_3 \left\{\log|z_1| \mathrm{dlog}|z_2| \wedge \mathrm{dlog}|z_3|\right\}.$$

On a component $\mathcal{W}_0$ on which, say, $z_1 = c$, the problem reduces to showing

$$\log|c| \int_{\mathcal{W}_0} \mathrm{dlog}|z_2| \wedge \mathrm{dlog}|z_3| \;=\; 0.$$

But this is clear because

$$\mathrm{d}[\log|z_2| \mathrm{dlog}|z_3|] \;=\; \mathrm{dlog}|z_2| \wedge \mathrm{dlog}|z_3|$$

(there are no residues).

Therefore $\int_{\mathcal{W}} r_{\square}^3 = 0$, and on $\mathcal{Z}_\xi = \bar{\rho}_2(\xi) + \mathcal{W} = \sum m_j \rho_2(a_j) + \mathcal{W}$ we have (using Lemma 3.1.3)

$$-\frac{1}{2\pi i}\int_{\mathcal{Z}_\xi} r_{\square}^3 \;=\; -\frac{1}{2\pi i}\int_{\sum m_j \rho_2(a_j)} r_{\square}^3 \;=\; i\sum m_j \mathcal{L}_2(a_j) \;=:\; i\mathcal{L}_2(\xi).$$

But $\mathcal{Z}_\xi$ is a $\partial_{\mathcal{B}}$-cycle and so

$$-\frac{1}{2\pi i}\int_{\mathcal{Z}_\xi} r_{\square}^3 = \pi_{\mathbb{R}}^2\left\{-\frac{1}{2\pi i}\int_{\mathcal{Z}_\xi} r_{\square}^3\right\} = \pi_{\mathbb{R}}^2\left\{R_{\mathcal{Z}_\xi}\right\};$$

and since $\pi_{\mathbb{R}}^2 = i\cdot\Im$, we have exactly

$$\Im\left\{R_{\mathcal{Z}_\xi}\right\} = \mathcal{L}_2(\xi),$$

which was the claim of §2.4.4.

3.1.3. Real Milnor regulator currents and $CH^n(X,n)$. Associated with the case $n = p$, $X = (n-1)$-dimensional projective variety, one has the following objects and maps:

$$\begin{tikzcd}
\left\{\begin{smallmatrix}\text{Steinberg}\\ \text{relations}\end{smallmatrix}\right\} \arrow[d, hook] & & \partial_{\mathcal{B}} Z^n(\eta_X, n+1) \arrow[d, hook] & \\
\otimes^n \mathbb{Z}\left[\mathbb{P}^1_{\mathbb{C}(X)} \setminus 0, \infty\right] \arrow[rr, "\gamma"', "\text{(graph)}"] \arrow[d] & & Z^n(\eta_X, n)) \arrow[d] & \\
K_n^M(\mathbb{C}(X)) \arrow[rr, "\bar{\gamma}", "\cong"'] & & CH^n(\eta_X, n) \arrow[r, "\mathcal{R} = R'"] & H^{n-1}(\eta_X, \mathbb{C}/\mathbb{Z}(n)) \\
K_n^M(X) \arrow[u, hook] \arrow[rr, phantom, "\bar{\gamma}^{-1}"] & & CH^n(X, n) \arrow[u, "\jmath_{\eta_X}^*"'] \arrow[r, "\mathcal{R} = AJ"] & H^{n-1}(X, \mathbb{C}/\mathbb{Z}(n)) \arrow[u, "\jmath_{\eta_X}^*"']
\end{tikzcd}$$

where $K_n^M(X)$ is by definition $\bar{\gamma}^{-1}\left\{\mathrm{im}(CH^n(X,n))\right\}$, and R' and AJ were constructed in §2.2 and §2.4. If $\mathbf{f} = \sum m_\alpha f_{1\alpha} \otimes \ldots \otimes f_{n\alpha} \in \otimes^n \mathbb{Z}\left[\mathbb{P}^1_{\mathbb{C}(X)} \setminus \{0,\infty\}\right]$ is such that $\{\mathbf{f}\} \in K_n^M(X)$, then there exists $\Gamma \in Z^n(X,n)$, $\partial_{\mathcal{B}}$-closed and "completing" $\overline{\gamma_{\mathbf{f}}}$ to a higher Chow cycle ($\gamma_{\mathbf{f}} = \jmath_{\eta_X}^*\Gamma$). Writing $R_{\mathbf{f}} = R_{\gamma_{\mathbf{f}}} = \sum m_\alpha R(f_{1\alpha}, \ldots, f_{n\alpha})$, we have also that $R_{\mathbf{f}} = \jmath_{\eta_X}^* R_\Gamma$, and these currents have respective d-closed lifts to η_X and X: $R'_{\mathbf{f}} = R_{\mathbf{f}} + (2\pi i)^n \partial^{-1}_{(X,V_{\mathbf{f}})} T_{\mathbf{f}}$, $R'_\Gamma = R_\Gamma + (2\pi i)^n \partial_X^{-1} T_\Gamma$. From §2.2.2, $R_{\mathbf{f}} - Alt_n R_{\mathbf{f}}$ is a d-coboundary on η_X, as is (using the beginning of §3.1.1) $R_\Gamma - (Alt_n R)_\Gamma$ on X, possibly modulo $(2\pi i)^n T_0$ for some topological cycle $T_0 \in Z_{n-1}(X, \mathbb{Q})$.

We slightly change $\pi_{\mathbb{R}}^n$ (by a factor of ε_n) so that now $\pi_{\mathbb{R}}^n = \begin{cases} \Re\,(n\,\text{odd}) \\ \Im\,(n\,\text{even}) \end{cases}$: $\mathbb{C} \to \mathbb{R}$; clearly this induces also a map $\mathbb{C}/\mathbb{Q}(n) \to \mathbb{R}$ of coefficients. Applying $\pi_{\mathbb{R}}^n$ kills $(2\pi i)^n T_0$, $(2\pi i)^n \partial^{-1} T_{\mathbf{f}}$, $(2\pi i)^n \partial^{-1} T_\Gamma$; in particular note that $\pi_{\mathbb{R}}^n R'_{\mathbf{f}} [= \pi_{\mathbb{R}}^n R_{\mathbf{f}}]$ is closed *without* a membrane term. Applying $\pi_{\mathbb{R}}^n$ to the above coboundaries on η_X and X, and combining this with the Proposition we find that $\pi_{\mathbb{R}}^n R'_{\mathbf{f}} - \tilde{r}_{\mathbf{f}}$ and $\pi_{\mathbb{R}}^n R'_\Gamma - \tilde{r}_\Gamma$ are also $\mathrm{d}[\,\cdot\,]$ (on η_X, X respectively). We can "see" this in a diagram:

$$\begin{tikzcd} H^{n-1}(X, \mathbb{C}/\mathbb{Q}(n)) \arrow[r, "\gamma^*"] \arrow[d, "\pi_{\mathbb{R}}^n"] & H^{n-1}(\eta_X, \mathbb{C}/\mathbb{Q}(n)) \arrow[d, "\pi_{\mathbb{R}}^n"] \\ H^{n-1}(X, \mathbb{R}) \arrow[r, "\gamma^*"] & H^{n-1}(\eta_X, \mathbb{R}) \end{tikzcd} \qquad \begin{tikzcd} \overline{[R'_\Gamma]} \arrow[r, mapsto] \arrow[d, mapsto] & \overline{[R'_{\mathbf{f}}]} \arrow[d, mapsto] \\ [\tilde{r}_\Gamma] \arrow[r, mapsto] & [\tilde{r}_{\mathbf{f}}] \end{tikzcd}$$

Referring to §4.5.6, suppose X is a *very general* type [A] complete intersection on $\mathbb{P}^{n+r}$ (of codimension $r+1$). Then according to the vanishing theorem stated there (and proved in Chapter 4), $\overline{[R'_{\mathbf{f}}]} = 0$; so $[\tilde{r}_{\mathbf{f}}] = 0$ and

$$[\tilde{r}_\Gamma] \in \ker \left\{ \jmath_{\eta_X}^* : H^{n-1}(X, \mathbb{R}) \to H^{n-1}(\eta_X, \mathbb{R}) \right\}.$$

Now $\Omega_\Gamma = 0$ and so $\tilde{r}_\Gamma$ completely describes $\mathsf{r}(\Gamma) = (0, \varepsilon_n \cdot \tilde{r}_\Gamma)$; moreover, any $\Gamma \in Z^n(X, n)$ with $\partial_{\mathcal{B}} \Gamma = 0$ gives rise to $\mathbf{f} \in K_n^M(X)$, and so $[\tilde{r}_\Gamma] \in \ker(\jmath_{\eta_X}^*)$ for all $[\Gamma] \in CH^n(X, n)$! Furthermore by Proposition 4.5.5, for n even $\ker(\jmath_{\eta_X}^*) = 0$, and for $n = 2n-1$ odd $\ker(\jmath_{\eta_X}^*) = \mathrm{im} H^{n-1}(\mathbb{P}^{n+r}, \mathbb{R})$. In the latter case we can completely describe $\mathsf{r}(\Gamma)$ by integrating against the appropriate power $[H]^{m-1}$ of the hyperplane class, or equivalently over $X \cap \left\{ \mathbb{P}^{m+r} \subset \mathbb{P}^{2m+r-1} \right\}$ (where $2m + r - 1 = n + r$), since

$$\mathrm{im} \left\{ H^{n-1}(\mathbb{P}^{n+r}) \overset{\iota_X^*}{\to} H^{n-1}(X) \right\} \overset{\cong}{\longrightarrow} \mathrm{coim} \left\{ H^{n-1}(X) \overset{\iota_*^X}{\to} H^{2r+n+1}(\mathbb{P}^{n+r}) \right\}.$$

Writing $\Gamma \cdot [H]^{m-1}$ for the intersection of $\Gamma \subset X \times \square^{2m-1} \subset \mathbb{P}^{2m+r-1} \times \square^{2m-1}$ with $\mathbb{P}^{m+r} \times \square^{2m-1}$, this becomes

$$\int_{X \cap \mathbb{P}^{m+r}} \tilde{r}_\Gamma = \int_{X \cap \mathbb{P}^{m+r}} \pi_{X*}^\Gamma \pi_\square^{\Gamma *} \tilde{r}_\square^{2m-1} = \int_{\Gamma \cdot [H]^{m-1}} \pi_\square^{\Gamma *} \tilde{r}_\square^{2m-1} = \int_{\pi_\square(\Gamma \cdot [H]^{m-1})} \tilde{r}_\square^{2m-1},$$

where one can think of $\pi_\square \circ \{\cdot [H]^{m-1}\}$ as giving a map

$$CH^{2m-1}(X, 2m-1) \overset{\cdot [H]^{m-1}}{\longrightarrow} CH^m(X \cap \mathbb{P}^{m+r}, 2m-1) \overset{\pi_\square}{\longrightarrow} CH^m(\mathbb{C}2m-1)$$

and the integral as computing the Goncharov regulator on the image. According to Lemma 3.1.3, this is computed by $\mathcal{L}_m$ for special linear cycles; more generally he calls it the Chow m-logarithm and conjectures that it can be computed in terms of the $\mathcal{L}_m$. Since $\tilde{r}_\Gamma$ gives a class in $H^{n-1}(X, \mathbb{R})$, the value of $\int_{X \cap \mathbb{P}^{m+r}} \tilde{r}_\Gamma$ is independent of the particular linear subspace

$\mathbb{P}^{m+r} \subset \mathbb{P}^{2m+r-1}$ chosen, so we get a family of cycles in $CH^m(\mathbb{C}, 2m-1)$ with constant Chow m-logarithm.

Now any $\Gamma \in Z^n(X,n)$ is of the form $\Gamma = \gamma^0 + \sum_{i\geq 1}\gamma^i$ where $\mathrm{supp}(\pi_X\gamma^i)$ has codimension i in X, and we write $V_\Gamma := \pi_X(\sum_{i\geq 1}\gamma^i)$. With this in mind we summarize the above discussion:

PROPOSITION 3.1.5. *Assume X is a very general $(n-1)$-dimensional type* [A] *complete intersection, and* $[\Gamma] \in CH^n(X,n)$. *Then for n even,* $[\mathsf{r}(\Gamma)] \in H^n_{\mathcal{D}}(X, \mathbb{R}(n))$ *is zero, while for n odd it is computed by a current* $\tilde{r}_\Gamma$ *whose periods are*
(a) a value of Goncharov's $\left(\frac{n+1}{2}\right)$*-logarithm, on* $\pi_\square\left([H]^{\frac{n-1}{2}}\cdot\Gamma\right) \in Z^{\frac{n+1}{2}}(\mathbb{C},n)$,
(b) zero on all cycles avoiding $V_\Gamma \subset X$;
and according to Proposition 4.5.5 *these two types of cycles span the space* $H_{n-1}(X,\mathbb{Q})$.

A few remarks are in order here.

First, this illustrates clearly the value of constructing a $\mathbb{C}/\mathbb{Z}(n)$-lift of the real regulator as we did in §2.4; to the lift we may apply infinitesimal invariant techniques in Chapter 4, which lead to this vanishing theorem for the real regulator.

Moreover this result is consistent with the Beilinson conjecture, which predicts the nontriviality of [the covolume of] the image of r_{Be} for X defined over a *number field* (and therefore *not* general). Provided

$$Gr^0_N H^{2p-n}_{\mathcal{D}}(X,\mathbb{Q}(p)) \;\left[= \mathrm{im}\left\{H^{2p-n}_{\mathcal{D}}(X,\mathbb{Q}(p)) \to H^{2p-n}_{\mathcal{D}}(\eta_X,\mathbb{Q}(p))\right\}\right]$$

is nonzero, this "guarantees" interesting values for Gr^0_N of r and $\mathcal{R}$ for X over a number field. For $n=p$, the latter of these (Gr^0_N of $\mathcal{R}$) is the *holomorphic Milnor regulator* which is discussed in Chapter 4, and for which the vanishing theorem is proved for X very general (so definitely *not* over a number field).

The notion that the image of r_{Be} is a $\mathbb{Q}$-lattice (so that one may speak of its covolume), or at least topologically zero-dimensional, is "explained" by the fact that r_X is *constant on continuous families* in $CH^n(X,n)$ (for X *fixed*). This is proved in **[Be]**, using rigidity of deRham classes. For this same reason one can easily show the holomorphic Milnor regulator is also constant on continuous families in $K^M_n(X)$, for X fixed. So in that Chapter we investigate instead the situation when X *is allowed to vary* (and Proposition 3.1.5 above comes out of this).

3.2. Relative Regulators and Polylogarithms

We get many more opportunities for explicit Milnor regulator computations, if we generalize the maps and definitions of Chapter 2 to the case of a relative variety (X,Y) and its "generic point" $\eta_{(X,Y)}$. (This is obtained in the limit by removing divisors intersecting Y properly; such divisors are henceforth written "$V \subset (X,Y)$"). The resulting integrals $\int_{\mathcal{C}} R_{\mathbf{f}}$ are still periods in the sense of Kontsevich and Zagier [**KZ**], and we calculate some very basic ones here. However we first put them in their proper context with quite a bit of abstract nonsense; our approach may initially seem rather *ad hoc* and naive to readers of [**B2**].

3.2.1. Motivation and definitions. Let $\mathbb{C}(X,Y)$:=the multiplicative group of all functions $\equiv 1$ on Y, and assume for simplicity $\dim X = n-1$. Set

$$K_n^M(\mathbb{C}(X,Y)) := \frac{\mathbb{Z}[\mathbb{P}^1_{\mathbb{C}(X,Y)}] \otimes \left(\otimes^{n-1}\mathbb{Z}[\mathbb{P}^1_{\mathbb{C}(X)} \setminus \{0,\infty\}]\right)}{\{\text{Steinberg relations}\} \cap \text{num} =: \mathcal{S}}$$

$$= \frac{\mathbb{C}(X,Y) \otimes_{\mathbb{Z}} \bigwedge_{\mathbb{Z}}^{n-1} \mathbb{C}(X)^*}{\left\langle \begin{array}{c} f \otimes (1-f) \wedge g_2 \wedge \ldots \wedge g_{n-1}, \\ f \otimes g_1 \wedge (1-g_1) \wedge g_3 \wedge \ldots g_{n-1} \end{array} \right\rangle}$$

where the first function $(=f)$ must be $\equiv 1$ on Y; and

$$CH^n(\eta_{(X,Y)}, n) := \varinjlim_{V \subset (X,Y)} CH^n(X \setminus V \,,\, Y \cap X \setminus V \,;\, n).$$

In this situation there are still well-defined[9] (but perhaps not isomorphic) graph maps γ:

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[9](Well-definedness of γ.) For instance, recall that for $n=3$ one has the "Totaro cycles" $\mathcal{C}_{\mathbf{f}}$ – which are elements of $Z^3(\eta_X, 4)$, and $\partial_{\mathcal{B}}$ of which hits $\gamma_{\mathbf{f}}$ – for any $\mathbf{f}$ $\in$Steinberg relations. The question here is whether, if $\mathbf{f} \in \mathcal{S}$, the Totaro cycles can be chosen in $Z^3(\eta_{(X,Y)}, 4)$. If $f = f_1 f_2$ and f, f_1, $f_2 \equiv 1$ on Y, then to the generators of $\mathcal{S}$ we can associate such "relative" $\mathcal{C}_{\mathbf{f}}$ (with $\partial_{\mathcal{B}}\mathcal{C}_{\mathbf{f}} = \gamma_{\mathbf{f}}$) as follows:

$$\begin{array}{rr} \mathbf{f} = f \otimes g \otimes h - f_1 \otimes g \otimes h - f_2 \otimes g \otimes h & \mathcal{C}_{\mathbf{f}} = \left(z,\, \frac{(z-f(x))(z-1)}{(z-f_1(x))(z-f_2(x))},\, g(x),\, h(x)\right) \\ \mathbf{f} = f \otimes g_1 g_2 \otimes h - f \otimes g_1 \otimes h - f \otimes g_2 \otimes h & \mathcal{C}_{\mathbf{f}} = \left(f(x),\, \frac{(z-g_1(x)g_2(x))(z-1)}{(z-g_1)(z-g_2)},\, z,\, h(x)\right) \\ \mathbf{f} = f \otimes g \otimes h + f \otimes h \otimes g & \mathcal{C}_{\mathbf{f}} = \left(f(x),\, \frac{(z-g(x))(z-h(x))}{(z-g(x)h(x))(z-1)},\, \frac{g(x)h(x)}{z},\, z\right) \\ \mathbf{f} = f \otimes (1-f) \otimes h & \mathcal{C}_{\mathbf{f}} = \left(z,\, \frac{z-f(x)}{z-1},\, 1-z,\, h(x)\right) \\ \mathbf{f} = f \otimes g \otimes (1-g) & \mathcal{C}_{\mathbf{f}} = \left(f(x),\, \frac{z-g(x)}{z-1},\, z,\, 1-z\right) \end{array}$$

where each $\mathcal{C}_{\mathbf{f}}$ is parametrized by $z \in \mathbb{P}^1$ and $x \in X$, and has one coordinate$\equiv 1$ always when $x \in Y$ (and so $\mathcal{C}_{\mathbf{f}} \in Z^3(\eta_{(X,Y)}, 4)$).

Notice that if, in the fourth $\mathcal{C}_{\mathbf{f}}$, we put only f *or* $1-f \equiv 1$ on each component of Y, i.e. allow some "sharing of responsibility" between the functions, then on some Y_i we get $\frac{z-0}{z-1}$ rather than $\frac{z-1}{z-1} \equiv 1$, which is a problem. (Hopefully this gives some insight into our otherwise arbitrary-looking definition of the relative Milnor K-groups.)

$$\begin{tikzcd} K_n^M(\mathbb{C}(X,Y)) \arrow[r, "\gamma"] & CH^n(\eta_{(X,Y)}, n) \\ K_n^M(X,Y) \arrow[u, hook] & CH^n(X,Y;n), \arrow[u] \end{tikzcd}$$

where $K_n^M(X,Y) := \gamma^{-1}(\mathrm{im} CH^n(X,Y;n))$. However, the dotted arrow, which represents the concept of moving[10] and completing $\gamma_{\mathbf{f}}$ (trivially possible for $\{\mathbf{f}\} \in K_n^M(X,Y)$, by definition), is *not* well-defined.[11] We draw it because its composition with other maps can be.

In particular, recalling that $\square^n = (\mathbb{P}^1 \backslash \{1\})^n$, for $(X,Y) = (\mathbb{P}^1, \{0,\infty\})^{n-1} =: (\bar{\square}^{n-1}, \partial\bar{\square}^{n-1})$ consider

$$CH^n(\bar{\square}^{n-1}, \partial\bar{\square}^{n-1}; n) \to CH^n(\square^{n-1}, \partial\square^{n-1}; n) \to CH^n(\mathbb{C}, 2n-1) \overset{\mathcal{R}}{\to} \mathbb{C}/\mathbb{Z}(n)$$

where the second map is induced by a map of $\partial_{\mathcal{B}}$-complexes. We can essentially (modulo some further restrictions on $\mathbf{f}$) show that the resulting composition

$$K_n^M(\bar{\square}^{n-1}, \partial\bar{\square}^{n-1}) \to \mathbb{C}/\mathbb{Z}(n)$$

is independent of the choices made along $V_{\mathbf{f}}$ in completing $\gamma_{\mathbf{f}}$ to Γ. Say $\Gamma = \Gamma_{V_{\mathbf{f}}} + \overline{\gamma_{\mathbf{f}}}$, then $\mathcal{R}(\Gamma)$ is computed by

$$\frac{1}{(2\pi i)^{n-1}} \int_{\gamma_{\mathbf{f}}} R_{\square}^{2n-1} \; + \; \frac{1}{(2\pi i)^{n-1}} \int_{\Gamma_{V_{\mathbf{f}}}} R_{\square}^{2n-1}.$$

Now write $\{w_1, \ldots, w_{n-1}; z_1, \ldots, z_n\}$ for the coordinates on $\square^{2n-1}$, and set $T_w^{n-1} = T_{w_1} \cap \ldots \cap T_{w_{n-1}} \subset \square^{2n-1}$. T_w^{n-1} will also denote its image under the projection to X, $\pi : \square^{2n-1} \to \square^{n-1}$, and we can show the second integral vanishes under the assumption[12] that (on X) $V_{\mathbf{f}} \cap T_w^{n-1} \subseteq \partial\bar{\square}^{n-1} (= Y)$. Namely, in

$$\int_{\Gamma_{V_{\mathbf{f}}}} R_{\square}^{2n-1} = \left\{ \int_{\Gamma_{V_{\mathbf{f}}}} \log w_1 \mathrm{dlog} w_2 \wedge \ldots \wedge \mathrm{dlog} w_{n-1} \wedge \mathrm{dlog} z_1 \wedge \ldots \wedge \mathrm{dlog} z_n + \ldots \right.$$

$$\left. + (2\pi i)^{n-2} \int_{\Gamma_V \cap T_{w_1} \cap \ldots \cap T_{w_{n-2}}} \log w_n \mathrm{dlog} z_1 \wedge \ldots \wedge \mathrm{dlog} z_n \right\}$$

---

[10] Note that in moving $\gamma_{\mathbf{f}}$ we have the added restriction that it remain a relative cycle $\in Z^n(\eta_{(X,Y)}, n)$; we have not pursued whether a version of the moving lemma still holds.

[11] (as usual the completion is not unique)

[12] T_w^{n-1} will play in this section the same role as $\mathcal{C}$ at the end of §2.2, namely, the topological cycle for a Milnor-regulator current to be integrated over, which must avoid the divisors $V_{\mathbf{f}}$. In the present argument, which is motivation for the relative Milnor regulator, T_w^{n-1} appears as a restriction on the function $\mathbf{f}$ (rather than vice-versa).

$$+(2\pi i)^{n-1}\left\{\int_{\Gamma_V\cap T_w^{n-1}}\log z_1 \mathrm{dlog} z_2\wedge\ldots\wedge\mathrm{dlog} z_n+\ldots\right.$$

$$\left. +(2\pi i)^{n-1}\sum_{\Gamma_V\cap T_w^{n-1}\cap T_{z_1}\cap\ldots\cap T_{z_{n-1}}}\log z_n\right\}$$

the first term is zero by type since $\dim_{\mathbb{C}}\Gamma_V = n-1$. The second term vanishes under the above assumption, because Γ_V is a *relative* higher Chow cycle and so as a cycle $\Gamma_V\cap\pi^{-1}(Y)\left(\supseteq \Gamma_V\cap T_w^{n-1}\right)$ is zero. Similarly

$$\frac{1}{(2\pi i)^{n-1}}\int_{\gamma_{\mathbf{f}}}R_{\square}^{2n-1}=\{\text{1st } n-1 \text{ terms}=0\}+\left\{\int_{\gamma_{\mathbf{f}}\cap T_w^{n-1}}\log z_1\mathrm{dlog} z_2\wedge\ldots\wedge\mathrm{dlog} z_n\right.$$

$$\left. 2\pi i\int_{\gamma_{\mathbf{f}}\cap T_w^{n-1}\cap T_{z_1}}\log z_2\mathrm{dlog} z_3\wedge\ldots\wedge\mathrm{dlog} z_n+\ldots+(2\pi i)^{n-1}\sum_{\gamma_{\mathbf{f}}\cap T_w^{n-1}\cap T_{z_1}\cap\ldots\cap T_{z_{n-1}}}\log z_n\right\}$$

$$=(-1)^{n-1}\int_{T_w^{n-1}}R_{\mathbf{f}},$$

where $\{z_1,\ldots,z_n\}$ have been replaced by $f_1,\ldots,f_n$ because we are on $\gamma_{\mathbf{f}}$. this is just a Milnor regulator current integrated over a relative cycle $[T_w^{n-1}]\in H_{n-1}\left(\bar{\square}^{n-1}\setminus V_{\mathbf{f}},\partial\bar{\square}^{n-1}\cap\bar{\square}^{n-1}\setminus V_{\mathbf{f}}\right)$. It makes sense to try to turn this into a map on all of $K_n^M(\mathbb{C}(\square^{n-1},\partial\square^{n-1}))$, where one can choose cycles[13] that avoid $V_{\mathbf{f}}$ (instead of restricting $\mathbf{f}$), and to show well-definedness directly via coboundary currents as in §2.2.2.

3.2.2. Well-definedness issues. More generally, we start with a map

$$\mathbb{Z}[\mathbb{P}^1_{\mathbb{C}(X,Y)}]\otimes\left(\otimes^{n-1}\mathbb{Z}[\mathbb{P}^1_{\mathbb{C}(X)}\setminus\{0,\infty\}]\right)\longrightarrow$$

$$\varinjlim_{V\subset(X,Y)}\mathrm{Hom}\left\{H_{n-1}(X\setminus V_{\mathbf{f}},Y\cap X\setminus V_{\mathbf{f}};\mathbb{Z}),\mathbb{C}/\mathbb{Q}(n)\right\}=H^{n-1}(\eta_{(X,Y)},\mathbb{C}/\mathbb{Q}(n))$$

defined by

$$\mathbf{f}\longmapsto\left\{\mathcal{C}\mapsto\int_{\mathcal{C}}R_{\mathbf{f}}\right\},$$

where the chains $\mathcal{C}$ must avoid $V_{\mathbf{f}}$ but may bound on Y (they do *not* have to be topological cycles on X). To show that this map descends to $K_n^M(\mathbb{C}(X,Y))$, recall that the regulator currents associated to any Steinberg $\mathbf{f}$ are coboundaries on $X\setminus V_{\mathbf{f}}$: that is, $R'_{\mathbf{f}}=\mathrm{d}[S_{\mathbf{f}}]$; or simply $R_{\mathbf{f}}\equiv\mathrm{d}[S_{\mathbf{f}}]$ modulo $(2\pi i)^n\times$ {integral chains}. So if $\iota_Y^* S_{\mathbf{f}}=0$ then

$$\int_{\mathcal{C}}R_{\mathbf{f}}\equiv\int_{\mathcal{C}}\mathrm{d}[S_{\mathbf{f}}]=\int_{\partial\mathcal{C}\subseteq Y}S_{\mathbf{f}}=0 \qquad \mathrm{mod}\ \mathbb{Z}(n)$$

[13] e.g., some deformation of T_w^{n-1} for which $'T_w^{n-1}\cap V_{\mathbf{f}}\subseteq\partial\bar{\square}^{n-1}$.

So we then just write down $S_{\mathbf{f}}$ for every "trivial" *relative* $\mathbf{f}$ (i.e. $\mathbf{f} \in \mathcal{S}$), and verify that $\iota_Y^* S_{\mathbf{f}} = 0$.

We will do this for $n = 3$ on the two interesting such $\mathbf{f}$ in that case; it is crucial to remember that for us log *always* means the branch with $-\pi < \arg \leq \pi$. Let $f = f_1 f_2$and consider

$$\mathbf{f} = f \otimes g \otimes h - f_1 \otimes g \otimes h - f_2 \otimes g \otimes h \quad \longmapsto$$

$$R_{\mathbf{f}} = 2\pi i \Delta_f \mathrm{dlog} g \wedge \mathrm{dlog} h - 2\pi i \log g \mathrm{dlog} h \cdot \delta_{\partial \Delta_f} + 4\pi^2 \log h \cdot \delta_{\partial \Delta_f \cap T_g}$$

where

$$\Delta_f = \frac{1}{2\pi i} \{\log f - \log f_1 - \log f_2\}, \ \partial \Delta_f = T_{f_1} + T_{f_2} - T_f.$$

Then $\mathrm{d}[S_{\mathbf{f}}] = R_{\mathbf{f}}$ (up to $\mathbb{Z}(3)\times$integral chain) for the following

$$S_{\mathbf{f}} = 2\pi i \Delta_f \log g \mathrm{dlog} h + 4\pi^2 \Delta_f \log h \cdot \delta_{T_g}.$$

As for

$$\mathbf{f} = f \otimes (1-f) \otimes h \quad \longmapsto \quad R_{\mathbf{f}} = \log f \mathrm{dlog}(1-f) \wedge \mathrm{dlog} h + 2\pi i \log(1-f) \mathrm{dlog} h \cdot \delta_{T_f}$$

one has exactly $\mathrm{d}[S_{\mathbf{f}}] = R_{\mathbf{f}}$ with

$$S_{\mathbf{f}} = -\mathrm{Li}_2(1-f) \mathrm{dlog} h$$

where once again, for us $\mathrm{Li}_2(z)$ always means the "standard branch" (as a zero-current) with a cut at T_{1-z}. Both these $S_{\mathbf{f}}$'s clearly have $\iota_Y^* S_{\mathbf{f}} = 0$, since $f = 1$ on Y and using our given branches $\log 1$ and $\mathrm{Li}_2(0)$ are always exactly zero.[14]

In the case of $(X, Y) = (\bar{\square}^{n-1}, \partial \bar{\square}^{n-1})$ it turns out to be easier to compute certain integrals of the form $\int_X R'_{\mathbf{f}} \wedge \omega$ instead of $\int_{\mathcal{C}} R_{\mathbf{f}}$,. Indeed we shall use the former to compute the latter (which is what we are interested in). Assuming normal crossings in $Y \cup V_{\mathbf{f}}$, we have a well-defined regulator map like that above:

$$K_n^M(\mathbb{C}(X,Y)) \longrightarrow \varinjlim_{V \subset (X,Y)} \frac{\{H^{n-1}(X \setminus Y, V_{\mathbf{f}} \cap X \setminus Y; \mathbb{C})\}^\vee}{\mathrm{im} H_{n-1}(X \setminus Y, V_{\mathbf{f}} \cap X \setminus Y; \mathbb{Q}(n))} = H^{n-1}(\eta_{(X,Y)}, \mathbb{C}/\mathbb{Q}(n))$$

$$\mathbf{f} \quad \longmapsto \quad \left\{ \omega \mapsto \int R'_{\mathbf{f}} \wedge \omega \right\}$$

where we may represent $H^{n-1}(X \setminus Y, V_{\mathbf{f}} \cap X \setminus Y; \mathbb{C})$ by either forms $\omega \in \Gamma \left\{ \Omega^{n-1}_{(X \setminus V_{\mathbf{f}})^\infty} (\mathrm{dlog} Y) \right\}$ compactly supported away from $V_{\mathbf{f}}$, or forms $\omega \in \Gamma \left\{ \Omega^{n-1}_{X^\infty} (\mathrm{null} V_{\mathbf{f}}) (\mathrm{dlog} Y) \right\}$ with $\iota_V^* \omega = 0$. In particular, simply by *type* a

[14](In fact half of our work is to preserve branch-accuracy, since we have so many instances of a product of an Li_2 or log with something else; and one cannot go modulo, for example, $2\pi i$ times any current.)

form in $H^0(\Omega_X^{n-1}(\mathrm{dlog}Y))$ is fair game, and for $(X,Y) = (\bar{\square}^{n-1}, \partial\bar{\square}^{n-1})$ this means

$$\Omega_w^{n-1} := \frac{1}{(2\pi i)^{n-1}} \wedge^{n-1} \mathrm{dlog} w_i$$

is a candidate for ω.

Let $[T_w^{n-1}] \in H^{n-1}(\bar{\square}^{n-1} \setminus \partial\bar{\square}^{n-1}, V_{\mathbf{f}} \cap \bar{\square}^{n-1} \setminus \partial\bar{\square}^{n-1})$ denote also the topological cycle's Lefschetz dual class; it can be represented by a form ω_T with support on a tubular neighborhood of T_w^{n-1} and disjoint from $V_{\mathbf{f}}$. Then $[T_w^{n-1}] \neq [\Omega_w^{n-1}]$ but their images in $H^{n-1}(\bar{\square}^{n-1} \setminus \partial\bar{\square}^{n-1})$ *are* equal, and so $\omega_T - \Omega_w^{n-1} = \mathrm{d}\alpha$ on $\bar{\square}^{n-1} \setminus \partial\bar{\square}^{n-1}$. Since $\mathrm{d}[R'_{\mathbf{f}}] = 0$ *on* $\bar{\square}^{n-1} \setminus \partial\bar{\square}^{n-1}$, it seemingly should follow that

$$0 = \int_{\bar{\square}^{n-1}(\setminus\partial\bar{\square}^{n-1})} R'_{\mathbf{f}} \wedge \mathrm{d}\alpha \;=\; \int_{T_w^{n-1}} R'_{\mathbf{f}} - \int_{\bar{\square}^{n-1}} R'_{\mathbf{f}} \wedge \Omega_w^{n-1}\,,$$

but this is not quite right. There is a subtle flaw in the first equality: although by construction $\mathrm{d}\alpha$ is null$Y(=\partial\bar{\square})$, α itself may *not* be, and so $\int \mathrm{d}[R'_{\mathbf{f}}] \wedge \alpha$, where $\mathrm{d}[R'_{\mathbf{f}}]$ is supported on $V_{\mathbf{f}}$, may be a problem. The way to get around this (at least for $\{\mathbf{f}\} \in K_n^M(\bar{\square}^{n-1}, \partial\bar{\square}^{n-1})$) is to appeal to §2.3.3, which works out a map

$$K_n^M(X) \longrightarrow \varinjlim_{V_{\mathbf{f}} \subset X} \frac{\left(\mathrm{coim}\left\{H^{n-1}(X, V_{\mathbf{f}}; \mathbb{C}) \to H^{n-1}(X, \mathbb{C})\right\}\right)^{\vee}}{\mathrm{im} H_{n-1}(X, \mathbb{Q}(n))}$$

defined by

$$\mathbf{f} \longmapsto \left\{\omega \mapsto \int_X R''_{\mathbf{f}} \wedge \omega\right\},$$

where (in the language of §2.4) $R''_{\mathbf{f}} = \jmath^*_{X \setminus V_{\mathbf{f}}} R'_{\Gamma}$ for Γ a completion of $\gamma_{\mathbf{f}}$ (so that $\mathrm{d}[R'_{\Gamma}] = 0$ on *all* of X). That is, we have to choose our membrane $\partial^{-1}_{(X,V_{\mathbf{f}})} T_{\mathbf{f}}$ more "precisely" than usual: just any $R'_{\mathbf{f}} = R_{\mathbf{f}} + (2\pi i)^n \partial^{-1}_{(X,V_{\mathbf{f}})} T_{\mathbf{f}}$ is off by $R'_{\mathbf{f}} - R''_{\mathbf{f}} = (2\pi i)^n \zeta$, for some $\zeta \in H_{n-1}(X, V_{\mathbf{f}}; \mathbb{Z})$. This is the only way to ensure that, for ω compactly supported away from $V_{\mathbf{f}}$, and $\omega = \mathrm{d}\alpha$ (where α may have support on all of X),

$$\int_X R''_{\mathbf{f}} \wedge \omega \underset{\substack{\uparrow \\ \omega \text{ c.s. on } X \setminus V_{\mathbf{f}} \\ R'_{\Gamma} = R''_{\mathbf{f}} + R_V}}{=} \int_X R'_{\Gamma} \wedge \omega \;=\; \int_X R'_{\Gamma} \wedge \mathrm{d}\alpha = \pm \int_X \mathrm{d}[R'_{\Gamma}] \wedge \alpha = 0\,.$$

One extends this in an obvious way to get

$$K_n^M(X,Y) \longrightarrow \varinjlim_{V_{\mathbf{f}} \subset X} \frac{\left(\mathrm{coim}\left\{H^{n-1}(X \setminus Y, V_{\mathbf{f}} \cap X \setminus Y; \mathbb{C}) \to H^{n-1}(X \setminus Y, \mathbb{C})\right\}\right)^{\vee}}{\mathrm{im} H_{n-1}(X \setminus Y, \mathbb{Q}(n))}$$

by sending

$$\mathbf{f} \longmapsto \left\{\omega \mapsto \int_X R''_{\mathbf{f}} \wedge \omega\right\}$$

where $R''_{\mathbf{f}}(=\jmath^*_{X\setminus V_{\mathbf{f}}}R'_\Gamma) = R'_{\mathbf{f}} + (2\pi i)^n\zeta$ for $\zeta \in H_{n-1}(X\setminus Y, V_{\mathbf{f}}\cap X\setminus Y;\mathbb{Q})$, is consistent with the notion that $\iota^*_Y R''_{\mathbf{f}} = \iota^*_Y R'_\Gamma = 0$ (since ζ avoids Y).

Now returning to $(X,Y) = (\bar{\square}^{n-1}, \partial\bar{\square}^{n-1})$, we have $[T^{n-1}_w] = [\Omega^{n-1}_w]$ on $\bar{\square}^{n-1}\setminus\partial\bar{\square}^{n-1}$; and this is reflected by the fact that on $\bar{\square}^{n-1}$

$$\mathrm{d}[R^{n-1}_w] = \mathrm{d}[\frac{1}{(2\pi i)^{n-1}}R(w_1,\ldots,w_n)] = \Omega^{n-1}_w - T^{n-1}_w + \mathrm{Res}(R^{n-1})$$

where the residues are supported on $\partial\bar{\square}^{n-1}$. If $\{\mathbf{f}\}\in K^M_n(\bar{\square}^{n-1},\partial\bar{\square}^{n-1})$ so that $\Gamma = \overline{\gamma_{\mathbf{f}}} + \Gamma_V$ completes $\gamma_{\mathbf{f}}$, and $R'_\Gamma = R''_{\mathbf{f}} + R_V$ where R_V is supported on $V_{\mathbf{f}}$, then[15]

$$0 = \pm\int_{\bar{\square}^{n-1}} \mathrm{d}[R'_\Gamma]\wedge R^{n-1}_w \boxed{=} \int_{\bar{\square}} R'_\Gamma \wedge \mathrm{d}[R^{n-1}_w]$$

$$= \int (R_V + R''_{\mathbf{f}})\wedge(\Omega^{n-1}_w - T^{n-1}_w) \; + \; \int R'_\Gamma\wedge \mathrm{Res}(R^{n-1}_w);$$

the last integral here is zero because $\iota^*_Y R'_\Gamma = 0$ while the residues are supported on $Y = \partial\bar{\square}^{n-1}$. So we have

$$\int R_V\wedge\Omega^{n-1}_w \; - \; \int_{T^{n-1}_w} R_V \; + \; \int R''_{\mathbf{f}}\wedge\Omega^{n-1}_w \; - \; \int_{T^{n-1}_w} R''_{\mathbf{f}}$$

and the two leading terms in this are zero: the first because R_V is supported on V and Ω^{n-1}_w is null$V$, the second since $T^{n-1}_w\cap V \subset \partial\bar{\square}^{n-1} = Y$ and $\iota^*_Y R_V = 0$ (since Γ_V is relative). Therefore this reduces to

$$\int_{\bar{\square}^{n-1}} R''_{\mathbf{f}}\wedge\Omega^{n-1}_w - \int_{T^{n-1}_w} R_{\mathbf{f}} \qquad (\equiv 0 \mod \mathbb{Z}(n)).$$

We are interested in the latter integral, in particular its $\pi^n_{\mathbb{R}}$-part. We have justified the following idea: in order to get an idea of what this should be, at least for $\{\mathbf{f}\}\in K^M_n(\bar{\square}^{n-1},\partial\bar{\square}^{n-1})$, we can try to compute the former integral.

3.2.3. Linear factorization and classical polylogarithms. We will in fact compute a modified version of this integral. To illustrate the idea we turn to $n=2$, where $(\bar{\square}^1,\partial\bar{\square}^1) = (\mathbb{P}^1,\{0,\infty\})$. Any $\mathbf{f}[=\sum \ell_\alpha f_\alpha\otimes g_\alpha]$ involves functions $f_\alpha, g_\alpha\in\mathbb{C}(\mathbb{P}^1)$ which split into linear factors; however, breaking $\{\mathbf{f}\}$ up into the resulting "linear" symbols is not allowed in $K^M_2(\mathbb{C}(\mathbb{P}^1,\{0,\infty\}))$, because the linear factors of f_α do not $=1$ at $\{0\}$ and $\{\infty\}$. But this kind of symbol-factpring is necessary to do any computations. So we slightly change the definition of K^M_2 so that f_α and g_α need

[15]The boxed equailty is not automatic; it amounts to saying that if $\mathcal{T}_\epsilon$ is an ϵ-tube about $\partial\bar{\square}^{n-1}$ as in §1.3, then $\int_{\mathcal{T}_\epsilon} R_{\mathbf{f}}\wedge R^{n-1}_w \to 0$ with $\epsilon\to 0$. We will justify this carefully for $n=2$ below.

only "cover $\{0\}$ and $\{\infty\}$ between them", and define a regulator on it. Let

$$'K_2^M(\mathbb{C}(\mathbb{P}^1,\{0,\infty\})) := \frac{\left\{\begin{array}{c}\text{subgroup of } \otimes^2 \mathbb{Z}[\mathbb{P}^1_{\mathbb{C}(\mathbb{P}^1)} \setminus \{0,\infty\}] \text{ generated by} \\ f\otimes g \text{ with } (1-f)(1-g)=0 \text{ on } \{0,\infty\}\end{array}\right\}}{\{\text{Steinberg relations}\}\cap \text{num} =: {}'\mathcal{S}}.$$

The γ defined in the beginning of the section is not well-defined;[16] however it is defined on the numerator, and so the notion that a symbol $\{\mathbf{f}\} \in {}'K_2^M(\mathbb{C}(\mathbb{P}^1,\{0,\infty\}))$ has a representative $\mathbf{f}$ whose associated graph cycle $\gamma_{\mathbf{f}}$ (modulo $\mathrm{im}\partial_{\mathcal{B}}$) is completable to a relative higher Chow cycle in $Z^2((\mathbb{P}^1,\{0,\infty\}),2)$, is a well-defined one. So we have a $'K_2^M(\mathbb{P}^1,\{0,\infty\})$, and this contains any symbols represented by $\mathbf{f}=\sum \ell_\alpha f_\alpha\otimes g_\alpha$, $|(f_\alpha)|\cap|(g_\alpha)|=\emptyset$ for each α, in $\ker(\mathrm{Tame})$,[17] essentially by the construction in Chapter 1 (or see Example 1.3.1(b)). Such $\mathbf{f}$ will be called "nice". For example,

$$f_\alpha(w)=\prod_i\left(1-\frac{a_i^\alpha}{w}\right)^{m_i^\alpha},\ g_\alpha(w)=\prod_j\left(1-\frac{w}{b_j^\alpha}\right)^{n_j^\alpha} \qquad (*)$$

give an element of $'K_2^M(\mathbb{C}(\mathbb{P}^1,\{0,\infty\}))$; if $\{\mathbf{f}\}=\prod\{f_\alpha,g_\alpha\}^{\ell_\alpha}\in\ker(\mathrm{Tame})$, and for each fixed α the set of $\{a_i^\alpha\}$ and $\{b_j^\alpha\}$ are disjoint, then $\mathbf{f}$ is nice and $\overline{\gamma_{\mathbf{f}}}$ may be completed to a relative higher Chow cycle (by adding curves over points in $V_{\mathbf{f}}$).

To simplify notation we will usually work with elements of the form

$$\{\mathbf{f}\}=\left\{f=\prod\left(1-\frac{a_i}{w}\right)^{m_i},\ g=\prod\left(1-\frac{w}{b_j}\right)^{n_j}\right\},$$

so that $\{\mathbf{f}\}\in\ker(\mathrm{Tame})$ means for us $\prod_i\left(1-\frac{a_i}{b_j}\right)^{m_in_j}=1$ for each j and $\prod_j\left(1-\frac{a_i}{b_j}\right)^{m_in_j}=1$ for each fixed i. The proofs for $\{f_\alpha,g_\alpha\}^{\ell_\alpha}$ are then trivial generalizations of the ones we give here (with more complicated notation).

There is a regulator map

$$R:\ 'K_2^M(\mathbb{C}(\mathbb{P}^1,\{0,\infty\}))\to \varinjlim_{V\subset\mathbb{C}^*}\mathrm{Hom}\left(H_1(\mathbb{P}^1\setminus V,\{0,\infty\};\mathbb{Z}),\,\mathbb{C}/\mathbb{Q}(2)\right)$$

$$= H^1(\eta_{(\mathbb{P}^1,\{0,\infty\})},\mathbb{C}/\mathbb{Q}(2))$$

defined *a priori* on the generating elements $\mathbf{f}$ by $\mathbf{f}\mapsto\left\{\mathcal{C}\mapsto\int_{\mathcal{C}}R_{\mathbf{f}}\right\}$. There are three[18] interesting "coboundary currents" to check here (in order to prove

[16] This is just the $n=2$ analogue of the problem mentioned (for $n=3$) at the end of the long footnote there; γ does not kill the $'K_2^M$-relation: $f\otimes(1-f)$ if $f=0$ at $\{0\}$, 1 at $\{\infty\}$.

[17] In this case this means: for any $p\in\mathbb{P}^1$, $\prod_\alpha\left(\frac{f_\alpha(p)^{\nu_p(g_\alpha)}}{g_\alpha(p)^{\nu_p(f_\alpha)}}\right)^{\ell_\alpha}=1$.

[18] The third is, for $f_1f_2\otimes g$, $f_1\otimes g$, and $f_2\otimes g$ all in the numerator of the definition of $'K_2^M$,

$$R_{f_1f_2\otimes g-f_1\otimes g-f_2\otimes g}=(\log f_1f_2-\log f_1-\log f_2)\mathrm{dlog}g-2\pi i\log g\cdot\delta_{(T_{f_1f_2}-T_{f_1}-T_{f_2})}$$

this is well-defined):

$$R_{f\otimes g+g\otimes f}=\mathrm{d}[\log f\log g]\quad\text{and}\quad R_{f\otimes(1-f)}=\mathrm{d}[-\mathrm{Li}_2(1-f)];$$

ideally we want to show the quantities in brackets are 0 at $\{0,\infty\}$. Since either f or g is 1 at $\{0\}$ and at $\{\infty\}$, and always $\log 1$ =exactly zero for us, the former is easy. However the latter can be $-\mathrm{Li}_2(0)=0$ or $-\mathrm{Li}_2(1)=\pi^2/6$. In this case, say if $f=0$ at $\{0\}$ and 1 at $\{\infty\}$, one has for $\mathcal{C}=T_w$

$$-\int_{T_w}R_{f\otimes(1-f)}=\int_0^\infty \mathrm{d}[-\mathrm{Li}_2(f)]=\mathrm{Li}_2(f(0))-\mathrm{Li}_2(f(\infty))=\pi^2/6\in\mathbb{Q}(2),$$

which for a regulator modulo torsion is considered trivial. So the map is well-defined on cycles bounding on $\{0,\infty\}$.

It is also well-defined on 1-forms with dlog-poles at $\{0,\infty\}$. The fact which we must check, is that for nice $\mathbf{f}$ ($\in\ker(\mathrm{Tame})$ and without corners, so that $\overline{\gamma_{\mathbf{f}}}$ completes)

$$\int_{T_w^1}R_{\mathbf{f}}\equiv\int_{\mathbb{P}^1}R''_{\mathbf{f}}\wedge\Omega^1_w\mod\mathbb{Q}(2)$$

where T_w^1 is the path from ∞ to 0 along $\mathbb{R}^-$, $\Omega^1_w=\frac{1}{2\pi i}\mathrm{dlog}w$, and we are assuming $V_{\mathbf{f}}\cap\mathbb{R}^-=\emptyset$. First note that $R_V=0$ so $R''_{\mathbf{f}}=R'_\Gamma$, i.e. $\mathrm{d}[R''_{\mathbf{f}}]=0$ (on $\mathbb{P}^1$); then

$$0=-\int_{\mathbb{P}^1}\mathrm{d}[R''_{\mathbf{f}}]\wedge\frac{1}{2\pi i}\log w\boxed{\equiv}\int_{\mathbb{P}^1}R''_{\mathbf{f}}\wedge\mathrm{d}[\frac{1}{2\pi i}\log w]$$

$$=\int_{\mathbb{P}^1}R''_{\mathbf{f}}\wedge\left(\frac{1}{2\pi i}\mathrm{dlog}w-\delta_{T_w^1}\right)\equiv\frac{1}{2\pi i}\int R''_{\mathbf{f}}\wedge\mathrm{dlog}w-\int_{T_w^1}R_{\mathbf{f}}\mod\mathbb{Q}(2)$$

provided the boxed equality holds: this is the same as saying, for C_ϵ a circle about $\{0\}$ (or $\{\infty\}$),

$$0=\lim_{\epsilon\to0}\int_{\mathsf{C}_\epsilon}\log w\cdot R_{\mathbf{f}}=\lim_{\epsilon\to0}\left\{\int_{\mathsf{C}_\epsilon}\log w\log f\mathrm{dlog}g-2\pi i\sum_{\mathsf{C}_\epsilon\cap T_{\mathbf{f}}}\log w\log g\right\}.$$

The worst-case scenarios (at $\{0\}$) are:

(i) $f=0,\ g=1\implies$ $\epsilon\log^2\epsilon\to0$ limit bounds first term
second term $\to0$ as $\epsilon\log\epsilon$

$$=2\pi i(\Delta_f\mathrm{dlog}g+\log g\cdot\delta_{\partial\Delta_f})\equiv\mathrm{d}[2\pi i\Delta_f\log g]=\mathrm{d}[S]$$

modulo the $\mathbb{Z}(2)$-valued current $4\pi^2\Delta_f\cdot\delta_{T_g}$ (Δ_f takes only integral values). We must check that $S=0$ at $\{0\}$ and $\{\infty\}$. For example if $g=1$ at 0 and f_1, $f_2=1$ at ∞ then ($S=$)$2\pi i\Delta_f\log g$ is 0 at $\{0\}$ (here our policy of log having imaginary part between $-\pi i$ and $+\pi i$ is obviously important) while one has to show Δ_f is zero in a neighborhood of $\{\infty\}$. But this is clear: $\log f_1f_2=\log f_1=\log f_2=0$ at $\{\infty\}$. (What we have to worry about is something like $(e^{2\pi i/3})^3=1$, not $1\cdot1=1$ – this does not generate a branch change.)

(ii) $f=1,\ g=0 \quad \Longrightarrow \quad \begin{array}{c}\epsilon\log\epsilon \text{ bounds first term}\\ \mathsf{C}_\epsilon\cap T_f=\emptyset \text{ in second}\end{array}$

and so the above equality (mod $\mathbb{Q}(2)$) does hold for any nice $\mathbf{f}$.

Now if $\mathbf{f}=\prod\left(1-\frac{a_i}{w}\right)^{m_i}\otimes\prod\left(1-\frac{w}{b_j}\right)^{n_j}$ is nice then so is $\mathbf{f}_\prod=\sum m_i n_j\left(1-\frac{a_i}{w}\right)\otimes\left(1-\frac{w}{b_j}\right)$, and since $\mathbf{f}-\mathbf{f}_\prod\in{}'\mathcal{S}$, we have (mod $\mathbb{Q}(2)$) that

$$\int_{T^1_w}R_\mathbf{f}\equiv\int_{T^1_w}R_{\mathbf{f}_\prod}\equiv\frac{1}{2\pi i}\int_{\mathbb{P}^1}R''_{\mathbf{f}_\prod}\wedge \mathrm{dlog}w=\frac{1}{2\pi i}\int_{\mathbb{P}^1}R_{\mathbf{f}_\prod}\wedge\mathrm{dlog}w\,+\,2\pi i\int_\xi \mathrm{dlog}w$$

where $\xi\in H_1(\mathbb{C}^*,\bigcup\{a_i\}\cup\{b_j\})$ because $[T_\mathbf{f}\neq]\,T_{\mathbf{f}_\prod}=\sum m_i n_j T_{\left(1-\frac{a_i}{w}\right)}\cap T_{\left(1-\frac{w}{b_j}\right)}=\emptyset$, since no $a_i=b_j$ $(\forall i,j)$, and so there is no membrane. The second ($=\int_\xi$) term is a mysterious correction to residues of $R_{\mathbf{f}_\prod}$ on $V_\mathbf{f}$ involving $2\pi i\log a_i$, $2\pi i\log b_j$. The first is

$$\sum m_i n_j\left\{\frac{1}{2\pi i}\int_{\mathbb{P}^1}\log(1-\frac{a_i}{w})\mathrm{dlog}(1-\frac{w}{b_j})\wedge\mathrm{dlog}w-\int_{T_{\left(1-\frac{a_i}{w}\right)}}\log\left(1-\frac{w}{b_j}\right)\mathrm{dlog}w\right\}$$

$$=-\sum m_i n_j\int_0^{a_i}\log\left(1-\frac{w}{b_j}\right)\mathrm{dlog}w=-\sum m_i n_j\int_0^{\frac{a_i}{b_j}}\log(1-u)\mathrm{dlog}u$$

$$=\sum m_i n_j \mathrm{Li}_2(\frac{a_i}{b_j})=:\mathrm{Li}_2(\mathcal{N}_{f,g}),\quad \text{where } \mathcal{N}_{f,g}:=\sum m_i n_j\left(\frac{a_i}{b_j}\right)\in\mathbb{Z}[\mathbb{C}^*].$$

(This is even the standard branch of Li_2, as any branch-changes have apparently been pushed into the "mysterious" second term above.) So more generally, we have for $\mathbf{f}=\sum \ell_\alpha f_\alpha\otimes g_\alpha$ nice,

$$-\int_0^\infty R_\mathbf{f}\equiv\sum_\alpha \ell_\alpha\mathrm{Li}_2(\mathcal{N}_{f_\alpha,g_\alpha})+2\pi i\sum_{p\in V_\mathbf{f}}N(p)\log p \qquad \begin{array}{c}N \text{ some } \mathbb{Z}\text{-valued}\\ \text{function on } V_\mathbf{f}\end{array}$$

$$=:\ \mathrm{Li}_2(\mathcal{N}_\mathbf{f})\,+\,\{2\pi i\log\ \text{ terms}\}\qquad \mathrm{mod}\ \mathbb{Q}(2),$$

a minor result just as nice as $\mathbf{f}$ (and out of a whole lot of abstract nonsense).

Can we do something similar for $n\geq 3$? First of all, $(\bar{\square}^2,\partial\bar{\square}^2)=(\mathbb{P}^1\times\mathbb{P}^1,\#)$ so Y has *four* components. While we can define ${}'K_3^M(\mathbb{C}(\bar{\square}^2,\partial\bar{\square}^2))$,[19] it will not contain relative function *triples* $\{f,g,h\}$ factoring linearly in ${}'K_3^M$. If we switch to $(\mathbb{P}^2,\cup_{i=0}^2\{x_i=0\})=:(\mathbb{P}^2,\partial\mathbb{P}^2)$ so that Y has three components, the situation is partially repaired (see the triples below[20]). However, in order to get a well-defined regulator map we cannot quotient by some of the relations, e.g. $f\otimes(1-f)\otimes h$ since its regulator current is d of

[19](in terms of generators $f\otimes g\otimes h$ such that $(1-f)(1-g)(1-h)=0$ on each component of $\partial\bar{\square}^2$)

[20]Still, it seems strange to restrict to functions in $\mathbb{C}(\mathbb{P}^2)$ with linear factors; even though not all elements of ${}'K_2^M(\mathbb{C}(\mathbb{P}^1,\{0,\infty\}))$ factor linearly it seems more natural there since all functions do.

$S_{f\otimes(1-f)\otimes h} = -\mathrm{Li}_2(1-f)\mathrm{dlog}h$; and if $1-f=1$ on a component of $\partial\mathbb{P}^2$ then $S_{\mathbf{f}} = -\frac{\pi^2}{6}\mathrm{dlog}h$ there, which is unacceptable. Now above in the case $n=2$ we had *use* only for the branch-change relations $\mathbf{f}-\mathbf{f}_{\prod}$ (and no use for triviality of $f\otimes(1-f)$), so perhaps this is no matter.

The more serious problem is the existence of linearizable elements of $'K_3^M(\mathbb{P}^2,\partial\mathbb{P}^2)$, that is, for which $\overline{\gamma_{\mathbf{f}}}$ is completable to a higher *relative* Chow cycle. Say $\mathbf{f}=\sum \ell_\alpha f_\alpha\otimes g_\alpha\otimes h_\alpha$ is a sum of terms of the form

$$f\otimes g\otimes h \;=\; \prod\left(1-\frac{a_i}{w_1}\right)^{m_i}\otimes\prod\left(1-\frac{b_jw_1}{w_2}\right)^{n_j}\otimes\prod\left(1-\frac{w_2}{c_k}\right)^{p_k};$$

and even suppose that it is in ker(Tame): e.g. for one term, this means that on $\{w_2=b_jw_1\}\simeq(\mathbb{P}^1,\{0,\infty\})$, $f\otimes h\in{}'\mathcal{S}$. The trouble is that, for the same reason that γ was not well-defined on $'K_2^M(\mathbb{P}^1,\{0,\infty\})$, we cannot always complete such a $\gamma_{\mathbf{f}}$: suppose $h=1-f$, and h and f share the responsibility of being 1 on $\{0,\infty\}$ between them – then this is not completable by a relative cycle. One can remedy this extra discrepancy between ker(Tame) and $'K_3^M(\mathbb{P}^2,\partial\mathbb{P}^2)$ by *not* including Steinbergs of the form $f\otimes(1-f)$ in $'\mathcal{S}$ (and thereby making it harder to be in ker(Tame)).

As for producing linear elements of $'K_3^M(\mathbb{P}^2,\partial\mathbb{P}^2)$, which seems hard, we expect there is a connection with $\mathcal{B}_3(\mathbb{C})$; we will write one down in the example below. In the meantime, assuming their existence, we can replicate the situation on $\mathbb{P}^1$:

$$\int_{T_w^2} R_{\mathbf{f}} \equiv -\frac{1}{4\pi^2}\int_{\mathbb{P}^2} R_{\mathbf{f}_{\prod}}\wedge\mathrm{dlog}w_1\wedge\mathrm{dlog}w_2 \;+\; 2\pi i\int_{\xi}\mathrm{dlog}w_1\wedge\mathrm{dlog}w_2,$$

with $\xi\in H_2((\mathbb{C}^*)^2,\, V_{\mathbf{f}}\cap(\mathbb{C}^*)^2)$. The $\int_{\mathbb{P}^2}$ is easily computable (as is its analogue for all n):

$$\sum m_in_jp_k\left\{-\frac{1}{4\pi^2}\int_{\mathbb{P}^2}\log(1-\frac{a_i}{w_1})\mathrm{dlog}(1-\frac{b_jw_1}{w_2})\wedge\mathrm{dlog}(1-\frac{w_2}{c_k})\wedge\mathrm{dlog}w_1\wedge\;\mathrm{dlog}w_2\,[=0 \text{ by type}]\right.$$

$$+\frac{1}{2\pi i}\int_{T_{1-\frac{a_i}{w_1}}}\log(1-\frac{b_jw_1}{w_2})\mathrm{dlog}(1-\frac{w_2}{c_k})\wedge\mathrm{dlog}w_1\wedge\mathrm{dlog}w_2\;\;[=0\text{ by type}]$$

$$\left.+\int_{T_{(1-\frac{a_i}{w_1})}\cap T_{(1-\frac{b_jw_1}{w_2})}}\log(1-\frac{w_2}{c_k})\mathrm{dlog}w_1\wedge\mathrm{dlog}w_2+\begin{matrix}\text{more trivial terms,}\\ \text{since we may assume that}\\ T_{(1-\frac{a_i}{w_1})}\cap T_{(1-\frac{b_jw_1}{w_2})}\cap T_{(1-\frac{w_2}{c_k})}=\emptyset\end{matrix}\right\}$$

$$=-\sum m_in_jp_k\int_0^{a_i}\int_0^{b_jw_1}\log(1-\frac{w_2}{c_k})\mathrm{dlog}w_2\wedge\mathrm{dlog}w_1$$

$$=\sum m_in_jp_k\int_0^{a_i}\left(-\int_0^{\frac{b_jw_1}{c_k}}\log(1-u)\mathrm{dlog}u\right)\mathrm{dlog}w_1$$

$$= \sum m_i n_j p_k \int_0^{\frac{b_j a_i}{c_k}} \mathrm{Li}_2(v)\mathrm{dlog}v = \sum m_i n_j p_k \mathrm{Li}_3(\frac{b_j a_i}{c_k}) =: \mathrm{Li}_3(\mathcal{N}_{f,g,h}) =: \mathrm{Li}_3(\mathcal{N}_{\mathbf{f}}).$$

(This computation of the $\int_{\mathbb{P}^2}$ term generalizes to a result of the form "$\int_{\mathbb{P}^n} R'_{\mathbf{f}} \wedge \Omega_w^{n-1} = \mathrm{Li}_n(\mathcal{N}_{\mathbf{f}})$".)

EXAMPLE 3.2.1. Consider

$$\mathbf{f} = 8\,\mathsf{f} \otimes \mathsf{g} \otimes \mathsf{h} := 8\left(\frac{1-\frac{i}{w_1}}{1+\frac{i}{w_1}}\right) \otimes \left(\frac{1-i\frac{w_1}{w_2}}{1+i\frac{w_1}{w_2}}\right) \otimes \left(\frac{1-w_2}{1+w_2}\right)$$

where $(1-\mathsf{f})(1-\mathsf{g})(1-\mathsf{h}) = 0$ on $\partial\mathbb{P}^2$. Noting that $\{\cdot\}$ denotes going modulo fewer relations than usual, we have that

$$\{\mathbf{f}\} = \left\{\frac{1-\frac{i}{w_1}}{1+\frac{i}{w_1}}, \frac{1-i\frac{w_1}{w_2}}{1+i\frac{w_1}{w_2}}, \frac{1-w_2}{1+w_2}\right\}^8 \in \ker(\mathrm{Tame}) \subseteq {}'K_3^M(\mathbb{C}(\mathbb{P}^2, \partial\mathbb{P}^2)),$$

where ker(Tame) denotes the more restrictive sense mentioned above; and since $\mathbf{f}$ has no corners, $\overline{\gamma_{\mathbf{f}}}$ is an admissible element of $Z^3(\mathbb{P}^2, \partial\mathbb{P}^2; 3)$. Therefore it should be completable to a *relative* higher Chow ($\partial_{\mathcal{B}}$-)cycle, so that (by definition) $\{\mathbf{f}\} \in {}'K_3^M(\mathbb{P}^2, \partial\mathbb{P}^2)$. We will show how to complete it (by adding a relative cycle) along just one component of $V_{\mathbf{f}}$, namely $V_1 = \{w_2 = 1\}$, and leave the rest to the reader. Writing $w = w_1$ and $f(w) := \frac{1-\frac{i}{w}}{1+\frac{i}{w}}$, the component of Tame($\mathbf{f}$) there, by a simple calculation, is simply $8\,f(w) \otimes (-\frac{1}{f(w)})$, which one may express as a sum of Steinbergs *not* of the form $\xi \otimes (1-\xi)$. Alternately, the component of $\partial_{\mathcal{B}}\overline{\gamma_{\mathbf{f}}}$ with support over $(V_1, V_1 \cap \partial\mathbb{P}^2) \simeq (\mathbb{P}^1_w, \{w = 0, \infty\})$ is $8(f(w), -\frac{1}{f(w)})$. This is $\partial_{\mathcal{B}}$ of

$$-\Gamma_{V_1} := 4\left(f, \frac{(z+\frac{1}{f})^2}{(z-\frac{1}{f^2})(z-1)}, z\right) + 2\left(z, \frac{(z-f)^2}{(z-f^2)(z-1)}, \frac{1}{f^2}\right)$$

$$+2\left(f^2, \frac{(z-f^2)(z-\frac{1}{f^2})}{(z-1)^2}, z\right) - \left(z, \frac{(z-f^2)^2}{(z-f^4)(z-1)}, \frac{f^4}{z}\right) \in Z^2((\mathbb{P}^1_w, \{0,\infty\}), 3)$$

which is parametrized by $z \in$ an auxiliary $\mathbb{P}^1$ and $w = w_1$ on $V_1 = \mathbb{P}^1_w$ (through $f = f(w)$). It is easy to check that Γ_{V_1} is "relative": one coordinate in each component of Γ_{V_1} is 1 whenever $w = 0, \infty$ (since $f(0) = 1$ and $f(\infty) = -1$).

So the above computation applies, and noting that $\mathcal{N}_{\mathbf{f}} = 32(\{-1\} - \{1\}) \in \mathbb{Z}[\mathbb{C}^*]$ and the known values $\mathrm{Li}_3(-1) = -\frac{3}{4}\zeta(3)$, $\mathrm{Li}_3(1) = \zeta(3)$, we have apparently

$$\int_{T_w^2} R_{\mathbf{f}} \equiv -56\,\zeta(3) + \text{"lower-weight" terms}.$$

This checks with what one would expect from a Beilinson-type conjecture for relative varieties. We do not pursue this here.

3.2.4. Relative real regulators and Bloch-Wigner. Now we return to the $n = 2$ case and reinstate the $f \otimes (1-f)$ Steinbergs. One wonders if the $2\pi i \log$ terms added to $\mathrm{Li}_2(\mathcal{N}_{\mathbf{f}})$ are, like the integrals over elements of $\left\{C^1(\mathbb{C},1) \wedge C^1(\mathbb{C},2) = S^2(\mathbb{C},3)\right\}$ in §2.4.4, are corrections in order to make the $\Im$ part *exactly* $\mathcal{L}_2$ of something? In fact, the overall idea we want to suggest here is that the relative Milnor regulator on ${}'K_n^M(\mathbb{C}(\mathbb{P}^n, \partial\mathbb{P}^n))$ applied to Ω_w^{n-1}, *does* give classical n-logarithms; and that on ${}'K_n^M(\mathbb{P}^n, \partial\mathbb{P}^n)$ applied to T_w^{n-1} (followed by $\pi_{\mathbb{R}}^n$), *should* give modified Bloch-Wigner n-logarithms. So for these [relatively] completable linearly-factoring $\mathbf{f}$'s with "generalized Abelian symbol" $\mathcal{N}_{\mathbf{f}}$, one should get a $\mathbb{C}/\mathbb{Q}(n)$-lift of $\mathcal{L}_n(\mathcal{N}_{\mathbf{f}})$; the connection with §2.4.4 is that the $\mathcal{N}_{\mathbf{f}}$ of such functions, projected from $\mathbb{Z}[\mathbb{C}^*]$ to $\mathcal{B}_n(\mathbb{C})$, should give elements of $\ker(\delta)$. What we do now, is prove it all for $n = 2$, starting with the last statement.

First recall that $\{\mathbf{f}\}$ "nice" $\implies$

$$\{\mathbf{f}\} \in {}'K_2^M(\mathbb{P}^1, \{0,\infty\}) \subseteq \ker(\text{Tame}) \subseteq {}'K_2^M(\mathbb{C}(\mathbb{P}^1, \{0,\infty\})).$$

LEMMA 3.2.2. *Given $\boldsymbol{f} = \sum \ell_\alpha f_\alpha(w) \otimes g_\alpha(w)$ nice, with f_α and g_α of the form (*); then as an element of $\mathcal{B}_2(\mathbb{C})$, $\bar{\mathcal{N}}_{\boldsymbol{f}} = \sum \ell_\alpha m_i^\alpha n_j^\alpha \left\{\frac{a_i^\alpha}{b_j^\alpha}\right\}_2 \in \ker(st)$.*

PROOF. If

$$\mathbf{f} = f \otimes g = \prod_i (1 - \frac{a_i}{w})^{m_i} \otimes \prod_j (1 - \frac{w}{b_j})^{n_j}$$

so that $\mathbf{f}$ nice $\implies$ $\{a_i\}$ and $\{b_j\}$ are disjoint and $\prod_i (1 - \frac{a_i}{b_j})^{m_i n_j} = \prod_j (1 - \frac{a_i}{b_j})^{m_i n_j} = 1$, so that

$$\mathrm{st}(\bar{\mathcal{N}}_{f,g}) = \sum_{i,j} m_i n_j \cdot \frac{a_i}{b_j} \wedge (1 - \frac{a_i}{b_j})$$

$$= \sum_i a_i \wedge \prod_j (1 - \frac{a_i}{b_j})^{m_i n_j} - \sum_j b_j \wedge \prod_i (1 - \frac{a_i}{b_j})^{m_i n_j} = 0.$$

More generally, since $\{\mathbf{f}\} \in \ker(\text{Tame})$ and $\{a_i^\alpha\}$, $\{b_j^\alpha\}$ are disjoint for each α, $\mathrm{st}(\bar{\mathcal{N}}_{\mathbf{f}}) =$

$$\sum_{\alpha,i,j} \ell_\alpha m_i^\alpha n_j^\alpha \cdot \frac{a_i^\alpha}{b_j^\alpha} \wedge (1 - \frac{a_i^\alpha}{b_j^\alpha}) = \sum_\alpha \ell_\alpha \left\{ \sum_i m_i^\alpha \cdot a_i^\alpha \wedge g_\alpha(a_i^\alpha) - \sum_j n_j^\alpha \cdot b_j^\alpha \wedge f_\alpha(b_j^\alpha) \right\}$$

$$= \sum_\alpha \ell_\alpha \left\{ \sum_{p \in \mathbb{C}^*} p \wedge g_\alpha(p)^{\nu_p(f_\alpha)} - \sum_{p \in \mathbb{C}^*} p \wedge f_\alpha(p)^{\nu_p(g_\alpha)} \right\} = \sum_{\rho_i^e \in \mathbb{C}^*} p \wedge \prod_\alpha \left(\frac{g_\alpha(p)^{\nu_p(f_\alpha)}}{f_\alpha(p)^{\nu_p(g_\alpha)}} \right)^{\ell_\alpha}$$

$= 0.$ $\square$

We will retain the above conditions on $\mathbf{f}$ in the following, and[21] also assume that none of the $\{a_i^\alpha, b_j^\alpha\}$ are in[22] $\mathbb{R}^- = T_w^1$. Again assume $\mathbf{f} = f \otimes g$ to simplify notation. From the beginning of the section recall that

$$-\int_{T_w} R_{\mathbf{f}} = -\int_{T_w} \log f \mathrm{dlog} g + 2\pi i \sum_{T_f \cap T_w} \log g$$

$$= \frac{1}{2\pi i}\int_{\mathbb{P}^1} \log w \mathrm{dlog} f \wedge \mathrm{dlog} g - \int_{T_w} \log f \mathrm{dlog} g - 2\pi i \sum_{T_w \cap T_f} \log g = \frac{1}{2\pi i}\int_{\gamma_{\mathbf{f}}} R_\square^3.$$

Since the regulator sends branch changes $\mathbf{f} - \mathbf{f}_\prod$ to 0, and this last computation applies equally well to $\mathbf{f}_\prod$,

$$\int_{T_w} R_{\mathbf{f}} \equiv \int_{T_w} R_{\mathbf{f}_\prod} = \frac{-1}{2\pi i}\int_{\gamma_{\mathbf{f}_\prod}} R_\square^3.$$

Now $\overline{\gamma_{\mathbf{f}_\prod}}$ is completable (by the same additions as $\overline{\gamma_{\mathbf{f}}}$) since $\mathbf{f}$ is nice, and one has (with $V = \bigcup\{a_i\} \cup \{b_j\}$) $\Gamma_\prod = \Gamma_V + \overline{\gamma_{\mathbf{f}_\prod}}$, $\partial_{\mathcal{B}}\Gamma_\prod = 0$, $\Gamma_\prod \cap \left(\{0,\infty\} \times \square^2\right) \subseteq \{0,\infty\} \times \mathbb{I}^2$ (so that $\Gamma_\prod$ is a relative higher Chow cycle). Moreover $\int_{\Gamma_V} R_\square^3 = 0$ as $V \cap T_w = \emptyset$ and Γ_V consists of curves $\subset \square^2$ over points $\in V$. So

$$\int_{T_w} R_{\mathbf{f}} = \frac{-1}{2\pi i}\int_{\Gamma_\prod} R_\square^3,$$

and by the last section (and because $\partial_{\mathcal{B}}\Gamma_\prod = 0$)

$$\Im \int_{T_w} R_{\mathbf{f}} = \frac{1}{2\pi}\int_{\Gamma_\prod} \Re\{R_\square^3\} = \frac{1}{2\pi}\int_{\Gamma_\prod} r_\square^3.$$

Now $\int_{\Gamma_V} r_\square^3 = 0$ also (again, see §3.1.2) because one coordinate in $\square^3$ is constant, and

$$\gamma_{\mathbf{f}_\prod} = \sum m_i n_j \left(w, 1 - \frac{a_i}{w}, 1 - \frac{w}{b_j}\right) \underset{u = \frac{w}{b_j}}{\underset{\uparrow}{=}} \sum m_i n_j \left(u, 1 - \frac{a_i}{b_j u}, 1 - u\right).$$

Since $r_\square^3$ is alternating,

$$\int_{\Gamma_\prod} r_\square^3 = \int_{Alt(\Gamma_\prod)} r_\square^3 \underset{\text{discard } \Gamma_V}{\underset{\uparrow}{=}} \int_{Alt(\gamma_{\mathbf{f}_\prod})} r_\square^3 = \sum m_i n_j \int_{Alt(w, 1 - \frac{a_i/b_j}{w}, 1-w)} r_\square^3$$

$$= \sum m_i n_j \int_{\rho_2(a_i/b_j)} r_\square^3 = \sum m_i n_j \cdot 2\pi \mathcal{L}_2(a_i/b_j) = 2\pi \mathcal{L}_2(\mathcal{N}_{f,g}),$$

[21] though one may avoid it easily; as we will see in the computation at the end of the section.

[22] note: $\int_{T_w^1} = -\int_0^\infty$

where we have used Lemma 3.1.3 for $n = 2$, $\mathcal{L}_2(a) = \int_{\rho_2(a)} r^3_{\square}$. Therefore we have proved that

$$\Im \int_{T_w} R_{\mathbf{f}} \;=\; \mathcal{L}_2(\mathcal{N}_{f,g}).$$

Now we "test" our machinery by doing instead a comparatively "direct" computation, to the same effect, *not* using the Goncharov lemma (our 3.1.3).

Again the first key is to observe that (mod $\mathbb{Z}(2)$)

$$\int_{T_w} R_{\mathbf{f}} \;\equiv\; \int_{T_w} R_{\mathbf{f}_\Pi} \;=\sum_{i,j} m_i n_j \left\{ \int_{T_w} \log f_i \mathrm{dlog} g_j - 2\pi i \sum_{T_w \cap T_{f_i}} \log g_j \right\}$$

$$= \sum_{i,j} m_i n_j \int_{T_w} \log f_i \mathrm{dlog} g_i \,, \qquad \text{where } f_i = (1 - \frac{a_i}{w}),\; g_j = (1 - \frac{w}{b_j})$$

and the sum inside the brackets vanished because $T_w \cap V \subseteq \{0, \infty\}$. On $\mathbb{P}^1_\epsilon (= \mathbb{P}^1 \setminus N_\epsilon\{0,\infty\})$, since $\mathrm{dlog} f_i \wedge \mathrm{dlog} g_j \;=\; \mathrm{dlog} w \wedge \mathrm{dlog} g_j \;=\; 0$ by type,

$$\mathrm{d}\left[\frac{1}{2\pi i} \log w \log f_i \mathrm{dlog} g_j - \log g_j \log w \cdot \delta_{T_{f_i}} \right] =$$

$$\Big(\log w \mathrm{dlog} g_j \cdot \delta_{T_{f_i}} + \log f_i \mathrm{dlog} g_j \cdot \delta_{T_w} + \log w \log f_i \cdot \delta_{(g_j)} \Big) \;-\; \Big(\log g_j \mathrm{dlog} w \cdot \delta_{T_{f_i}} +$$

$$\log w \mathrm{dlog} g_j \cdot \delta_{T_{f_i}} + 2\pi i \log g_j \cdot \delta_{T_{f_i} \cap T_w} + 2\pi i \log w \cdot \delta_{T_{f_i} \cap T_{g_j}} + \log g_j \log w \cdot \delta_{(f_i)} \Big)$$

$$= \log f_i \mathrm{dlog} g_j \cdot \delta_{T_w} - \log g_j \mathrm{dlog} w \cdot \delta_{T_{f_i}} + \log w \log f_i \cdot \delta_{(g_j)} - \log w \log g_j \cdot \delta_{(f_i)},$$

where $\log g_j \;=\; 0$ on $T_{f_i} \cap T_w \subseteq \{0\}$, and $T_{f_i} \cap T_{g_j} \;=\; \emptyset$ because $\{a_i\}$ and $\{b_j\}$ are disjoint; this is one thing we have gained by switching to product branches.[23] Integrating the result on $\mathbb{P}^1_\epsilon$ (and taking a limit) we have[24]

$$\int_{T_w} \log f_i \mathrm{dlog} g_j \;=\; \int_{T_{f_i}} \log g_j \mathrm{dlog} w + \sum_{w \in (f_i) \cap \mathbb{C}^*} \log w \log g_j(w) - \sum_{w \in (g_j) \cap \mathbb{C}^*} \log w \log f_i(w)$$

---

[23] $T_{f_i} = f_i^{-1}(\mathbb{R}^-)$, $T_{g_j} = g_j^{-1}(\mathbb{R}^-)$. We have to make perhaps the stronger assumption here that $\{\arg a_i\}$ and $\{\arg b_j\}$ are disjoint, for the argument to go through with no extra work.

[24] in greater detail: if $S := \frac{1}{2\pi i} \log w \log f_i \mathrm{dlog} g_j - \log g_j \log w \cdot \delta_{T_f}$, we computed above $\mathrm{d}[S]$ on $\mathbb{P}^1_\epsilon$, and claim that "we have" $\lim_{\epsilon \to 0} \int_{\mathbb{P}^1_\epsilon} \mathrm{d}[S] = 0$, while *a priori* this only equals $\lim_{\epsilon \to 0} \int_{\partial \mathbb{P}^1_\epsilon} S$. Therefore, if $\mathcal{C}_\epsilon(0)$ is a circle around $\{0\}$ [and similarly with $\{\infty\}$], we must show the following tends to 0 with ϵ:

$$\lim_{\epsilon \to 0} \left\{ \int_{\mathcal{C}_\epsilon(0)} \frac{1}{2\pi i} \log w \log(1 - \frac{a_i}{w}) \mathrm{dlog}(1 - \frac{w}{b_j}) \;+ \sum_{\mathcal{C}_\epsilon(0) \cap T_{(1 - \frac{a_i}{w})}} \log(1 - \frac{w}{b_j}) \log w \right\}$$

where $T_{(1-\frac{a_i}{w})}$ is a path from 0 to a_i. if we approximate $\log(1 - \frac{a_i}{w})$ by $\log w$, $\mathrm{dlog}(1 - \frac{w}{b_j})$ by $\mathrm{d}w$ and the circumference of $\mathcal{C}_\epsilon(0)$ by $2\pi i \epsilon$, then the integral is bounded by essentially $\epsilon \log^2 \epsilon$ and the sum by $\epsilon \log \epsilon$, which is enough.

$$= \int_0^{a_i} \log(1 - \frac{w}{b_j})\mathrm{dlog}w \, + \, \log a_i \log(1 - \frac{a_i}{b_j}) \, - \, \log b_j \log(1 - \frac{a_i}{b_j}).$$

Adding all of these together, we have shown that

$$- \int_0^{\infty} R_{\mathbf{f}} \, = \, - \sum m_i n_j \mathrm{Li}_2(\frac{a_i}{b_j}) \, + \, \sum m_i n_j \left\{\log a_i - \log b_j\right\} \log(1 - \frac{a_i}{b_j}).$$

We cannot write $\log a_i - \log b_j = \log \frac{a_i}{b_j}$, because this would not be the standard branch of $\log \frac{a_i}{b_j}$. However if we take the imaginary part of the second term, we obtain

$$\sum_{i,j} m_i n_j \log \left|\frac{a_i}{b_j}\right| \arg(1 - \frac{a_i}{b_j}) \, + \, \sum_{i,j} m_i n_j \arg a_i \log \left|1 - \frac{a_i}{b_j}\right| - \sum_{i,j} m_i n_j \arg b_j \log \left|1 - \frac{a_i}{b_j}\right|$$

$$= \, \log |w| \arg(1-w) \,|_{\mathcal{N}_{f,g}} \, + \sum_i \arg a_i \log \left|\prod_j (1 - \frac{a_i}{b_j})^{m_i n_j}\right| - \sum_j \arg b_j \log \left|\prod_i (1 - \frac{a_i}{b_j})^{m_i n_j}\right|$$

where the second and third terms are zero since $\{\mathbf{f}\} \in \ker(\mathrm{Tame})$. This is beautiful: we have exactly the right branches of everything (Li_2, arg) involved and can state the conclusion, recalling that $\mathcal{L}_2(w) = -\Im\{\mathrm{Li}_2(w)\} + \arg(1-w)\log|w|$ (where Li_2, arg are assumed in the standard branches):

$$\Im \int_{T_w} R_{\mathbf{f}} \, = \, \sum_{i,j} m_i n_j \mathcal{L}_2 \left(\frac{a_i}{b_j}\right) \;\; =: \, \mathcal{L}_2(\mathcal{N}_{f,g}).$$

More generally for nice $\mathbf{f}$

$$\Im \int_{T_w} R_{\mathbf{f}} \, = \, \sum_{\alpha,i,j} \ell_\alpha m_i n_j \mathcal{L}_2 \left(\frac{a_i^\alpha}{b_j^\alpha}\right) \;\; =: \, \mathcal{L}_2(\mathcal{N}_{\mathbf{f}}).$$

We can make some progress towards a comparable result for $n = 3$ (the computation in this spirit is much more lengthy), but have trouble believing it until the following difficulty is resolved. In the argument using the Goncharov lemma for $n = 2$, we used that $r^3_\square$ integrated over an element of $S^2(\mathbb{C}, 3)$ is zero; we would need for $n = 3$ that $r^5_\square$ integrates to zero over elements of $S^3(\mathbb{C}, 5)$. Is this true?

3.2.5. The easiest nontrivial regulator computation. Finally for $n = 2$ we give a simple example where we can compute the entire $\mathbb{C}/\mathbb{Q}(2)$ regulator value, exhibiting the Catalan constant G (a famous transcendental number) as a period. Consider the functions

$$f = \frac{1 - \frac{i}{w}}{1 + \frac{i}{w}}, \quad g = \frac{1-w}{1+w}$$

and begin with the element $\{\mathbf{f}\} = \{f^2, g^2\} \in \ker(\mathrm{Tame}) \subseteq K_2^M(\mathbb{C}(\mathbb{P}^1, \{0, \infty\}))$. Let γ_ϵ be the path along $\mathbb{R}^-$ from $\{0\}$ to $\{\infty\}$, perturbed by $e^{-i\epsilon}$ to avoid $\{1\} \in |(g)|$. It gives a generator of $H_1(\mathbb{P}^1, \{0, \infty\})$.

We want to evaluate

$$\int_{\gamma_\epsilon} R_{\mathbf{f}} = \int_{\gamma_\epsilon} \log f^2 \mathrm{dlog} g^2 - \sum_{\gamma_\epsilon \cap T_{f^2}} 2\pi i \log g^2$$

but do not want to deal with the branch $\log f^2$ or [the point] $\gamma_\epsilon \cap T_{f^2}$. (Here T_{f^2} is the preimage of $\mathbb{R}^-$ under f^2, which in this case is the unit circle; $\log g$ blows up very near the intersection.) In order to change to a more convenient (but equivalent) $'\mathbf{f}$, we enlarge our available generating elements (i.e., use a larger subset of $\otimes^2\mathbb{Z}[\mathbb{C}(\mathbb{P}^1)^*]$) by working in $'K_2^M(\mathbb{C}(\mathbb{P}^1, \{0,\infty\}))$. If we set $'\mathbf{f} = 4 \cdot f \otimes g$, then $'\mathbf{f} - \mathbf{f} = 4 \cdot f \otimes g - f^2 \otimes g^2 \in' \mathcal{S}$ and so $\{'\mathbf{f}\} = \{f, g\}^4 = \{f^2, g^2\} = \{\mathbf{f}\}$ in $'K_2^M$; therefore the above integral is identical (mod $\mathbb{Q}(2)$) to

$$\int_{\gamma_\epsilon} R_{'\mathbf{f}} = 4 \int_{\gamma_\epsilon} \log f \mathrm{dlog} g$$

(where the second term is not written because $\gamma_\epsilon \cap T_f = \{0\}$ and $\log g = 0$ there). One can see even more directly that $\int_{\gamma_\epsilon} R_{'\mathbf{f}} \equiv \int_{\gamma_\epsilon} R_{\mathbf{f}}$ by noticing that up to $\mathbb{Z}(2)$-valued currents [=integral chains $\times\ 4\pi^2$] one has on $\mathbb{P}^1$ (or just on γ_ϵ) $R_{'\mathbf{f}} - R_{\mathbf{f}} = \mathrm{d}[2\pi i \Delta_f^2 \log g^2]$, where the 0-current $\Delta_f^2 := \frac{1}{2\pi i}(2\log f - \log f^2)$ and $\mathrm{d}[\Delta_f^2] = T_{f^2} - 2T_f$.

Now we make a rare exception, and use the slightly altered branch of $\log z$ corresponding to the cut $-\gamma_\epsilon$; we also perturb T_f and T_g (and the accompanying branches of $\log f$ and $\log g$) slightly so they avoid $\{0\}$, $\{\infty\}$, and γ_ϵ, and reason that one gets the same answer in the limit as this perturbation$\to 0$. The following computation mimics what we did on $\mathbb{P}^1_\epsilon$ above:

$$0 = \int_{\mathbb{P}^1} \mathrm{d}\left[-\frac{1}{2\pi i} \log z \log f \mathrm{dlog} g + \log z \log g|_{T_f}\right]$$

$$= \int_{\mathbb{P}^1} \big\{\log f \mathrm{dlog} g|_{\gamma_\epsilon} - \log z \mathrm{dlog} g|_{T_f} + \log z \mathrm{dlog} g|_{T_f} + \log g \mathrm{dlog} z|_{T_f}$$

$$- \log z \log f|_{(g)} + \log z \log g|_{(f)} + 2\pi i \log z|_{T_f \cap T_g} - 2\pi i \log g|_{T_f \cap \gamma_\epsilon}\big\}$$

$$= \int_{\gamma_\epsilon} \log f \mathrm{dlog} g + \int_{T_f} \log g \mathrm{dlog} z - \sum_{(g)} \log z \log f + \sum_{(f)} \log z \log g,$$

since the two intersections are empty. Multiplying by 4, the last two terms are $\mathbb{Z}(2)$. Now $\log g = \log(1-z) - \log(1+z)$ exactly (i.e., *not* mod $\mathbb{Z}(1)$), and so

$$\int_{\gamma_\epsilon} \log f \mathrm{dlog} g = -\int_{T_f} \log(1-z) \mathrm{dlog} z + \int_{T_f} \log(1+z) \mathrm{dlog} z.$$

Since the dilogarithm has no monodromy about 0, this

$$= -2\int_{T_f} \log(1-z)\mathrm{dlog} z \;=\; 2\,(\mathrm{Li}(i) - \mathrm{Li}(-i))$$

$$= 2\left(\sum_{k=1}^{\infty} \frac{i^k}{k^2} - \sum_{k=1}^{\infty} \frac{(-i)^k}{k^2}\right) \;=\; 4i \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2},$$

and 4 times this is the final result, $16i \cdot G$.

3.3. Comparison to the Milnor-Sheaf Regulator

In this section we define a regulator on the Milnor sheaf[25] compatible with the cup product in Deligne cohomology; it can be shown its real part thus necessarily agrees with Beilinson's regulator. Thus comparing our Milnor regulator to this one would give an alternative method for proving its compatibility with Beilinson (if we did not have Goncharov around), and is interesting in itself.

Let $\underline{K^M_{n,X}}$ and $\underline{\mathcal{H}^n_{\mathcal{D}}(n)}$, respectively, be the sheaves on X associated to the presheaves

$$\mathcal{U} \to K^M_n(\Gamma(\mathcal{U}, \mathcal{O}^*_X)), \quad \mathcal{U} \to H^n_{\mathcal{D}}(\mathcal{U}, \mathbb{Z}(n))\,;$$

an alternative definition for the Milnor sheaf is

$$\underline{K^M_{n,X}} := \frac{\mathcal{O}^*_X \otimes \ldots \otimes \mathcal{O}^*_X}{\mathcal{S}} \tag{*}$$

where $\mathcal{S}$ is the subsheaf of the tensor product generated by the Steinberg relations. Modulo torsion it is resolved by the Gersten sequence

$$0 \to \underline{K^M_{n,X}} \to \coprod_{x \in X^0} \iota^x_* K^M_n(\mathbb{C}(x)) \overset{\text{Tame}}{\to} \coprod_{x \in X^1} \iota^x_* K^M_{n-1}(\mathbb{C}(x)) \to \ldots$$

$$\to \coprod_{x \in X^{n-1}} \iota^x_* \mathbb{C}(x)^* \to \coprod_{x \in X^n} \iota^x_* \mathbb{Z} \to 0$$

so that in particular

$$\underline{K^M_{n,X}} \simeq \mathrm{im}\left\{\underline{K^M_{n,X}} \to \coprod_{x\in X^0} K^M_n(\mathbb{C}(x))\right\} = \ker\left\{\coprod_{x \in X^0} K^M_n(\mathbb{C}(x)) \to \coprod_{x\in X^1} K^M_{n-1}(\mathbb{C}(x))\right\},$$

and taking sections

$$H^0(X, \underline{K^M_{n,X}}) \cong_{\otimes\mathbb{Q}} \ker(\mathrm{Tame}) \subseteq K^M_n(\mathbb{C}(X)), \tag{**}$$

$$H^0(\eta_X, \underline{K^M_{n,X}}) \cong_{\otimes\mathbb{Q}} K^M_n(\mathbb{C}(X)).$$

[25](which has appeared for example in **[Es]**)

For a proof of the Gersten sequence (and more details) the reader may consult **[So]**.

REMARK 3.3.1. It is interesting to observe that, starting with an element $\{\mathbf{f}\} \in \ker(\text{Tame})$, (*) + (**) means that one may choose a cover $\{\mathcal{U}_\xi\}$ of X and $\mathbf{f}_\xi \in H^0(\mathcal{U}_\xi, \mathcal{O}_X^* \otimes \ldots \otimes \mathcal{O}_X^*)$ so that $\mathbf{f} - \mathbf{f}_\xi \in H^0(\mathcal{U}_\xi, \mathcal{S})$ is a "local Steinberg relation." Thus, while it is possible (according to the moving lemma) *globally* to remove "corners" from any element of $K_n^M(\mathbb{C}(X))$ via a Steinberg, one may also *locally* remove *all* singularities from an element of $\ker(\text{Tame})$.

Using local-global spectral sequences as in §1.2, one may show modulo torsion for $n \geq 2$

$$\begin{tikzcd}
K_n^M(X) \arrow[r, hook, "\cong \text{for} n=2,3"'] & \ker(\text{Tame}) \arrow[r, hook] & K_n^M(\mathbb{C}(X)) \\
CH^n(X,n) \arrow[u, "\cong \text{for} n=2"'] \arrow[r, "\cong\, (n=2)", "\twoheadrightarrow\, (n=3)"'] & H^0(X, \underline{K_{n,X}^M}) \arrow[u, "\cong_{\otimes\mathbb{Q}}"'] &
\end{tikzcd}$$

We will sketch how to construct a map of sheaves[26]

$$\underline{K_{n,X}^M} \to \underline{\mathcal{H}_{\mathcal{D}}^n(n)}$$

so that the induced regulator on sections fits into our picture as follows (mod torsion):

$$\begin{tikzcd}
CH^n(X,n) \arrow[r, two heads] \arrow[d] & K_n^M(X) \arrow[r, hook] \arrow[d] & H^0(X, \underline{K_{n,X}^M}) \arrow[r, hook] & K_n^m(\mathbb{C}(X)) \arrow[d] \\
H_{\mathcal{D}}^n(X, \mathbb{Z}(n)) \arrow[r, two heads] & \operatorname{im}\{H_{\mathcal{D}}^n(X, \mathbb{Z}(n)) \to H_{\mathcal{D}}^n(\eta_X, \mathbb{Z}(n))\} \arrow[r, hook] & H^0(X, \underline{\mathcal{H}_{\mathcal{D}}^n(n)}) \arrow[r, hook] & H_{\mathcal{D}}^n(\eta_X, \mathbb{C}/\mathbb{Z}(n))
\end{tikzcd}$$

where $H^0(X, \underline{\mathcal{H}_{\mathcal{D}}^n(n)})$ is essentially $\ker(\text{Res}^1)$. What we will actually prove below, is that this "sheaf regulator" agrees with our Milnor regulator over the generic point.

3.3.1. Formula for the regulator on $\underline{K_{n,X}^M}$. Given an element $\{\mathbf{f}\} \in H^0(X, \underline{K_{n,X}^M})$ we can again find a cover by Zariski open sets $\{\mathcal{U}_\xi\}$ and $\mathbf{f}_\xi \in H^0(\mathcal{U}_\xi, \mathcal{O}_X^*) \otimes \ldots \otimes H^0(\mathcal{U}_\xi, \mathcal{O}_X^*)$ whose images $\{\mathbf{f}_\xi\} \in H^0(\mathcal{U}_\xi, \underline{K_{n,X}^M})$ agree with the restictions of $\{\mathbf{f}\}$. Set

$$\mathbb{Z}(1)^\bullet_{\mathcal{D},X} := \mathbb{Z}(1) \to \mathcal{O}_X\,, \qquad H_{\mathcal{D}}^*(\mathcal{U}_\xi, \mathbb{Z}(1)) := \mathbb{H}^*(\mathcal{U}_\xi, \mathbb{Z}(1)^\bullet_{\mathcal{D},X}),$$

[26] The map $\bigwedge^n \text{dlog} : \underline{K_{n,X}^M} \to \Omega_X^n$ induced by sending $\{f_1, \ldots, f_n\} \in K_n^M(\Gamma(\mathcal{U}, \mathcal{O}_X^*))$ to $\text{dlog} f_1 \wedge \ldots \wedge \text{dlog} f_n \in \Omega_X^n(\mathcal{U})$, which is familiar to readers of **[EP]** or **[Gr4]**, factors through this map.

and map

$$H^0(\mathcal{U}_\xi, \mathcal{O}_X^*) \overset{\theta}{\to} H^1_{\mathcal{D}}(\mathcal{U}_\xi, \mathbb{Z}(1))$$

by sending functions[27]

$$\mathbf{f} \longmapsto \{\mathrm{Log}_\alpha f\,,\ \mathrm{Log}_\alpha f - \mathrm{Log}_\beta f\} =: \theta(f) \in \left\{\check{C}^0(\mathcal{U}_\xi, \mathcal{O}_{\mathcal{X}})\,,\ \check{C}^1(\mathcal{U}_\xi, \mathbb{Z}(1))\right\}$$

to cocycles in the double complex

$$E_0^{p,q}(1) := \check{C}^p(\mathcal{U}_\xi, \mathbb{Z}(1)^q_{\mathcal{D},X})$$

of Cech cochains. (Here α, β reflect a *real analytic* refinement of $\mathcal{U}_\xi$ by a collection $\{\mathcal{W}_{\alpha\beta}\}$; $\check{C}^0$ is given by sections over each $\mathcal{W}_{\alpha\beta}$, $\check{C}^1$ by sections over $\mathcal{W}_{\alpha\beta} \cap \mathcal{W}_{\beta\gamma}$, and so on.) So $\mathbf{f}_\xi$ maps to $\otimes^n H^1_{\mathcal{D}}(\mathcal{U}_\xi, \mathbb{Z}(1))$; we may now employ the cup-product on Deligne cohomology (see below) to reach $H^n_{\mathcal{D}}(\mathcal{U}_\xi, \mathbb{Z}(n))$. Provided the composite kills Steinbergs $H^0(\mathcal{U}_\xi, \mathcal{S})$, $H^0(\mathcal{U}_\xi \cap \mathcal{U}_\eta, \mathcal{S})$ locally we wind up with a well-defined section in $H^0(X, \underline{\mathcal{H}^n_{\mathcal{D}}(n)})$.

To simplify notation we pass to the generic point η_X and give explicit formulas for this regulator, in terms of Cech cochains (again using a system of analytic neighborhoods $\{\mathcal{W}_\alpha\}$ on η_X). To begin with, define

$$\mathbb{Z}(n)^\bullet_{\mathcal{D},\mathcal{X}} := \mathbb{Z}(n) \to \mathcal{O}_X \overset{\mathrm{d}}{\to} \Omega^1_X \overset{\mathrm{d}}{\to} \ \ldots\ \overset{\mathrm{d}}{\to} \Omega^{n-1}_X \to 0 \left[= \mathrm{Cone}\left\{\mathbb{Z}(n) \to \Omega^{\bullet<n}_X\right\}[-1]\right]$$

(with $\mathbb{Z}(n)$ placed in degree 0) and

$$H^*_{\mathcal{D}}(\eta_X, \mathbb{Z}(n)) := \mathbb{H}^*(\eta_X, \mathbb{Z}(n)^\bullet_{\mathcal{D},X})$$

where the hypercohomology map be computed by means of the double complex (either as a total complex or as a spectral sequence)

$$E_0^{p,q}(n) := \check{C}^p(\eta_X, \mathbb{Z}(n)^q_{\mathcal{D},X}) \ \text{ with total differential } D = \mathrm{d} + (-1)^{n-q}\delta.$$

According to **[EV]**, there is a well-defined cup-product

$$H^p_{\mathcal{D}}(\eta_X, \mathbb{Z}(\ell)) \wedge H^q_{\mathcal{D}}(\eta_X, \mathbb{Z}(m)) \overset{\cup}{\to} H^{p+q}_{\mathcal{D}}(\eta_X, \mathbb{Z}(\ell+m)),$$

induced by the map of sheaves

$$\cup :\ \mathbb{Z}(\ell)^a_{\mathcal{D},X} \otimes \mathbb{Z}(m)^b_{\mathcal{D},X} \to \mathbb{Z}(\ell+m)^{a+b}_{\mathcal{D},X}$$

taking

$$x \otimes y \ \longmapsto \left\{\begin{array}{cc} x \cdot y & a = 0 \\ x \wedge \mathrm{d}y & a > 0,\, b = m \\ 0 & a > 0,\, b \neq m \end{array}\right\} =: x \cup y.$$

[27]we write Log_α for these branches of log, which are chosen so as to be continuous (and single-valued) on each $\mathcal{W}_\alpha$, instead of $\log_\alpha$, reserving the lower-case log-notation for our preferred branch with argument between π and $-\pi$.

(In particular, it takes cocycles to cocyles because the product obeys a Leibniz rule under both differentials.) For example, for f, $g \in \mathbb{C}(X)^*$ consider the two cocycles

$$\theta(f) = \{\mathrm{Log}_\alpha f,\, \mathrm{Log}_\alpha f - \mathrm{Log}_\beta f\}\,,\ \theta(g) = \{\mathrm{Log}_\alpha g,\, \mathrm{Log}_\alpha g - \mathrm{Log}_\beta g\}$$

$$\in \left\{\check{C}^0(\eta_X,\, \mathcal{O}_X = \mathbb{Z}(1)^1_{\mathcal{D},X}\,,\ \check{C}^1(\eta_X, \mathbb{Z}(1) = \mathbb{Z}(1)^0_{\mathcal{D},X}\right\}$$

in the double complex defining elements of $H^1_{\mathcal{D}}(\eta_X, \mathbb{Z}(1))$. Then $\theta(f) \cup \theta(g) =$

$$\{\mathrm{Log}_\alpha f\, \mathrm{dlog} g\,,\ (\mathrm{Log}_\alpha f - \mathrm{Log}_\beta f)\cdot \mathrm{Log}_\beta g\,,\ (\mathrm{Log}_\alpha f - \mathrm{Log}_\beta f)\cdot(\mathrm{Log}_\beta g - \mathrm{Log}_\gamma g)\}$$

$$\in \left\{\check{C}^0(\eta_X, \Omega^1_X = \mathbb{Z}(2)^2_{\mathcal{D},X})\,,\ \check{C}^1(\eta_X, \mathcal{O}_X = \mathbb{Z}(2)^1_{\mathcal{D},X})\,,\ \check{C}^2(\eta_X, \mathbb{Z}(2)^0_{\mathcal{D},X})\right\}$$

defines an element of $\mathbb{H}^2(\eta_X, \mathbb{Z}(2)^\bullet_{\mathcal{D}\,X}) = H^2_{\mathcal{D}}(X, \mathbb{Z}(2))$; the reader may wish to check that this element is closed in the double complex (under D).

To state the generalization we introduce the notation

$$\mathrm{Log} f := \mathrm{Log}_\alpha f \in \check{C}^0(\eta_X, \mathbb{Z}(1))\ , \qquad \Delta\mathrm{Log} f := \mathrm{Log}_\alpha f - \mathrm{Log}_\beta f\ \in \check{C}^1(\eta_X, \mathbb{Z}(1))$$

(the latter defined on intersections $\mathcal{W}_{\alpha\beta} := \mathcal{W}_\alpha \cap \mathcal{W}_\beta \subset \eta_X$), and

$$\Delta\mathrm{Log} f\, \Delta\mathrm{Log} g := (\mathrm{Log}_\alpha f - \mathrm{Log}_\beta f)\cdot(\mathrm{Log}_\beta g - \mathrm{Log}_\gamma g)\ \in \check{C}^2(\eta_X, \mathbb{Z}(2))$$

for Cech cup-products, and so on. Then

$$\theta(f_1) \cup \ldots \cup \theta(f_n)\ =\ \{\log f_1 \mathrm{dlog} f_2 \wedge \ldots \wedge \mathrm{dlog} f_n\,,\ \Delta\mathrm{Log} f_1 \mathrm{Log} f_2 \mathrm{dlog} f_3 \wedge \ldots \wedge \mathrm{dlog} f_n\,,$$

$$\ldots\,,\ \Delta\mathrm{Log} f_1 \Delta\mathrm{Log} f_2 \cdot \ldots \cdot \Delta\mathrm{Log} f_{n-1} \mathrm{Log} f_n\,,\ \Delta\mathrm{Log} f_1 \cdot \ldots \cdot \Delta\mathrm{Log} f_n\}$$

is a cocycle in $(E_0^{*,*}(n), D)$ defining an element of $\mathbb{H}^n(\eta_X, \mathbb{Z}(n)^\bullet_{\mathcal{D},X})$. The check that this map from

$$\otimes^n \mathbb{Z}[\mathbb{P}^1_{\mathbb{C}(X)} \setminus \{0, \infty\}] \to H^n_{\mathcal{D}}(\eta_X, \mathbb{Z}(n))$$

factors through $H^0(\eta_X, \underline{K^M_{n,X}}) \cong K^M_n(\mathbb{C}(X))$, uses monodromy of Li_2 (not Li_n) for all n. This will also follow from the comparison with our regulator on which we now embark.

3.3.2. Comparison with Milnor regulator currents. We will need rules for differentiating in the double complex, beginning with the obvious Leibniz rule

$$\delta\{K \cup K'\}\ =\ \delta K \cup K'\ +\ (-1)^{\deg K} K \cup \delta K'$$

for Cech cochains (where $\deg K$ is Cech degree). We will enlarge our complex to include currents: the 0-current $\log f$ (whose imaginary part is always $\in (-\pi, \pi]$), the 1-current [=codimension-1 chain] δ_{T_f}, and

$$\mathbb{D}\mathrm{Log} f\ :=\ \mathrm{Log} f - \log f\ :=\ \mathrm{Log}_\alpha f - \log f\ \in \check{C}^0(\eta_X,\, {}'\mathcal{D}^0_X)$$

(a strange Cech-current amalgam). We then have the rules

$$\delta \log f = \delta(\delta_{T_f}) = 0\,, \quad \delta(\Delta\mathrm{Log} f) = 0\,,$$

$$\delta\mathrm{Log} f = \delta(\mathbb{D}\log f) = -\Delta\mathrm{Log} f\,; \quad \text{and}$$

$$\mathrm{d}[\log f] \,=\, \mathrm{dlog} f - 2\pi i \delta_{T_f},$$

$$\mathrm{d}[\mathrm{Log} f] \,=\, \mathrm{dlog} f \quad (\text{on } \mathcal{W}_\alpha,\ \mathrm{Log}_\alpha f \text{ is continuous})$$

$$\mathrm{d}[\mathbb{D}\mathrm{Log} f] \,=\, \mathrm{dlog} f - (\mathrm{dlog} f - 2\pi i \delta_{T_f}) \,=\, 2\pi i \delta_{T_f}\,,$$

$$\mathrm{d}[\Delta \mathrm{Log} f] = 0 \qquad \left(\begin{array}{l} \text{on } \mathcal{W}_{\alpha\beta} \text{ where } \Delta\mathrm{Log} f \\ \text{gives a constant value} \end{array} \right).$$

There are no residues on η_X so $\mathrm{d}[\mathrm{dlog} f] = \mathrm{d}[\delta_{T_f}] = 0$.

Recall that the cohomological complex[28]

$$\mathcal{C}(n)^\bullet_{\mathcal{D},X} := \mathrm{Cone}\left\{ \mathcal{C}^X_{2d-\bullet}(\mathbb{Z}(n)) \oplus F^n{}'\mathcal{D}^\bullet_X \to {}'\mathcal{D}^\bullet_X \right\}[-1]$$

$$= \mathcal{C}^X_{2d-\bullet}(\mathbb{Z}(n)) \oplus F^n{}'\mathcal{D}^\bullet_X \oplus {}'\mathcal{D}^{\bullet-1}_X$$

with differential

$$\mathrm{d}(a,\, b,\, c) \,=\, (-\partial a,\, -\mathrm{d}[b],\, \mathrm{d}[c] - b + a)$$

computes Deligne (co)homology:

$$\mathbb{H}^*(\eta_X, \mathcal{C}(n)^\bullet_{\mathcal{D},X}) \cong H^*_{\mathcal{D}}(\eta_X, \mathbb{Z}(n)).$$

Moreover all its terms are acyclic, i.e.

$$H^*(\eta_X, \mathcal{C}(n)^q_{\mathcal{D},X}) = 0 \qquad \text{for } * > 0 \quad (\text{for each } q),$$

and so

$$\mathbb{H}^*(\eta_X, \mathcal{C}(n)^\bullet_{\mathcal{D},X}) \cong H^*(\mathcal{C}(n)^\bullet_{\mathcal{D},X}(\eta_X)) \;=\; \text{cohomology of}$$

$$\mathrm{Cone}\left\{ \mathcal{C}_{2d-\bullet}(\eta_X, \mathbb{Z}(n)) \oplus F^n{}'\mathcal{D}^\bullet_X(\eta_X) \to {}'\mathcal{D}^\bullet_X(\eta_X) \right\}[-1]$$

(which is familiar from §2.4). Now we can map

$$\mathbb{Z}(n)^\bullet_{\mathcal{D},X} \to \mathcal{C}(n)^\bullet_{\mathcal{D},X}$$

as complexes, as follows:

$$\begin{tikzcd}
 & (\deg 0) & (\deg 1) & (\deg 2) & \\
0 \arrow[r] & \mathbb{Z}(n) \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[r, "\mathrm{d}"] \arrow[d] & \Omega^1_X \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \mathcal{C}^X_{2d}(\mathbb{Z}(n)) \arrow[r] & \mathcal{C}^X_{2d-1}(\mathbb{Z}(n)) \oplus {}'\mathcal{D}^0_X \arrow[r] & \mathcal{C}^X_{2d-2}(\mathbb{Z}(n)) \oplus {}'\mathcal{D}^1_X \arrow[r] & \cdots
\end{tikzcd}$$

$$\Delta \longmapsto (-\partial\Delta, \Delta)$$

$$(\mathcal{C}, f) \longmapsto (-\partial\mathcal{C}, \mathrm{d}[f] + \mathcal{C})$$

[28] Here $\mathcal{C}^X_*(\mathbb{Z}(n))$ is the sheaf of germs of $*$-dimensional real-analytic chains with coefficients in $\mathbb{Z}(n)$.

$$\begin{array}{ccccc}
(\deg n-1) & & (\deg n) & & (\deg n+1)
\end{array}$$

$$\begin{tikzcd}
\cdots \arrow[r] & \Omega_X^{n-2} \arrow[r, "\mathrm{d}"] \arrow[d] & \Omega_X^{n-1} \arrow[r] \arrow[d, "(*)"] & 0 \arrow[d] \\
\cdots \arrow[r] & \mathcal{C}^X_{2d-n+1}(\mathbb{Z}(n)) \oplus {}'\mathcal{D}_X^{n-2} \arrow[r] & \mathcal{C}^X_{2d-n}(\mathbb{Z}(n)) \oplus F^n{}'\mathcal{D}_X^n \oplus {}'\mathcal{D}_X^{n-1} \arrow[r] & \mathcal{C}^X_{2d-n+1}(\mathbb{Z}(n)) \oplus F^n{}'\mathcal{D}_X^{n+1} \oplus {}'\mathcal{D}_X^n \arrow[r] & \cdots
\end{tikzcd}$$

$$(\mathcal{C}, S) \longmapsto (-\partial\mathcal{C}, 0, \mathrm{d}[S] + \mathcal{C})$$

$$(\mathcal{C}, \Omega, R) \longmapsto (-\partial\mathcal{C}, -\mathrm{d}[\Omega], \mathrm{d}[R] - \Omega + \mathcal{C})$$

where we have left out the $(2\pi i)^n$'s multiplying Δ and the $\mathcal{C}$'s, and (in order that the diagram commute) $(*)$ is given by sending

$$\omega \mapsto (0,\, \mathrm{d}\omega,\, \omega).$$

(Apart from that, $\mathbb{Z}(n) \hookrightarrow \mathcal{C}^X_{2d}(\mathbb{Z}(n))$ while $\mathcal{O}_X \hookrightarrow {}'\mathcal{D}^0_X$, $\Omega^1_X \hookrightarrow {}'\mathcal{D}^1_X$, and so on.) We now have a composite

$$\otimes^n \mathbb{Z}[\mathbb{P}^1_{\mathbb{C}(X)} \setminus \{0, \infty\}] \longrightarrow \mathbb{H}^n(\eta_X, \mathbb{Z}(n)^\bullet_{\mathcal{D},X}) \xrightarrow{\cong} \mathbb{H}^n(\eta_X, \mathcal{C}(n)^\bullet_{\mathcal{D},X})$$

$$\begin{array}{ccccc}
\mathbf{f} = f_1 \otimes \ldots \otimes f_n & \longmapsto & \theta(f_1) \cup \ldots \cup \theta(f_n) & \longmapsto & D\text{-cycle in } {}'E_0^{p,q}(n) \\
 & & D\text{-cycle in } E_0^{*,*}(n) & & := \check{C}^p(\eta_X, \mathcal{C}(n)^q_{\mathcal{D},X})
\end{array}$$

computed by a D-cycle in a new double complex, namely

$$\left\{(0, \Omega^n_{\mathbf{f}}, S^n_{\mathbf{f}}),\, (0, S^{n-1}_{\mathbf{f}}), \ldots,\, (0, S^1_{\mathbf{f}})\,,\, S^0_{\mathbf{f}}\right\} :=$$

$$\{(0\,,\, \mathrm{dlog} f_1 \wedge \ldots \wedge \mathrm{dlog} f_n\,,\, \mathrm{Log} f_1 \mathrm{dlog} f_2 \wedge \ldots \wedge \mathrm{dlog} f_n)\ ,$$

$$(0\,,\, \Delta\mathrm{Log} f_1 \mathrm{Log} f_2 \mathrm{dlog} f_3 \wedge \ldots \wedge \mathrm{dlog} f_n)\ ,\ \ldots\ ,\ \Delta\mathrm{Log} f_1 \cdot \ldots \cdot \Delta\mathrm{Log} f_n\}$$

$$\in \left\{{}'E_0^{0,n}(n), {}'E_0^{1,n-1}(n), \ldots, {}'E_0^{n,0}(n)\right\},$$

where we note the peculiarity that $S^0_{\mathbf{f}} \in \check{C}^n(\eta_X, \mathcal{C}^X_{2d}(\mathbb{Z}(n)))$ while the remaining $S^i_{\mathbf{f}} \in \check{C}^{n-i}(\eta_X, {}'\mathcal{D}_X^{i-1})$. We compose with one more (isomorphic) map

$$\mathbb{H}^n(\eta_X, \mathcal{C}(n)^\bullet_{\mathcal{D},X}) \to H^n(\mathcal{C}(n)^\bullet_{\mathcal{D},X}(\eta_X))\, [\cong H^n_{\mathcal{D}}(\eta_X, \mathbb{Z}(n))]$$

by using acyclicity of $\mathcal{C}(n)^q_{\mathcal{D},X}$ to pull the whole D-cocycle (by adding D-coboundaries) up to a $\mathrm{d}_{\mathrm{Cone}}$- and δ_{Cech}- closed element of ${}'E_0^{0,n}(n)$. (This is abstractly possible, by an easy diagram chase.)

The picture we would like to have is the following:

$$
\begin{tikzcd}
(0,\Omega^n_{\mathbf{f}},S^n_{\mathbf{f}}) & & & & & \\
(\mathcal{C}^{n-1}_{\mathbf{f}},Q^{n-1}_{\mathbf{f}}) \arrow[r, "(-1)^{n-1}\delta"'] & (0,S^{n-1}_{\mathbf{f}}) & & & & \\
& (\mathcal{C}^{n-2}_{\mathbf{f}},Q^{n-2}_{\mathbf{f}}) \arrow[u] \arrow[r, "(-1)^{n-2}\delta"'] & \ddots & & & \\
& & (\mathcal{C}^{2}_{\mathbf{f}},Q^{2}_{\mathbf{f}}) \arrow[u] \arrow[r, "(-1)^{2}\delta"'] & (0,S^{2}_{\mathbf{f}}) & & \\
& & & (\mathcal{C}^{1}_{\mathbf{f}},Q^{1}_{\mathbf{f}}) \arrow[u, "{(-\partial\mathcal{C},\mathrm{d}[Q]+\mathcal{C})}"'] \arrow[r, "-\delta"'] & (0,S^{1}_{\mathbf{f}}) & \\
& & & & \mathcal{C}^{0}_{\mathbf{f}} \arrow[u, "{(-\partial\mathcal{C},\mathcal{C})}"'] \arrow[r, "\delta"'] & S^{0}_{\mathbf{f}}
\end{tikzcd}
$$

where all arrows but the dotted one represent two quantities summing to give another (or $\delta\mathcal{C}^0_{\mathbf{f}} = \pm S^0_{\mathbf{f}}$). The "answer" (after subtracting the D-coboundary from the D-cocycle S, so that all other entries are zero) is then just

$$\left(\partial\mathcal{C}^{n-1}_{\mathbf{f}}\,,\,\Omega_{\mathbf{f}}\,,\,S^n_{\mathbf{f}}-\mathrm{d}[Q^{n-1}_{\mathbf{f}}]-\mathcal{C}^{n-1}_{\mathbf{f}}\right)\;\in\;{}'E_0^{0,n}(n)$$

in the upper left-hand entry, and it had better equal $((2\pi i)^n T_{\mathbf{f}}\,,\,\Omega_{\mathbf{f}}\,,\,R_{\mathbf{f}})$ in order to agree with $\mathcal{R}$. More precisely we want

$$\mathcal{C}^i_{\mathbf{f}}\in\check{C}^{n-i-1}\left(\eta_X,\mathcal{C}^X_{2d-i}(\mathbb{Z}(n))\right)\,,\quad Q^i_{\mathbf{f}}\in\check{C}^{n-i-1}\left(\eta_X,{}'\mathcal{D}^{i-1}_X\right),$$

$$\text{i.e.}\qquad\left(\mathcal{C}^i_{\mathbf{f}}\,,\,Q^i_{\mathbf{f}}\right)\in{}'E_0^{n-i-1,i}(n)\,,$$

such that:

$$(-1)^n\delta\mathcal{C}^0_{\mathbf{f}}\;=\;S^0_{\mathbf{f}}\,,\;\;\mathcal{C}^0_{\mathbf{f}}+(-1)^{n-1}\delta Q^1_{\mathbf{f}}\;=\;S^1_{\mathbf{f}}\,,$$

$$-\partial\mathcal{C}^{i-1}_{\mathbf{f}}+(-1)^{n-i}\delta\mathcal{C}^i_{\mathbf{f}}\;=\;0\qquad\text{for }\;1\leq i\leq n-1\,,$$

$$\mathrm{d}[Q^{i-1}_{\mathbf{f}}]+\mathcal{C}^{i-1}_{\mathbf{f}}+(-1)^{n-i}\delta Q^i_{\mathbf{f}}\;=\;S^i_{\mathbf{f}}\qquad\text{for }\;2\leq i\leq n-1\,,$$

$$\partial\mathcal{C}^{n-1}_{\mathbf{f}}\;=\;(2\pi i)^n T_{\mathbf{f}}\,,\;\;S^n_{\mathbf{f}}-\mathrm{d}[Q^{n-1}_{\mathbf{f}}]-\mathcal{C}^{n-1}_{\mathbf{f}}\;=\;R_{\mathbf{f}}.$$

Very explicitly now, the terms of the D-cocycle which we are modifying by a coboundary are

$$S^i_{\mathbf{f}}\;=\;\Delta\mathrm{Log}f_1\cdot\ldots\cdot\Delta\mathrm{Log}f_{n-i}\mathrm{Log}f_{n-i+1}\mathrm{dlog}f_{n-i+2}\wedge\ldots\wedge\mathrm{dlog}f_n.$$

The fact that the $\{S^i_{\mathbf{f}}\}$ comprise a double-cocycle is expressed by the relations

$$0 = \mathrm{d}[S^{i-1}_{\mathbf{f}}] + (-1)^{n-i}\delta S^i_{\mathbf{f}} \quad \text{and} \quad \partial S^0_{\mathbf{f}} = 0.$$

Also, recall the notation

$$R(f) = \log f\,, \quad R(f,g) = \log f \operatorname{dlog} g - 2\pi i \log g \cdot \delta_{T_f}\,, \quad \text{etc.;}$$

the properties of these currents we shall use are that

$$\mathrm{d}[R(f_1,\ldots,f_m)] = \Omega(f_1,\ldots,f_m) - (2\pi i)^m \delta_{T_{f_1}\cap\ldots\cap T_{f_m}}$$

and (somewhat less obviously)

$$R(f_1,\ldots,f_m) \;=\; \log f_1 \operatorname{dlog} f_2 \wedge \ldots \wedge \operatorname{dlog} f_m - 2\pi i \delta_{T_{f_1}} \cdot R(f_2,\ldots,f_m).$$

We find that

$$\boxed{\begin{array}{c} \mathcal{C}^i_{\mathbf{f}} := (2\pi\sqrt{-1})^i \Delta\mathrm{Log} f_1 \cdot \ldots \cdot \Delta\mathrm{Log} f_{n-i-1} \mathbb{D}\mathrm{Log} f_{n-i} \cdot \delta_{T_{f_{n-i+1}}\cap\ldots\cap T_{f_n}} \\ Q^i_{\mathbf{f}} := \Delta\mathrm{Log} f_1 \cdot \ldots \cdot \Delta\mathrm{Log} f_{n-i-1} \mathbb{D}\mathrm{Log} f_{n-i} \cdot R(f_{n-i+1},\ldots,f_n) \end{array}}$$

do the job. Absolutely crucial here is the fact that the $\mathcal{C}^i_{\mathbf{f}}$ as we have defined them *are* real analytic chains with $\mathbb{Z}(n)$-coefficients (and $\mathbb{D}\mathrm{Log}$ is important in this connection). We state our conclusion:

PROPOSITION 3.3.2. *There is a well-defined homomorphism of sheaves* $\underline{K^M_{n,X}} \to \underline{\mathcal{H}^n_{\mathcal{D}}(n)}$. *On* η_X *it induces a map*

$$K^M_n(\mathbb{C}(X)) \cong H^0(\eta_X, \underline{K^M_{n,X}}) \longrightarrow H^0(\eta_X, \underline{\mathcal{H}^n_{\mathcal{D}}(n)}) \cong H^n_{\mathcal{D}}(\eta_X, \mathbb{Z}(n))$$

which coincides with that induced by the Milnor regulator currents, namely the assignment $\boldsymbol{f} \mapsto ((2\pi i)^n T_{\boldsymbol{f}}, \Omega_{\boldsymbol{f}}, R_{\boldsymbol{f}})$.

3.3.3. A note on product structure. If

$$\mathbf{f} = \sum_\alpha k_\alpha f_{1\alpha} \otimes \ldots \otimes f_{n\alpha} \;\in\; \otimes^n \mathbb{Z}[\mathbb{P}^1_{\mathbb{C}(X)} \setminus \{0,\infty\},$$

$$\mathbf{g} = \sum_\beta \ell_\beta g_{1\beta} \otimes \ldots \otimes g_{m\beta} \;\in\; \otimes^m \mathbb{Z}[\mathbb{P}^1_{\mathbb{C}(X)} \setminus \{0,\infty\}]$$

set

$$\mathbf{f} \otimes \mathbf{g} := \sum_{\alpha,\beta} k_\alpha \ell_\beta f_{1\alpha} \otimes \ldots \otimes f_{n\alpha} \otimes g_{1\beta} \otimes \ldots \otimes g_{m\beta} \;\in\; \otimes^{n+m} \mathbb{Z}[\mathbb{P}^1_{\mathbb{C}(X)} \setminus \{0,\infty\}].$$

This induces a "tensor product"

$$K^M_n(\mathbb{C}(X)) \otimes K^M_m(\mathbb{C}(X)) \to K^M_{n+m}(\mathbb{C}(X))$$

on Milnor K-theory by sending

$$\{\mathbf{f}\} \otimes \{\mathbf{g}\} \mapsto \{\mathbf{f} \otimes \mathbf{g}\}.$$

It is obviously well-defined; if $\mathbf{f}$ or $\mathbf{g}$ is a Steinberg then so is $\mathbf{f} \otimes \mathbf{g}$.

By construction, the sheaf homomorphism is compatible with the product structure on Zariski stalks; therefore the map in the Proposition sends products to products. Namely, we have (as a corollary of the Proposition) that

$$((2\pi i)^n T_{\mathbf{f}}, \Omega_{\mathbf{f}}, R_{\mathbf{f}}) \cup ((2\pi i)^m T_{\mathbf{g}}, \Omega_{\mathbf{g}}, R_{\mathbf{g}}) = \left((2\pi i)^{n+m} T_{\mathbf{f}\otimes\mathbf{g}}, \Omega_{\mathbf{f}\otimes\mathbf{g}}, R_{\mathbf{f}\otimes\mathbf{g}}\right)$$

in $H^{n+m}_{\mathcal{D}}(\eta_X, \mathbb{Z}(n+m))$. We expect similar product formulas hold for all the maps of §2.4.

CHAPTER 4

Rigidity and Vanishing

4.1. Gauss-Manin Connection (How to Differentiate Periods)

4.1.1. Why a "Vanishing" Result? Let E_t be a family of smooth elliptic curves parametrized by a Zariski-open subset of $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$. Collino **[C1]** has demonstrated the existence[1] of a "continuous family" of holomorphic symbols

$$\{\mathbf{f}_t\} = \prod_\alpha \{f_\alpha(t), g_\alpha(t)\}^{m_\alpha} \in \ker(\mathrm{Tame}) = K_2^M(E_t) \subseteq K_2^{[M]}(\mathbb{C}(E_t))$$

with nontorsion regulator image

$$[R_{\mathbf{f}_t}] \in \mathrm{im}\left\{H^1(E_t, \mathbb{C}/\mathbb{Z}(2)) \to H^1(\eta_{E_t}, \mathbb{C}/\mathbb{Z}(2))\right\} \cong H^1(E_t, \mathbb{C}/\mathbb{Z}(2))$$

for general t, as well as nontrivial infinitesimal invariant. The family of symbols is constructed geometrically,[2] by starting with a family of hyperelliptic curves C_t of genus 3, which produce Ceresa cycles $C_t^+ - C_t^-$ in the corresponding (hyperelliptic) Jacobians; degenerating these Jacobians twice then yields (from the Ceresa cycles) the desired family of symbols.

This is very much in the same (geometric) spirit as the work of Müller-Stach, Lewis, and Collino ([**MS1**], [**AM**], [**C1**], [**C2**], [**GL**], etc.) on constructing indecomposable elements of $CH^2(X,1)$ for X a general $K3$ surface.[3] On the other hand, Collino's construction is the only case of an interesting holomorphic Milnor-regulator image on a general *curve*: if we replace $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$ by $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(D \neq 3))$ (Collino, [**C1**]) or by the universal family of curves of genus g (Griffiths-Green, [**GG1**]), such an $\{\mathbf{f}_t\}$ as above cannot exist. So why not try higher dimension?

The aim of the project represented in this chapter, was to start with a family X_t of smooth complete intersections parametrized by a Zariski open subset of $H^0(\mathbb{P}^{n+r}, \mathcal{O}_{\mathbb{P}}(D_0) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}}(D_k))$, and produce *geometrically* a family of holomorphic symbols

$$\{\mathbf{f}_t\} \in K_n^M(X_t) \subseteq K_n^M(\mathbb{C}(X_t)) \quad (\implies \Omega_{\mathbf{f}_t} = 0)$$

with interesting regulator image

[1](really on a cover of this family)

[2]as opposed to arithmetically for special t

[3]The above amounts to an interesting element of $CH^2(E,2)$ for general E; the work on $CH^2(X,1)$ also involves computing regulator images.

$$[R_{\mathbf{f}_t}] \in \mathrm{im}\{H^{n-1}(X_t, \mathbb{C}/\mathbb{Q}(n)) \to H^{n-1}(\eta_X, \mathbb{C}/\mathbb{Q}(n))\} \tag{4.1.1}$$

for general t.[4] We always take $\dim X_t = n-1$ in order that $H^{n-1}(X_t)$ be interesting; it follows that the target space (4.1.1)[$\cong H^{n-1}_{pr}(X_t)$ for very general t] is too.

Instead we wound up proving that this situation is impossible *unless* $n = 2$ *and* X_t *is an elliptic curve*. Otherwise, nontrivial images (in $H^{n-1}(\eta_X, \mathbb{C}/\mathbb{Q}(n))$) are possible (for general t) only for symbols with nontrivial residue, i.e. not in $K_n^M(X_t)$. For $n \geq 4$, $\{\mathbf{f}_t\} \in \ker(\mathrm{Tame})$ is a less restrictive assumption but already "rigidifies" the image, i.e. makes it flat with respect to the integral structure. (See the Theorem at the end of §4.5 for a formal statement of the results.) Nontrivial "geometric" images – i.e. the cohomology classes – probably always have residues (for $X \neq E$).

The main idea behind the proof is a computation of the "infinitesimal invariant" (of $[R_{\mathbf{f}_t}]$), which is a sort of derivative of a family of cohomology classes. This boils down to the differentiation of their periods on $(n-1)$-cycles (on X_t), since these cycles give a "rigid" integral or rational structure that one can differentiate against. So in the following we establish precisely, the sense in which such a derivation (or "Gauss-Manin" connection) is possible. Much of what follows (in this section) owes something to Chapter 3 of [**Gr3**].

4.1.2. Setup and Notation. Gauss-Manin globally (from Leray spectral sequence). Let Y be smooth projective and E [the sheaf of sections of] a vector bundle on Y; the families we study will always be of the form $\mathcal{X} \to \mathcal{S}$, $\mathcal{X}$ and $\mathcal{S}$ smooth quasi-projective and π smooth, where $\mathcal{S}$ is Zariski open in [a cover of] a complete linear system $\bar{\mathcal{S}} = \mathbb{P}H^0(Y, E)$. All $X_s := \pi^{-1}(s \in \mathcal{S})$ are smooth projective.

Let $0 \in \mathcal{S}$ be a base point and $D \subset \mathcal{S}$ be a 1-parameter[5] disk containing 0 (and parametrized by t). We fix a homeomorphism

$$F : X_0 \times D \to \pi^{-1}(D) = \mathcal{X}_D$$

auch that the restriction to $X_0 \times \{t\}$ maps it to X_t; this induces a map

$$F_t^* : H^*(X_t, \mathbb{C}) \to H^*(X_0, \mathbb{C}) \qquad (F_0^* = id)$$

by pulling back forms, which does not respect Hodge type. However this gives a way of recording the periods we want to differentiate in a fixed space.[6] In a diagram:

[4] or equivalently to produce an interesting regulator image on X_t, t very general.

[5] ($\mathcal{S}$ is *not* 1-dimensional)

[6] One can also see F_t^* as recording the (varying) Hodge structures on $H^*(X_0, \mathbb{C})$ as a vector space.

$$\begin{tikzcd}
\bar{\mathcal{X}} \arrow[d] & \mathcal{X} \arrow[l, hook'] \arrow[d, "\pi"] & \pi^{-1}(D) \arrow[l, hook'] \arrow[d] & X_0 \times D \arrow[l, "F"'] \arrow[dl, "p"] \\
\bar{\mathcal{S}} & \mathcal{S} \arrow[l, hook'] & D \arrow[l, hook'] &
\end{tikzcd}$$

A choice of F (= choice of local product structure) carries with it a choice of horizontal vectors $F_*(\partial/\partial t) =: \widetilde{\partial/\partial t}$ and vertical forms[7] (annihilating these) on $\mathcal{X}_D$. Gauss-Manin (and Kodaira-Spencer) is independent of this choice, so F will not become relevant until we seek to realize explicitly the abstract/invariant version we now start with.

On all of $\mathcal{X}$ (without F) we can define vertical forms *as a quotient*, via

$$0 \to \pi^*\Omega^1_{\mathcal{S}} \to \Omega^1_{\mathcal{X}} \to \Omega^1_{\mathcal{X}/\mathcal{S}} \to 0, \qquad\qquad \Omega^\ell_{\mathcal{X}/\mathcal{S}} := \bigwedge^\ell \Omega^1_{\mathcal{X}/\mathcal{S}},$$

or

$$0 \to \mathrm{im}\{\pi^*\Omega^1_{\mathcal{S}} \otimes \Omega^{\ell-1}_{\mathcal{X}} \to \Omega^\ell_{\mathcal{X}}\} \to \Omega^\ell_{\mathcal{X}} \to \Omega^\ell_{\mathcal{X}/\mathcal{S}} \to 0. \tag{4.1.2}$$

Consider a *Leray* filtration on the complex $\Omega^\bullet_{\mathcal{X}}$ by

$$\mathcal{L}^p\Omega^\bullet_{\mathcal{X}} := \mathrm{im}\left\{\pi^*\Omega^p_{\mathcal{S}} \otimes \Omega^{\bullet-p}_{\mathcal{X}} \to \Omega^\bullet_{\mathcal{X}}\right\} \qquad \to \qquad Gr^p_{\mathcal{L}}\Omega^\bullet_{\mathcal{X}} = \pi^*\Omega^p_{\mathcal{S}} \otimes \Omega^{\bullet-p}_{\mathcal{X}/\mathcal{S}},$$

which gives rise to a spectral sequence

$$E_1^{p,q} = \mathbb{R}^{p+q}_{\pi_*}(Gr^p_{\mathcal{L}}\Omega^\bullet_{\mathcal{X}}) = \Omega^p_{\mathcal{S}} \otimes \mathbb{R}^{p+q}_{\pi_*}\Omega^{\bullet-p}_{\mathcal{X}/\mathcal{S}} = \Omega^p_{\mathcal{S}} \otimes \mathbb{R}^q_{\pi_*}\Omega^\bullet_{\mathcal{X}/\mathcal{S}}$$

computing $E_\infty^{p,q} = Gr^p_{\mathcal{L}}\mathbb{R}^{p+q}_{\pi_*}\Omega^\bullet_{\mathcal{X}}$. We shall sometimes write $\mathcal{H}^q_{X_s}$ for $\mathbb{R}^q_{\pi_*}\Omega^\bullet_{\mathcal{X}/\mathcal{S}} = R^q_{\pi_*}\mathbb{C} \otimes \mathcal{O}_{\mathcal{S}}$, $\mathcal{F}^k\mathcal{H}^q_{X_s}$ for $\mathbb{R}^q_{\pi_*}\Omega^{\bullet \geq k}_{\mathcal{X}/\mathcal{S}}$, $\mathcal{H}^{k,q-k}_{X_s}$ for $\mathbb{R}^q_{\pi_*}\Omega^k_{\mathcal{X}/\mathcal{S}}[-k] = R^{q-k}_{\pi_*}\Omega^k_{\mathcal{X}/\mathcal{S}}$; though in the latter case one must often specify whether one is thinking of $\mathcal{H}^{k,q-k}_{X_s}$ as a subspace of $\mathcal{H}^q_{X_s}$ or as the quotient $\mathcal{F}^k/\mathcal{F}^{k-1}$.

Pasting together the long exact sequences arising from

$$\mathcal{L}^{p+1} \to \mathcal{L}^p \to Gr^p_{\mathcal{L}}\ , \qquad \mathcal{L}^{p+2} \to \mathcal{L}^{p+1} \to Gr^{p+1}_{\mathcal{L}}\ , \qquad \text{etc.}$$

we have from

$$\begin{tikzcd}
\to \mathbb{R}^{p+q}_{\pi_*}Gr^p_{\mathcal{L}}\Omega^\bullet_{\mathcal{X}} \arrow[r, "\delta"] \arrow[drr, bend right, "d_1 =: \nabla"] & \mathbb{R}^{p+q+1}_{\pi_*}\mathcal{L}^{p+1}\Omega^\bullet_{\mathcal{X}} \arrow[r] \arrow[d, equal] & \mathbb{R}^{p+q+1}_{\pi_*}\mathcal{L}^{p}\Omega^\bullet_{\mathcal{X}} & & \\
& \mathbb{R}^{p+q+1}_{\pi_*}\mathcal{L}^{p+1}\Omega^\bullet_{\mathcal{X}} \arrow[r] & \mathbb{R}^{p+q+1}_{\pi_*}Gr^{p+1}_{\mathcal{L}}\Omega^\bullet_{\mathcal{X}} \arrow[r, "\delta"] \arrow[drr, bend right, "\nabla"] & \mathbb{R}^{p+q+2}_{\pi_*}\mathcal{L}^{p+2}\Omega^\bullet_{\mathcal{X}} \arrow[d, equal] & \\
& & & \mathbb{R}^{p+q+2}_{\pi_*}\mathcal{L}^{p+2}\Omega^\bullet_{\mathcal{X}} \arrow[r] & {}
\end{tikzcd}$$

[7](vertical vectors and horizontal forms on $\mathcal{X}$ were well-defined to begin with)

our first definition of the Gauss-Manin connection

$$\nabla : \Omega^p_{\mathcal{S}} \otimes \mathcal{H}^q_{X_s} \to \Omega^{p+1}_{\mathcal{S}} \otimes \mathcal{H}^q_{X_s}$$

in such a way that $\nabla \circ \nabla = 0$ is obvious.

In the same way, a Leray filtration on $\Omega^{\bullet \geq k+p}_{\mathcal{X}}$ (instead of $\Omega^\bullet_{\mathcal{X}}$) leads to Hodge-filtered pieces of Gauss-Manin

$$\nabla : \Omega^p_{\mathcal{S}} \otimes \mathcal{F}^k \mathcal{H}^q_{X_s} \to \Omega^{p+1}_{\mathcal{S}} \otimes \mathcal{F}^{k-1} \mathcal{H}^q_{X_s}$$

while taking instead $\Omega^{k+p}_{\mathcal{X}}[-(k+p)]$ leads to maps

$$\nabla^{(0,1)} : \Omega^p_{\mathcal{S}} \otimes \mathcal{H}^{k,q-k}_{X_s} \to \Omega^{p+1}_{\mathcal{S}} \otimes \mathcal{H}^{k-1,q-k+1}_{X_s}$$

which may be viewed as *graded pieces*

$$\bar{\nabla} : \Omega^p_{\mathcal{S}} \otimes Gr^k_{\mathcal{F}} \mathcal{H}^q_{X_s} \to \Omega^{p+1}_{\mathcal{S}} \otimes Gr^{k-1}_{\mathcal{F}} \mathcal{H}^q_{X_s}$$

of ∇. In particular $\bar{\nabla} \circ \bar{\nabla} = 0$ is again immediate.

REMARK 4.1.1. We emphasize that ∇ itself does *not* take a section of $\Omega^p_{\mathcal{S}} \otimes \mathcal{H}^{k,q-k}_{X_s} \subseteq \Omega^p_{\mathcal{S}} \otimes \mathcal{H}^q_{X_s}$ to $\Omega^{p+1}_{\mathcal{S}} \otimes \mathcal{H}^{k-1,q-k+1}_{X_s} \subseteq \Omega^{p+1}_{\mathcal{S}} \otimes \mathcal{H}^q_{X_s}$! The map "$\nabla^{(0,1)}$" is to ∇ what $\bar{\partial}$ is to d on C^∞ differential forms; they coincide on Hodge-graded pieces (i.e., in a quotient).

4.1.3. Gauss-Manin locally (with 1 parameter). In §4.1.4 we will "compute" ∇ explicitly (for $p = 0$) on D, where (4.1.2) reduces to[8]

$$0 \to \pi^* \Omega^1_D \otimes \Omega^{\ell-1}_{\mathcal{X}/D} \to \Omega^\ell_{\mathcal{X}} \to \Omega^\ell_{\mathcal{X}/D} \to 0 \ ,$$

or

$$0 \to \pi^* \Omega^1_D \otimes \Omega^\bullet_{\mathcal{X}/D}[-1] \to \Omega^\bullet_{\mathcal{X}} \to \Omega^\bullet_{\mathcal{X}/D} \to 0. \tag{4.1.3}$$

This gives a long-exact sequence with connecting homomorphism

$$\mathbb{R}^q_{\pi_*} \Omega^\bullet_{\mathcal{X}/D} \to \mathbb{R}^{q+1}_{\pi_*} \left(\pi^* \Omega^1_D \otimes \Omega^\bullet_{\mathcal{X}/D}[-1] \right) = \Omega^1_D \otimes \mathbb{R}^q_{\pi_*} \Omega^\bullet_{\mathcal{X}/D}$$

which is of course $\nabla : \mathcal{H}^q_{X_t} \to \Omega^1_D \otimes \mathcal{H}^q_{X_t}$; we will show this differentiates the periods. For $\nu_t \in H^0(D, \mathcal{H}^q_{X_t})$ we write $\nabla \nu_t =: \mathrm{d}t \otimes \nabla_{\partial/\partial t} \nu_t$; replacing D by an open ball in $\mathcal{S}$ and t by $\{t_i\}_{i=1}^{\dim \mathcal{S}}$ would just give

$$\nabla \nu_{\mathbf{t}} = \sum_i \mathrm{d}t_i \otimes \nabla_{\partial/\partial t_i} \nu_{\mathbf{t}}.$$

Briefly, the filtered and graded pieces go as follows:

$$0 \to \pi^* \Omega^1_D \otimes \Omega^{\bullet \geq \ell-1}_{\mathcal{X}/D}[-1] \to \Omega^{\bullet \geq \ell}_{\mathcal{X}} \to \Omega^{\bullet \geq \ell}_{\mathcal{X}/D} \to 0$$

induces

$$\nabla : \mathbb{R}^q_{\pi_*} \Omega^{\bullet \geq \ell}_{\mathcal{X}/D} \to \Omega^1_D \otimes \mathbb{R}^q_{\pi_*} \Omega^{\bullet \geq \ell-1}_{\mathcal{X}/D} \ ;$$

and

$$0 \to \pi^* \Omega^1_D \otimes \Omega^{\ell-1}_{\mathcal{X}/D}[-\ell] \to \Omega^\ell_{\mathcal{X}}[-\ell] \to \Omega^\ell_{\mathcal{X}/D}[-\ell] \to 0 \tag{4.1.4}$$

[8] where we continue writing $\mathcal{X}$ for $\mathcal{X}_D$ (and F, $X_0 \times D$ do not yet enter)

gives rise to

$$\bar{\nabla} : Gr^{\ell}_{\mathcal{F}}\mathcal{H}^q_{X_t} \to \Omega^1_D \otimes Gr^{\ell-1}_{\mathcal{F}}\mathcal{H}^q_{X_t}$$

via the connecting homomorphism

$$\mathbb{R}^q_{\pi_*}\Omega^{\ell}_{\mathcal{X}/D}[-\ell] \to \Omega^1_D \otimes \mathbb{R}^{q+1}_{\pi_*}\Omega^{\ell-1}_{\mathcal{X}/D}[-\ell] = \Omega^1_D \otimes R^{q-\ell+1}_{\pi_*}\Omega^{\ell-1}_{\mathcal{X}/D}.$$

This connecting homomorphism must be given by cup-product with the extension class of (4.1.4), and so $\bar{\nabla}$ is just $\cup$ with

$$\epsilon \in R^1_{\pi_*}\mathrm{Hom}\left\{\Omega^{\ell}_{\mathcal{X}/D}, \pi^*\Omega^1_D \otimes \Omega^{\ell-1}_{\mathcal{X}/D}\right\} = \Omega^1_D \otimes R^1_{\pi_*}\mathrm{Hom}\left(\Omega^{\ell}_{\mathcal{X}/D}, \Omega^{\ell-1}_{\mathcal{X}/D}\right)$$

(the $[-\ell]$ shifts make no difference). On the other hand the Kodaira-Spencer class is the extension class

$$e \in R^1_{\pi_*}\mathrm{Hom}\left(\pi^*\theta^1_D,\, \theta^1_{\mathcal{X}/D}\right) = \Omega^1_D \otimes R^1_{\pi_*}\theta^1_{\mathcal{X}/D}$$

of

$$0 \to \theta^1_{\mathcal{X}/D} \to \theta^1_{\mathcal{X}} \to \pi^*\theta^1_D \to 0\,.$$

Dualizing this sequence and tensoring with $\Omega^{\ell-1}_{\mathcal{X}/D}$ (which doesn't change e) gives the bottom row in

$$\begin{tikzcd}
\pi^*\Omega^1_D \otimes \Omega^{\ell-1}_{\mathcal{X}/D} \arrow[r] \arrow[d, equal] & \Omega^{\ell}_{\mathcal{X}} \arrow[r] \arrow[d, "\omega \mapsto" left, "\iota^*_{\mathcal{X}/D} < \omega, \cdot >_{\mathcal{X}}" right] & \Omega^{\ell}_{\mathcal{X}/D} \arrow[d, "\omega \mapsto" left, "< \omega, \cdot >_{\mathcal{X}/D}" right] \\
\mathrm{Hom}(\pi^*\theta^1_D, \Omega^{\ell-1}_{\mathcal{X}/D}) \arrow[r] & \mathrm{Hom}(\theta^1_{\mathcal{X}}, \Omega^{\ell-1}_{\mathcal{X}/D}) \arrow[r] & \mathrm{Hom}(\theta^1_{\mathcal{X}/D}, \Omega^{\ell-1}_{\mathcal{X}/D}),
\end{tikzcd}$$

which shows that e naturally pulls up to ϵ, or "$\bar{\nabla} = \cup\bar{e}$". An immediate consequence is that (unlike ∇) $\bar{\nabla}$ is $\mathcal{O}_D$-linear;[9] we sometimes call it the $\mathcal{O}_D$-linear [graded] piece of ∇. That is , the value at $t = 0$ of a section $\bar{\nu}_t$ of $Gr^{\ell}_{\mathcal{F}}\mathcal{H}^q_{X_t}$ is enough to compute the value at $t = 0$ of $\bar{\nabla}_{\partial/\partial t}\bar{\nu}_t \in Gr^{\ell-1}_{\mathcal{F}}\mathcal{H}^q_{X_t}$. Of course one needs to know the local behavior of the family X_t (as encoded in $\bar{e}$), but not that of the section.[10] We'll pursue the computation of $\bar{\nabla}$ further in §4.2.

4.1.4. ∇ is computed by the Cartan formula. Returning to ∇ and (4.1.3), we now obtain the connecting homomorphism explicitly, on forms representing sections of the hyperderived sheaves. To this end we need to resolve the complexes of holomorphic sheaves by complexes of *acyclic* C^{∞}

[9]of course $\bar{\nabla}$ on $\mathcal{S}$ is $\mathcal{O}_{\mathcal{S}}$-linear

[10]though one should note that ν_t is implicitly restricted by having to be a (holomorphic) section of the holomorphic sheaf $R^{q-\ell}_{\pi_*}\Omega^{\ell}_{\mathcal{X}/D}$.

sheaves, e.g. $(\Omega^\bullet_{\mathcal{X}}, \partial) \hookrightarrow (\Omega^\bullet_{\mathcal{X}^\infty}, \mathrm{d})$ so that an abstract section of $\mathbb{R}^q_{\pi_*}\Omega^\bullet_{\mathcal{X}}(\cong \mathbb{R}^q_{\pi_*}\Omega^\bullet_{\mathcal{X}^\infty})$ over D can be represented by a form in (the numerator of)

$$H^0(D, \mathbb{R}^q_{\pi_*}\Omega^\bullet_{\mathcal{X}^\infty}) \cong \frac{\ker\left\{\mathrm{d} : \Gamma(\mathcal{X}_D, \Omega^q_{\mathcal{X}^\infty)}) \to \Gamma(\mathcal{X}_D, \Omega^{q+1}_{\mathcal{X}^\infty})\right\}}{\mathrm{im}\left\{\Gamma(\mathcal{X}_D, \Omega^{q-1}_{\mathcal{X}^\infty}) \to \Gamma(\mathcal{X}_D, \Omega^q_{\mathcal{X}^\infty})\right\}}.$$

However we run into the difficulty that (unlike $\Omega^\bullet_{\mathcal{X}^\infty} \simeq \Omega^\bullet_{\mathcal{X}} \simeq \mathbb{C}$) $\Omega^\bullet_{(\mathcal{X}/D)^\infty} \simeq \pi^*\mathcal{O}_{D^\infty} \otimes \mathbb{C}$ is not quasi-isomorphic to (and so does not resolve) $\Omega^\bullet_{\mathcal{X}/D} \simeq \pi^*\mathcal{O}_D \otimes \mathbb{C}$. Here is how to get around this: from

$$(*) \qquad \begin{tikzcd} \pi^*\Omega^1_D \otimes \Omega^{\bullet-1}_{\mathcal{X}/D} \arrow[r, "\cong"] \arrow[d] & \ker(\Omega^\bullet_{\mathcal{X}} \to \Omega^\bullet_{\mathcal{X}/D}) \arrow[r] \arrow[d] & \Omega^\bullet_{\mathcal{X}} \arrow[r] \arrow[d, "\simeq"'] & \Omega^\bullet_{\mathcal{X}/D} \arrow[d] & (\ \cdot\ , \partial) \\ \pi^*\Omega^1_{D^\infty} \otimes \Omega^{\bullet-1}_{(\mathcal{X}/D)^\infty} \arrow[r, "\simeq"] & \ker(\Omega^\bullet_{\mathcal{X}^\infty} \to \Omega^\bullet_{(\mathcal{X}/D)^\infty}) \arrow[r] & \Omega^\bullet_{\mathcal{X}^\infty} \arrow[r] & \Omega^\bullet_{(\mathcal{X}/D)^\infty} & (\ \cdot\ , \mathrm{d}) \end{tikzcd}$$

one has a commuting diagram

$$(**) \qquad \begin{tikzcd} \mathcal{O}_D \otimes R^q_{\pi_*}\mathbb{C} \cong \mathbb{R}^q_{\pi_*}\Omega^\bullet_{\mathcal{X}/D} \arrow[r] \arrow[d, hook] \arrow[rr, bend left] & \mathbb{R}^{q+1}_{\pi_*}\ker(\Omega^\bullet_{\mathcal{X}} \to \Omega^\bullet_{\mathcal{X}/D}) \arrow[d] & \Omega^1_D \otimes \mathbb{R}^q_{\pi_*}\Omega^\bullet_{\mathcal{X}/D} \cong \Omega^1_D \otimes R^q_{\pi_*}\mathbb{C} \arrow[l, "\cong"'] \arrow[d, hook] \\ \mathcal{O}_{D^\infty} \otimes R^q_{\pi_*}\mathbb{C} \cong \mathbb{R}^q_{\pi_*}\Omega^\bullet_{(\mathcal{X}/D)^\infty} \arrow[r, "\delta"] \arrow[rr, bend right, "\nabla^\infty"'] & \mathbb{R}^{q+1}_{\pi_*}\ker(\Omega^\bullet_{\mathcal{X}^\infty} \to \Omega^\bullet_{(\mathcal{X}/D)^\infty}) & \Omega^1_{D^\infty} \otimes \mathbb{R}^q_{\pi_*}\Omega^\bullet_{(\mathcal{X}/D)^\infty} \cong \Omega^1_{D^\infty} \otimes R^q_{\pi_*}\mathbb{C}. \arrow[l, "\cong"'] \end{tikzcd}$$

So for our purposes, given a section $\nu_t \in H^0(D, \mathcal{O}_D \otimes R^q_{\pi_*}\mathbb{C})$, it will suffice to compute $\nabla^\infty \nu_t$ and throw away the $\mathrm{d}\bar{t}$-part (to get $\nabla\nu_t$).

Recall that our choice of F induces a (local) notion of vertical forms and horizontal vectors (as subspaces rather than quotients); in particular we have a "preferred" $\widetilde{\partial/\partial t}$ on $\mathcal{X}_D$, and also for any section $\alpha \in H^0(\mathcal{X}_D, \Omega^q_{(\mathcal{X}/D)^\infty})$ a lift $\tilde{\alpha} \in \Omega^q_{\mathcal{X}^\infty}(\mathcal{X}_D)$ with $\left\langle \widetilde{\partial/\partial t}, \tilde{\alpha} \right\rangle = 0$. It is easy to see that on the image of $\mathbb{R}^q_{\pi_*}\Omega^\bullet_{\mathcal{X}/D}$, ∇^∞ is given by composing δ with the projection

$$\mathrm{d}t \otimes \iota^*_{\mathcal{X}/D} \left\langle \widetilde{\partial/\partial t}, \cdot \right\rangle_{\mathcal{X}}.$$

This will land in

$$\mathrm{im}\left\{\Omega^1_D \otimes \mathbb{R}^q_{\pi_*}\Omega^\bullet_{\mathcal{X}/D} \hookrightarrow \Omega^1_{D^\infty} \otimes \mathbb{R}^q_{\pi_*}\Omega^\bullet_{(\mathcal{X}/D)^\infty}\right\}$$

and yield a *representative* for $\nabla\nu_t$. To compute $\delta(\nu_t)$ we snake through the bottom row of $(*)$, first turning the image of ν_t in $\mathbb{R}^q_{\pi_*}\Omega^\bullet_{(\mathcal{X}/D)^\infty}$ into a $\mathrm{d}_{\mathcal{X}/D}$-closed $\mathbb{C}^\infty$ q-form α on $\mathcal{X}/D$ (or a family of q-forms α_t on X_t), then taking a "vertical" (non-closed) lift $\tilde{\alpha}$, then differentiating to get $\mathrm{d}\tilde{\alpha} \in \ker(\Omega^{q+1}_{\mathcal{X}^\infty} \to \Omega^{q+1}_{(\mathcal{X}/D)^\infty})$. Composing with the projection we have a formula which we write

$$\nabla\alpha_t = \mathrm{d}t \otimes \iota^*_{X_T} \left\langle \widetilde{\partial/\partial t}, \mathrm{d}\tilde{\alpha} \right\rangle_{\mathcal{X}}$$

in terms of the representative forms. On the other hand, if we let $\tilde{\alpha}'$ be *any* lift then we can "verticalize" it with respect to $\widetilde{\partial/\partial t}$ with

$$\tilde{\alpha} = \tilde{\alpha}' - (\pi^*\mathrm{d}t) \wedge \left\langle \widetilde{\partial/\partial t}, \tilde{\alpha}' \right\rangle \qquad (\implies \left\langle \widetilde{\partial/\partial t}, \tilde{\alpha} \right\rangle = 0)$$

so that

$$\left\langle \widetilde{\partial/\partial t}, \mathrm{d}\tilde{\alpha} \right\rangle = \left\langle \widetilde{\partial/\partial t} \, , \mathrm{d}\tilde{\alpha}' + (\pi^*\mathrm{d}t) \wedge \mathrm{d}\left\langle \widetilde{\partial/\partial t}, \tilde{\alpha}' \right\rangle \right\rangle$$

$$= \left\langle \widetilde{\partial/\partial t}, \mathrm{d}\tilde{\alpha}' \right\rangle + \mathrm{d}\left\langle \widetilde{\partial/\partial t}, \tilde{\alpha}' \right\rangle$$

which is the Cartan formula for the Lie derivative. This also shows that the cohomology class of $\iota^*_{X_t} \left\langle \widetilde{\partial/\partial t}, \mathrm{d}\tilde{\alpha}' \right\rangle$ is independent of the choice of lift of α. (Changing $\widetilde{\partial/\partial t}$ by a section of $\theta^1_{\mathcal{X}/D}$ wouldn't even change the form, since $\mathrm{d}\tilde{\alpha}' \in \ker(\Omega^{q+1}_{\mathcal{X}^\infty} \to \Omega^{q+1}_{(\mathcal{X}/D)^\infty})$.) Of course, for cohomology only the first term of Cartan is important.

4.1.5. The Cartan formula differentiates periods. Although our goal was merely to differentiate periods of α_t on a continuous family[11] of cycles $\{\mathcal{C}_t\}_{t\in D}$, with the choice of $(F, \widetilde{\partial/\partial t}, \tilde{\alpha})$ we can do "more". For an arbitrary q-chain $\mathcal{Q}$ on X_0 set $\tilde{\mathcal{Q}}_t := F_t(\mathcal{Q} \times \{t\})$ on $\mathcal{X}_D$ and on $X_0 \times D$, $\Gamma_\epsilon := \mathcal{Q} \times P_\epsilon$ for $P_\epsilon = [0, \epsilon]$ a (short) path in D so that $\partial\Gamma_\epsilon = \partial\mathcal{Q} \times P_\epsilon \pm (\mathcal{Q} \times \{\epsilon\} - \mathcal{Q} \times \{0\})$. Note that $\iota^*_{X_0 \times \{t\}} F^* \tilde{\alpha}^{(\prime)} = F_t^* \alpha_t$. We define a functional $(\mathcal{L}_{\widetilde{\partial/\partial t}} \alpha_t)_{t=0}$ on all such $\mathcal{Q}$, and compute a form representing it, by

$$\int_{\mathcal{Q}} (\mathcal{L}_{\widetilde{\partial/\partial t}} \alpha_t)_{t=0} := \left[\frac{\partial}{\partial t} \int_{\tilde{\mathcal{Q}}_t} \alpha_t \right]_{t=0} = \left[\frac{\partial}{\partial t} \int_{\mathcal{Q} \times \{t\}} F_t^* \alpha_t \right]$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[\int_{\mathcal{Q}} F_t^* \alpha_t - \int_{\mathcal{Q}} F_0^* \alpha_0 \right] = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[\int_{\mathcal{Q} \times \{\epsilon\}} F^* \tilde{\alpha}' - \int_{\mathcal{Q} \times \{0\}} F^* \tilde{\alpha}' \right]$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[\int_{\mathcal{Q} \times P_\epsilon} \mathrm{d}(F^* \tilde{\alpha}') + \int_{\partial\mathcal{Q} \times P_\epsilon} F^* \tilde{\alpha}' \right]$$

$$= \int_{\mathcal{Q}} \langle \partial/\partial t, \mathrm{d}(F^* \tilde{\alpha}') \rangle_{t=0} + \int_{\partial\mathcal{Q}} \langle \partial/\partial t, F^* \tilde{\alpha}' \rangle$$

[11] This might be given by $F_t(\mathcal{C}_0)$ for $\mathcal{C}_0$ a fixed cycle on X_0, but for (closed) cycles this is purely cosmetic; on the other hand "how" we vary a *non*-closed chain $\mathcal{Q}_0$ makes a difference.

$$= \int_{\mathcal{Q}} \left\{ \langle \partial/\partial t, \mathrm{d}(F^* \tilde{\alpha}') \rangle + \mathrm{d} \langle \partial/\partial t, F^* \tilde{\alpha}' \rangle \right\}$$

$$= \int_{\mathcal{Q}} \left\langle \widetilde{\partial/\partial t}, \mathrm{d}\tilde{\alpha}' \right\rangle + \mathrm{d} \left\langle \widetilde{\partial/\partial t}, \tilde{\alpha}' \right\rangle \;=\; \int_{\mathcal{Q}} \left\langle \widetilde{\partial/\partial t}, \mathrm{d}\tilde{\alpha} \right\rangle.$$

We state our conclusion that this version of the Lie derivative is given by the Cartan formula:

$$(\mathcal{L}_{\widetilde{\partial/\partial t}} \alpha_t)_{t=0} \;=\; \iota_{X_0}^* \left\langle \widetilde{\partial/\partial t}, \mathrm{d}\tilde{\alpha} \right\rangle \;=\; \iota_{X_0}^* \left\{ \left\langle \widetilde{\partial/\partial t}, \mathrm{d}\tilde{\alpha}' \right\rangle + \mathrm{d} \left\langle \widetilde{\partial/\partial t}, \tilde{\alpha}' \right\rangle \right\}$$

where as before $\tilde{\alpha}$ is an F-vertical lift and $\tilde{\alpha}'$ an arbitrary lift of α_t. Of course if we take $\mathcal{Q} = \mathcal{C}$ a (closed) q-cycle such distinctions don't matter, as $\int_{\mathcal{C}} \mathrm{d} < \widetilde{\partial/\partial t}, \tilde{\alpha}' >= 0$; combining the above computation with our prior unraveling of ∇,

$$\left[\frac{\partial}{\partial t} \int_{\mathcal{C}_t} \alpha_t \right]_{t=0} \;=\; \int_{\mathcal{C}} \iota_{X_0}^* \left\langle \widetilde{\partial/\partial t}, \mathrm{d}\tilde{\alpha}^{(\prime)} \right\rangle \;=\; \int_{\mathcal{C}} (\nabla_{\partial/\partial t} \alpha_t)_{t=0}$$

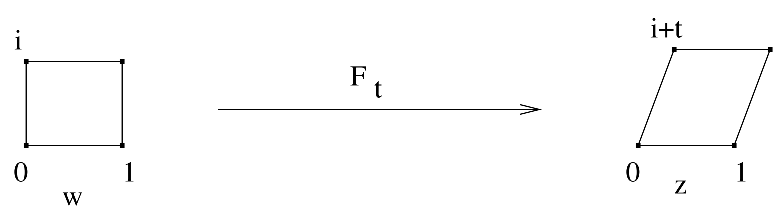
and so $\nabla_{\partial/\partial t}$ "differentiates the periods", i.e. yields a form with periods equal to the derivatives of α_t's periods in the $\partial/\partial t$-direction.

4.1.6. ∇ on a family of elliptic curves. Here is the easiest nontrivial example we could cook up. Let $\mathcal{X}_D$ be be the family

$$X_t \;=\; \mathbb{C}/\{\mathbb{Z} \oplus (t+i)\mathbb{Z}\} \qquad , \qquad t \in D[\ni 0] \subset \mathbb{C}$$

of elliptic curves; we use $z \in \mathbb{C}$ as coordinate on each X_t (of course identifying $z = 0$ with $z \in \mathbb{Z} + (t+i)\mathbb{Z}$) and take simply $\alpha_t = \mathrm{d}z$ (on each X_t) as [the representative of] our section of $\mathcal{H}^1_{X_t}$. Define

$$F_t : \, X_0 \times \{t\} \to X_t$$



by the "linear" map[12]

$$z(t, w) \;=\; \frac{1}{2} \{(2 - it)w \,+\, it\bar{w}\}$$

and pull the $\mathcal{X}/D$-forms α_t back to "$X_0 \times D/D$"-forms

$$F_t^* \alpha_t = F_t^* \mathrm{d}z = \frac{1}{2} \{(2 - it)dw \,+\, it\mathrm{d}\bar{w}\}$$

which can also be thought of (using the product structure) as a vertical form $\widetilde{F_t^* \alpha_t}$ on $X_0 \times D$. Then $(F^{-1})^* (\widetilde{F_t^* \alpha_t})$ is the desired "F-vertical" lift $\tilde{\alpha}$ of

[12]Of course this also defines $F : \, X_0 \times D \to \mathcal{X}_D$.

$\alpha_t = \mathrm{d}z$ [from $\mathcal{X}/D$] to $\mathcal{X}_D$ (simply taking $\mathrm{d}z$ on $\mathcal{X}_D$ isn't what we want[13]). So in fact the above is a formula for $F^*\tilde{\alpha}$, that is

$$F^*\tilde{\alpha} = \frac{1}{2}\{(2-it)\mathrm{d}w + it\mathrm{d}\bar{w}\}.$$

In order to compute $\nabla_{\partial/\partial t}\alpha_t = \iota^*_{X_t}\left\langle \widetilde{\partial/\partial t}, \mathrm{d}\tilde{\alpha}\right\rangle$ for the local family (not just at $t=0$) we write

$$\langle \partial/\partial t, \mathrm{d}F^*\tilde{\alpha}\rangle \;=\; F^*\left\langle \widetilde{\partial/\partial t}, \mathrm{d}\tilde{\alpha}\right\rangle$$

so that

$$\iota^*_{X_0\times\{t\}}\langle \partial/\partial t, \mathrm{d}F^*\tilde{\alpha}\rangle \;=\; \iota^*_{X_0\times\{t\}}F^*\left\langle \widetilde{\partial/\partial t}, \mathrm{d}\tilde{\alpha}\right\rangle \;=\; F_t^*\iota^*_{X_t}\left\langle \widetilde{\partial/\partial t}, \mathrm{d}\tilde{\alpha}\right\rangle$$

and

$$\iota^*_{X_t}\left\langle \widetilde{\partial/\partial t}, \mathrm{d}\tilde{\alpha}\right\rangle \;=\; (F_t^{-1})^*\iota^*_{X_0\times\{t\}}\langle \partial/\partial t, \mathrm{d}F^*\tilde{\alpha}\rangle$$

$$= (F_t^{-1})^*\;\iota^*_{X_0\times\{t\}}\left\langle \partial/\partial t, \mathrm{d}t\wedge\left(\frac{\mathrm{d}w-\mathrm{d}\bar{w}}{2i}\right)\right\rangle$$

$$= (F_t^{-1})^*\frac{1}{2i}(\mathrm{d}w - \mathrm{d}\bar{w}).$$

The inverse to $z(t,w)$ is

$$w(t,z) = \frac{(2+i\bar{t})z - it\bar{z}}{2+i\bar{t}-it} \qquad \longrightarrow \qquad \bar{w}(t,z) = \frac{(2-it)\bar{z}+i\bar{t}z}{2+i\bar{t}-it}$$

so

$$\frac{1}{2i}(w-\bar{w}) \;=\; \frac{z-\bar{z}}{2i+t-\bar{t}}$$

and

$$\iota^*_{X_t}\left\langle \widetilde{\partial/\partial t}, \mathrm{d}\tilde{\alpha}\right\rangle \;=\; \frac{\mathrm{d}z-\mathrm{d}\bar{z}}{2i+t-\bar{t}}.$$

If on X_t we let α be the path from $(z=)0$ to 1 and β be the path from 0 to $t+i$, the periods of $\mathrm{d}z$ are

$$\int_\alpha \mathrm{d}z \;=\; 1 \text{ (constant)} \qquad \text{and} \qquad \int_\beta \mathrm{d}z \;=\; t+i\;,$$

where the derivative of the latter by $\partial/\partial t$ is 1. And indeed

$$\int_\alpha \frac{\mathrm{d}z-\mathrm{d}\bar{z}}{2i+t-\bar{t}} \;=\; \frac{1-1}{2i+t-\bar{t}} = 0 \;\text{ while }\; \int_\beta \frac{\mathrm{d}z-\mathrm{d}\bar{z}}{2i+t-\bar{t}} \;=\; \frac{(t+i)-(\bar{t}-i)}{2i+t-\bar{t}} = 1$$

as promised.

Another phenomenon (mentioned abstractly at the beginning of the section) worth pointing out, is that (in this example) $\alpha_t \mapsto \nabla_{\partial/\partial t}\alpha_t$ drops from

[13] What we want is $\left\langle \widetilde{\partial/\partial t}, \tilde{\alpha}\right\rangle = 0$, of course; the actual expression is messy (we have worked around it here).

$\mathcal{F}^1$ to $\mathcal{F}^0$. In fact for $\nu_t \in \mathcal{F}^k \mathcal{H}^q_{X_t}$ we can always choose $\tilde{\alpha}$ in $\mathcal{F}^k \Omega^q_{\mathcal{X}^\infty}(\mathcal{X}_D)$ so that $\mathrm{d}\tilde{\alpha} \in \ker \left\{ \mathcal{F}^k \Omega^{q+1}_{\mathcal{X}^\infty} \to \Omega^{q+1}_{(\mathcal{X}/D)^\infty} \right\} (\mathcal{X}_D)$, this is essentially because $\Omega^{\bullet \geq k}_{\mathcal{X}}$ can be resolved by C^∞ (p,q)-forms with $p \geq k$. Any confusion as to how one can have $\tilde{\alpha} \in \mathcal{F}^k$, $\left\langle \widetilde{\partial/\partial t}, \tilde{\alpha} \right\rangle = 0$ and $\nabla_{\partial/\partial t} \alpha_t \notin \mathcal{F}^k$ can also be resolved, by this local "example": if $\widetilde{\partial/\partial t}$ looks like $\partial/\partial t - \bar{z}\partial/\partial z$ and $\tilde{\alpha}$ like $\mathrm{d}z + \bar{z}\mathrm{d}t$, then $< \widetilde{\partial/\partial t}, \tilde{\alpha} >= 0$, $\mathrm{d}\tilde{\alpha} = \mathrm{d}\bar{z} \wedge \mathrm{d}t$ and $\left\langle \widetilde{\partial/\partial t}, \mathrm{d}\bar{z} \wedge \mathrm{d}t \right\rangle = -\mathrm{d}\bar{z}$.

4.2. Cohomology of a Smooth Hypersurface $X \subset \mathbb{P}^n$

4.2.1. Meromorphic forms and homogeneous polynomials. We consider a smooth $X \subset \mathbb{P}^n$ $(n \geq 2, \dim X = n-1)$ cut out by $F \in S^D$ homogeneous of degree D. Let $\pi : U = \mathbb{C}^{n+1} \setminus \{0\} \twoheadrightarrow \mathbb{P}^n$ be the quotient by the action

$$t \cdot (z_0, \ldots, z_n) \;=\; (tz_0\,,\, \ldots\,,\, tz_n)$$

of $\mathbb{C}^*$; this is infinitesimally generated by

$$\vec{e} = \sum_{i=0}^n z_i \frac{\partial}{\partial z_i}.$$

Set

$\mathcal{A}^m_0 = H^0(\mathbb{C}^{n+1}, \Omega^m_{\mathbb{C}^{n+1}})$, $\quad \mathcal{A}^m =$ rational m-forms on $\mathbb{C}^{n+1}$,

$A^m_k = A^m_k(X) :=$ rational m-forms on $\mathbb{P}^n$ with poles of order $\leq k$ along X, (and no other poles)

$$A^m = A^m(X) := \bigcup_k A^m_k.$$

By Chow's theorem/GAGA, meromorphic forms are rational:

$$A^m_k = H^0(\Omega^m_{\mathbb{P}^n}(kX)), \qquad A^m = H^0(\Omega^m_{\mathbb{P}^n}(*X));$$

in particular, for $m = n$

$$A^n_k = H^0(K_{\mathbb{P}^n} \otimes \mathcal{O}(kD)) \cong S^{kD-n-1}.$$

We realize this $\cong$ as follows: if $\omega \in A^m_k$, then (by definition of a rational form) it lifts to

$$\pi^*\omega \;=:\; \Psi \;=\; \frac{1}{F^k} \sum_{I,\,|I|=m} A_I(\mathbf{z})\mathrm{d}\mathbf{z}_I \;\in\; \mathcal{A}^m$$

where $A_I(\mathbf{z})$ are polynomials. We characterize those $\Psi \in \mathcal{A}^m$ that descend to $\mathbb{P}^n$ (i.e. arise in this fashion). Let $i_{\vec{e}}(\cdot) = \langle \vec{e}, \cdot \rangle$ be interior product and for a *monomial* Ψ define its degree as an eigenvalue:

$$\deg(\Psi) \cdot \Psi := [\mathrm{d}, i_{\vec{e}}]\Psi := \langle \vec{e}, \mathrm{d}\Psi \rangle + \mathrm{d}\,\langle \vec{e}, \Psi \rangle\,.$$

For example, $\deg(\mathrm{d}z_{i_1} \wedge \ldots \wedge \mathrm{d}z_{i_k}) = k$; the definition extends to sums of monomials of the same degree. If $\deg(\Psi) = 0$ then Ψ is *invariant*; an invariant Ψ descends ($= \pi^*\omega$) iff it is *horizontal* with respect to π, i.e. $\langle \vec{e}, \Psi \rangle = 0$.

For $m = n$, set

$$\mathrm{d}\mathbf{z} = \mathrm{d}z_0 \wedge \ldots \wedge \mathrm{d}z_n \in \mathcal{A}_0^{n+1} \quad , \quad \Omega := \langle \vec{e}, \mathrm{d}\mathbf{z} \rangle \in \mathcal{A}_0^n$$

and compute

$$\deg(\Omega) \cdot \Omega = \mathrm{d} \langle \vec{e}, \langle \vec{e}, \mathrm{d}\mathbf{z} \rangle \rangle + \langle \vec{e}, \mathrm{d} \langle \vec{e}, \mathrm{d}\mathbf{z} \rangle \rangle$$

$$= 0 + \deg(\mathrm{d}\mathbf{z}) \langle \vec{e}, \mathrm{d}\mathbf{z} \rangle - \langle \vec{e}, \langle \vec{e}, \mathrm{d}(\mathrm{d}\mathbf{z}) \rangle \rangle = (n+1)\Omega.$$

One also finds $\deg(\alpha \wedge \beta) = \deg(\alpha) + \deg(\beta)$ (where α, β are forms or functions of well-defined degree), and so if $P \in S^a$ is a homogeneous polynomial them

$$\deg\left(\frac{P\Omega}{F^k}\right) = a - kD + (n+1) \quad \text{and} \quad \left\langle \vec{e}, \frac{P\Omega}{F^k} \right\rangle = \frac{P}{F^k} \langle \vec{e}, \Omega \rangle = 0$$

so that $(P\Omega / F^k) = \pi^*\omega$ if $a = kD - n - 1$.

In fact all $\Psi \in \mathcal{A}^n$ which descend to A_k^n take this form: that is, assuming $\Psi = \Psi_0 / F^k$ for $\Psi_0 \in \mathcal{A}_0^n$, and $\langle \vec{e}, \Psi \rangle = 0 = \langle \vec{e}, \mathrm{d}\Psi \rangle$, we have $\langle \vec{e}, \Psi_0 \rangle = 0$ and (using this together with $\langle \vec{e}, \mathrm{d}F \rangle = \deg(F) \cdot F$)

$$0 = F^{k+1} \langle \vec{e}, \mathrm{d}\Psi \rangle = \langle \vec{e}, F \mathrm{d}\Psi_0 - k \mathrm{d}F \wedge \Psi_0 \rangle = F \langle \vec{e}, \mathrm{d}\Psi_0 \rangle - k \langle \vec{e}, \mathrm{d}F \wedge \Psi_0 \rangle$$

$$= F \langle \vec{e}, \mathrm{d}\Psi_0 \rangle - k \langle \vec{e}, \mathrm{d}F \rangle \wedge \Psi_0 = F \cdot \deg(\Psi_0) \cdot \Psi_0 - kD \cdot F \cdot \Psi_0$$

so that $\deg(\Psi_0) = kD$ and $\Psi_0 = \left\langle \vec{e}, \frac{1}{kD} \mathrm{d}\Psi_0 \right\rangle = \langle \vec{e}, \varphi \rangle$ for $\varphi \in \mathcal{A}_0^{n+1}$. The only option for such a φ is $P\mathrm{d}\mathbf{z}$, so that $\Psi_0 = P\Omega$ and

$$kD = (\deg(\Psi_0) =) \deg P + \deg \Omega = \deg P + (n+1).$$

We note that the inclusion $A_{k-1}^n \hookrightarrow A_k^n$ corresponds to $F \cdot S^{(k-1)D-n-1} \subseteq S^{kD-n-1}$ via $\frac{P\Omega}{F^{k-1}} = \frac{(PF)\Omega}{F^k}$.

4.2.2. Meromorphic forms and cohomology of X.

Now viewing

$$\mathcal{G}^k H^0(\Omega_{\mathbb{P}^n}^m(*X)) := A_k^m(X)$$

as a filtration by order of pole on meremorphic forms, the corresponding graded pieces

$$Gr_{\mathcal{G}}^k H^0(\Omega_{\mathbb{P}^n}^m(*X)) = H^0(\Omega_{\mathbb{P}^n}^m(kX)) \big/ H^0(\Omega_{\mathbb{P}^n}^m((k-1)X))$$

may be shown to coincide with those derived from the standard spectral-sequence procedure for m=n, n-1 (the cases we shall use).[14] Namely, set

$$\mathcal{P}^k \Omega_{\mathbb{P}^n}^m(*X) = \Omega_{\mathbb{P}^n}^m(kX) \text{ for } k \geq 0 \text{ (and } \mathcal{P}^{-1} := 0)$$

[14] with the sole exception in the case $m = n-1$, $n = 2$ (which doesn't matter)

$$\left(Gr_{\mathcal{G}}^1 H^0 = \frac{H^0(\Omega_{\mathbb{P}^2}^1(X))}{H^0(\Omega_{\mathbb{P}^2}^1) = 0} \right) \to Gr_{\mathcal{P}}^1 H^0(\Omega_{\mathbb{P}^2}^1(*X)) \to H^1(\Omega_{\mathbb{P}^2}^1)[\neq 0].$$

and use

$$E_1^{p,q} = H^{p+q}(Gr^p_{\mathcal{P}}\Omega^m_{\mathbb{P}^n}(*X)) \implies E_\infty^{p,q} =: Gr^p_{\mathcal{P}} H^{p+q}(\Omega^m_{\mathbb{P}^n}(*X))$$

with $p+q=0$; then the assertion is that (for $m = n$, $n-1$) $Gr^p_{\mathcal{P}} H^0(\Omega^m_{\mathbb{P}^n}(*X)) = Gr^p_{\mathcal{G}} H^0(\Omega^m_{\mathbb{P}^n}(*X))$.

Griffiths **[G1]** considered the "rational de Rham cohomology" groups

$$\mathfrak{H}^n(X) := \frac{A^n(X)}{\mathrm{d}A^{n-1}(X)} \quad \text{with filtration} \quad \mathcal{G}_k\mathfrak{H}^n(X) := \frac{A^n_k(X)}{\mathrm{d}A^{n-1} \cap \{\mathrm{num}\}},$$

and graded pieces

$$Gr^k_{\mathcal{G}}\mathfrak{H}^n(X) = \frac{\mathcal{G}_k\mathfrak{H}^n(X)}{\mathcal{G}_{k-1}\mathfrak{H}^n(X)} = \frac{A^n_k}{A^n_{k-1} + \mathrm{d}A^{n-1} \cap \{\mathrm{num}\}}.$$

Clearly $\mathcal{G}_0\mathfrak{H}^n(X) = H^0(\Omega^n_{\mathbb{P}^n}) = 0$. We will quote the following results:

LEMMA 4.2.1. *(Griffiths)*
(i) $\mathcal{G}_n\mathfrak{H}^n(X) = \mathfrak{H}^n(X)$.
(ii) $Gr^k_{\mathcal{G}}\mathfrak{H}^n(X) = A^n_k / \left(A^n_{k-1} + dA^{n-1}_{k-1}\right)$.

It follows easily from *(ii)* that $\mathcal{G}_k\mathfrak{H}^n(X) = A^n_k(X)/dA^{n-1}_{k-1}(X)$.

The following simple argument suggests that to understand the interesting part of $H^*(X)$ we sould study the $Gr^k_{\mathcal{G}}\mathfrak{H}^n(X)$. First of all (working with coefficients in $\mathbb{Q}$ or $\mathbb{C}$), by the Lefschetz hyperplane theorem $H^\ell(X) \underset{\iota_X^*}{\overset{\cong}{\leftarrow}} H^\ell(\mathbb{P}^n)$ for $\ell < n-1$ and dually $H^\ell(X) \underset{\mathrm{Gy}}{\overset{\cong}{\rightarrow}} H^{\ell+2}(\mathbb{P}^n)$ for[15] $\ell > n-1$. Define

$$H^{n-1}_{var}(X) := \operatorname{coker}\left\{H^{n-1}(\mathbb{P}^n) \to H^{n-1}(X)\right\} \qquad \text{and}$$

$$H^{n-1}_{pr}(X) := \ker\left\{H^{n-1}(X) \underset{\cup D[H]}{\rightarrow} H^{n+1}(X)\right\} = \ker\left\{H^{n-1}(X) \underset{\mathrm{Gy}}{\rightarrow} H^{n+1}(\mathbb{P}^n)\right\},$$

noting that since the composition

$$H^{n-1}(\mathbb{P}^n) \to H^{n-1}(X) \to H^{n+1}(\mathbb{P}^n)$$

is $\cup D[H]$ and is therefore an isomorphism (over $\mathbb{Q}$), the natural map

$$H^{n-1}_{pr}(X) \to H^{n-1}_{var}(X)$$

is also an isomorphism. From

$$H^{n-2}(X) \underset{\cong}{\overset{\mathrm{Gy}}{\rightarrow}} H^n(\mathbb{P}^n) \to H^n(\mathbb{P}^n - X) \overset{\mathrm{Res}}{\rightarrow} H^{n-1}(X) \overset{\mathrm{Gy}}{\twoheadrightarrow} H^{n+1}(\mathbb{P}^n)$$

[15]In fact both maps are isomorphisms for all $\ell \neq n-1$ (except for ι_X^* at $\ell = 2n, 2n-1$), by factoring $\cup D[H] : H^\ell(\mathbb{P}^n) \to H^{\ell+2}(\mathbb{P}^n)$ through $H^\ell(X)$.

we have $H^n(\mathbb{P}^n - X) \underset{\cong}{\overset{\text{Res}}{\to}} H^{n-1}_{pr}(X)$, which is the "interesting part" of $H^*(X)$, and similarly

$$H^q(\Omega^{p+1}_{\mathbb{P}^n}(\mathrm{dlog}X)) \underset{\cong}{\overset{\text{Res}}{\to}} H^q(\Omega^p_X)_{pr} = \ker\left\{H^q(\Omega^p_X) \to H^{q+1}(\Omega^{p+1}_{\mathbb{P}^n})\right\}$$

for $p+q = n-1$.

Now here is the main point. Recall that "Bott vanishing" (see [**Gr3**]) says that $H^i(\Omega^j_{\mathbb{P}^n}(kX)) = 0$ for $i, k > 0$ (and $j \geq 0$). There is a long exact sequence of sheaves (see [**L1**])

$$0 \to \Omega^{p+1}_{\mathbb{P}^n}(\mathrm{dlog}X) \hookrightarrow \Omega^{p+1}_{\mathbb{P}^n}(X) \overset{\mathrm{d}}{\to} Gr^2_{\mathcal{P}}\Omega^{p+2}_{\mathbb{P}^n}(*X) \overset{\mathrm{d}}{\to} \ldots \overset{\mathrm{d}}{\to} Gr^{n-p}_{\mathcal{P}}\Omega^n_{\mathbb{P}^n}(*X) \to 0;$$

in fact (using Bott[16]) all terms but the first have trivial H^i, $i > 0$. So we have an acyclic resolution of the dlog forms, and

$$H^{n-p-1}(\Omega^{p+1}_{\mathbb{P}^n}(\mathrm{dlog}X)) \cong \frac{H^0(Gr^{n-p}_{\mathcal{P}}\Omega^n_{\mathbb{P}^n}(*X))}{\mathrm{d}H^0(Gr^{n-p-1}_{\mathcal{P}}\Omega^{n-1}_{\mathbb{P}^n}(*X))}$$

$$\cong \frac{H^0(\mathcal{P}^{n-p}\Omega^n_{\mathbb{P}^n}(*X))\,/\,H^0(\mathcal{P}^{n-p-1}\Omega^n_{\mathbb{P}^n}(*X))}{\mathrm{d}\left\{H^0(\mathcal{P}^{n-p-1}\Omega^{n-1}_{\mathbb{P}^n}(*X))\,/\,H^0(\mathcal{P}^{n-p-2}\Omega^{n-2}_{\mathbb{P}^n}(*X))\right\}}$$

$$= \frac{H^0(\Omega^n_{\mathbb{P}^n}((n-p)X))}{H^0(\Omega^n_{\mathbb{P}^n}((n-p-1)X)) + \mathrm{d}H^0(\Omega^{n-1}_{\mathbb{P}^n}((n-p-1)X))} = Gr^{n-p}_{\mathcal{G}}\mathfrak{H}^n(X)$$

(where the second $\cong$ again uses Bott to ensure $H^1(\mathcal{P}^{n-p-1}\Omega^n_{\mathbb{P}^n}(*X)) = 0$). So we obtain

$$H^{n-p-1}(\Omega^p_X)_{pr} \cong Gr^{n-p}_{\mathcal{G}}\mathfrak{H}^n(X),$$

but this (efficient) incarnation of the proof does not lend itself to studying how periods change[17] (as X varies), as easily as Griffiths' original argument does. This does everything explicitly on forms.

[16]together with the long exact sequences associated to $\mathcal{P}^{k-1} \to \mathcal{P}^k \to Gr^k_{\mathcal{P}}$. A more formal way of proceeding than the above, is a spectral sequence argument taking F_a, $0 \leq a \leq n-p$, to be the sheaves of the long exact sequence and $E_1^{a,b} := H^b(\mathbb{P}^n, \mathsf{F}_a) \implies 0 = E_\infty$. (This is because the spectral sequence computes $\mathbb{H}^*(\mathsf{F}_\bullet)$, and $\mathsf{F}_\bullet \simeq 0$ since it is exact; see [**GH**]) Then the first $\cong$ above is given by $d_{n-p} : E_1^{0,n-p-1} \to E_2^{n-p,0}$, because by Bott vanishing all $E_1^{a,b}$ are 0 except for $a = 0$ or $b = 0$.

[17]It can be made to yield up this information, basically by using the spectral sequence version in the above footnote and tracing through d_{n-p} (and this is exactly how we will do it for complete intersections in §4.5).

4.2.3. Griffiths' approach revisited; Jacobi rings. There are inclusions of sheaves

$$\Omega^\bullet_{\mathbb{P}^n}(*X) \hookrightarrow \jmath_*\Omega^\bullet_{(\mathbb{P}^n\setminus X)^\infty} \hookleftarrow \Omega^\bullet_{(\mathbb{P}^n)^\infty}(\mathrm{dlog}X)$$

in particular for $\bullet = n, n-1$; we denote sections by H^0 and closed sections by Z^0. The inclusions induce maps on cohomology

$$\begin{array}{ccccc} \mathfrak{H}^n(X) & \longrightarrow & H^n(\mathbb{P}^n - X, \mathbb{C}) & \longleftarrow & "H^n(\mathbb{P}^n(\mathrm{dlog}X))" \\ \shortparallel & & \ddot{\shortparallel} & & \ddot{\shortparallel} \\ \frac{H^0(\Omega^n_{\mathbb{P}^n}(*X))}{\mathrm{d}H^0(\Omega^{n-1}_{\mathbb{P}^n}(*X))} & & \frac{Z^0(\Omega^n_{(\mathbb{P}^n\setminus X)^\infty})}{\mathrm{d}H^0(\Omega^{n-1}_{(\mathbb{P}^n\setminus X)^\infty})} & & \frac{Z^0(\Omega^n_{(\mathbb{P}^n)^\infty}(\mathrm{dlog}X))}{\mathrm{d}H^0(\Omega^{n-1}_{(\mathbb{P}^n)^\infty}(\mathrm{dlog}X))} \end{array}$$

which are in fact both isomorphisms. If we define a filtration

$$F^{p+1}H^n(\mathbb{P}^n(\mathrm{dlog}X)) := \frac{Z^0(F^{p+1}\Omega^n_{(\mathbb{P}^n)^\infty}(\mathrm{dlog}X))}{\left\{\mathrm{d}H^0(\Omega^{n-1}_{(\mathbb{P}^n)^\infty}(\mathrm{dlog}X))\right\}\cap \mathrm{num}} = \frac{Z^0(F^{p+1}\Omega^n_{(\mathbb{P}^n)^\infty}(\mathrm{dlog}X))}{\mathrm{d}H^0(F^p\Omega^{n-1}_{(\mathbb{P}^n)^\infty}(\mathrm{dlog}X))},$$

Griffiths proved that the (isomorphic) images in $H^n(\mathbb{P}^n - X, \mathbb{C})$ of the two filtrations

$$\mathrm{im}\,\{\mathcal{G}_{n-p}\mathfrak{H}^n(X)\} = \mathrm{im}\left\{F^{p+1}H^n(\mathbb{P}^n(\mathrm{dlog}X))\right\}$$

coincide. Even if we take graded pieces here, this is more than the previous result (out of the long exact sequence), because it tells us the representatives differ by a coboundary on $\mathbb{P}^n - X$ (while the previous approach just gives "an" isomorphism), so that their periods coincide. Taking residues on the right hand side, we get Griffiths' map

$$\mathcal{G}_{n-p}\mathfrak{H}^n(X) \xrightarrow{\cong} F^pH^{n-1}(X) \cap H^{n-1}_{pr}(X). \tag{4.2.1}$$

In the following we shall always identify $k = n - p$ (this simplifies notation). We briefly explain how to change a rational form $\omega \in A^n_k = H^0(\Omega^n_{\mathbb{P}^n}(kX))$ by a C^∞ coboundary on $\mathbb{P}^n - X$ to get $\beta \in Z^0(F^{p+1}\Omega^n_{(\mathbb{P}^n)^\infty}(\mathrm{dlog}X))$. The key is that one can always reduce the pole *locally* by adding a *holomorphic* coboundary (which is *not* always possible globally). For instance, taking $\mathsf{U}_0 := \mathbb{P}^n \setminus \{z_0 = 0\} \subset \mathbb{P}^n$, if $\mathcal{U}_\alpha \subseteq \mathsf{U}_0$ is a small enough neighborhood of any point on $X \cap \mathsf{U}_0$, there is a $j(\alpha)$ for which $\partial F/\partial z_{j(\alpha)} \neq 0$ on $\mathcal{U}_\alpha$ (because X is smooth). On U_0, $\Omega = \mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}z_n$ (by taking $z_0 = 1$); and so on $\mathcal{U}_\alpha$, using $\mathrm{d}F = \sum_{i=1}^n \frac{\partial F}{\partial z_i}\mathrm{d}z_i$, we have

$$\omega = \frac{P\Omega}{F^k} = \frac{P\mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}z_n}{F^k} = \frac{P\mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}F \wedge \ldots \wedge \mathrm{d}z_n}{\partial F/\partial z_{j(\alpha)} \cdot F^k} =: \frac{\mathrm{d}F \wedge \theta_\alpha}{F^k}$$

where

$$\theta_\alpha := \frac{\pm P\mathrm{d}z_1 \wedge \ldots \wedge \widehat{\mathrm{d}z_{j(\alpha)}} \wedge \ldots \wedge \mathrm{d}z_n}{\partial F/\partial z_{j(\alpha)}}$$

is holomorphic on $\mathcal{U}_\alpha$. So

$$\mathrm{d}\eta_\alpha := \mathrm{d}\left(\frac{\theta_\alpha}{(k-1)F^{k-1}}\right) = \frac{\mathrm{d}\theta_\alpha}{(k-1)F^{k-1}} - \frac{\mathrm{d}F \wedge \theta_\alpha}{F^k} =: \tau_\alpha - \omega\,;$$

repeating this procedure on a system of neighborhoods $\mathcal{U}_\alpha$ covering X (and doing nothing[18] on a $\mathcal{U}_0$ with $\mathcal{U}_0 \cap X = \emptyset$, $\mathcal{U}_0 \cup_\alpha \mathcal{U}_\alpha = \mathbb{P}^n$) we piece the result together with a partition of unity $\{\rho_\alpha\}$:

$$\mathrm{d}\eta := \mathrm{d}\left(\sum_{0,\alpha} \rho_\alpha \eta_\alpha\right) = \sum (\mathrm{d}\rho_\alpha \wedge \eta_\alpha + \rho_\alpha \tau_\alpha) - \omega =: \tau - \omega$$

where (because of the $\{\mathrm{d}\rho_\alpha\}$) τ may have a $\mathrm{d}\bar{z}$ part: $\tau \in Z^0(F^{n-1}\Omega^n_{(\mathbb{P}^n)^\infty}((k-1)X))$. In fact we can always say something slightly stricter about the poles of τ: the "last" pole is $\mathrm{dlog}F$, so that repeating the procedure eventually lands us in $Z^0(F^{n-k+1}\Omega^n_{(\mathbb{P}^n)^\infty}(\mathrm{dlog}X))$ rather than $H^0(F^{n-k+1}\Omega^n_{(\mathbb{P}^n)^\infty}(X))$. See **[G1]** for details.

If, for some j, P has a factor of $\partial F/\partial z_j$, one can "globally" (at least on U_0) *holomorphically* reduce the pole (no $\mathrm{d}\bar{z}$'s come into play): if $P = P_0 \partial F/\partial z_j$ then

$$\theta := \frac{P\mathrm{d}z_1 \wedge \ldots \wedge \widehat{\mathrm{d}z_j} \wedge \ldots \wedge \mathrm{d}z_n}{\partial F/\partial z_j} = P_0 \mathrm{d}z_1 \wedge \ldots \wedge \widehat{\mathrm{d}z_j} \wedge \ldots \wedge \mathrm{d}z_n$$

is holomorphic on all of U_0.

To do this correctly on all of $\mathbb{P}^n$ we again employ $\mathbb{C}^*$-invariant forms on $U = \mathbb{C}^{n+1} \setminus \{0\}$. More precisely, we would like to show that $\omega = \frac{P\Omega}{F^k} \in A^n_{k-1} + \mathrm{d}A^{n-1}_{[k-1]} \subseteq A^n_k$ if P is in the Jacobi ideal

$$J_F^{kD-n-1} := \left(\frac{\partial F}{\partial z_0}, \ldots, \frac{\partial F}{\partial z_n}\right) \subseteq S^{kD-n-1}$$

(and the converse as well). For $\eta \in A^{n-1}_{k-1}$, $\pi^*\eta =: \Psi = \frac{\Psi_0}{F^{k-1}}$ with $\Psi_0 \in \mathcal{A}^{n-1}_0$; exactly as before we use $\langle \vec{e}, \Psi \rangle = \langle \vec{e}, \mathrm{d}\Psi \rangle = 0$ to show $\Psi_0 = \left\langle \vec{e}, \frac{1}{(k-1)D}\mathrm{d}\Psi_0 \right\rangle =: \langle \vec{e}, \varphi \rangle$, where $\varphi \in \mathcal{A}^n_0$ has degree $(k-1)D$ ($\deg(\mathrm{d}\Psi_0) = \deg(\Psi_0)$ is easily shown). So $\deg\left(\frac{\varphi}{F^{k-1}}\right) = 0$ and

$$\mathrm{d}\Psi = \mathrm{d}\left(\frac{\langle \vec{e}, \varphi \rangle}{F^{k-1}}\right) = \mathrm{d}\left\langle \vec{e}, \frac{\varphi}{F^{k-1}} \right\rangle = \deg\left(\frac{\varphi}{F^{k-1}}\right)\frac{\varphi}{F^{k-1}} - \left\langle \vec{e}, \mathrm{d}\left(\frac{\varphi}{F^{k-1}}\right)\right\rangle$$

$$= \frac{\langle \vec{e}, (k-1)\mathrm{d}F \wedge \varphi - F\mathrm{d}\varphi \rangle}{F^k}.$$

Now the only options for φ (= holomorphic n-form on $\mathbb{C}^{n+1}$ of degree $(k-1)D$) are to take $\mathrm{d}\mathbf{z}^{(i)} = \mathrm{d}z_0 \wedge \ldots \wedge \widehat{\mathrm{d}z_i} \wedge \ldots \wedge \mathrm{d}z_n$ and any $R_i \in S^{(k-1)D-n}$

[18] i.e., taking $\eta_\alpha =$ a primitive of ω on U_0.

and write

$$\varphi = \sum_{i=0}^{n}(-1)^i R_i \mathrm{d}\mathbf{z}^{(i)} \quad \text{so that}$$

$$\mathrm{d}\varphi = \sum_{i=0}^{n} \frac{\partial R_i}{\partial z_i}\mathrm{d}\mathbf{z}\,, \quad \mathrm{d}F \wedge \varphi = \sum_{i=0}^{n} R_i \frac{\partial F}{\partial z_i}\mathrm{d}\mathbf{z},$$

$$(\pi^*\mathrm{d}\eta =)\,\mathrm{d}\Psi = \frac{\left\langle \vec{e}, \sum\left((k-1)R_i\frac{\partial F}{\partial z_i} - F\frac{\partial R}{\partial z_i}\right)\mathrm{d}\mathbf{z}\right\rangle}{F^k} = \frac{\left\{\sum_i\left((k-1)R_i\frac{\partial F}{\partial z_i} - F\frac{\partial R}{\partial z_i}\right)\right\}\Omega}{F^k} \in A^n_k,$$

where the homogeneous polynomial in brackets is $\in J^{kD-n-1}$. Conversely, given any $P \in J$, say $P = \sum(k-1)R_i\frac{\partial F}{\mathrm{d}z_i}$, clearly [subtracting] the above $\mathrm{d}\Psi \in \mathrm{d}A^{n-1}_{k-1}$ will "reduce the poles" of $P\Omega/F^k$ to $(\sum \frac{\partial R}{\partial z_i})\Omega/F^{k-1} \in A^n_{k-1}$.

What we have shown is that the map (4.2.1) has *graded pieces* (identifying $k = n-p$)

$$R_F^{kD-n-1} := \frac{S^{kD-n-1}}{J_F} \xrightarrow{\cong} Gr^k_{\mathcal{G}}\mathfrak{H}^n(X) \xrightarrow{\cong} H^{n-k,k-1}(X)_{pr},$$

where the first isomorphism is induced by sending $P \mapsto \frac{P\Omega}{F^k} =: \omega$, and the second by modifying ω by a coboundary on $\mathbb{P}^n - X$ and taking Res_X. We emphasize that the $H^{n-k,k-1}(X)_{pr}$ are *not* to be thought of as subspaces of $H^{n-1}_{pr}(X)$, but as quotients $Gr_F^{n-k}H^{n-1}_{pr}(X)$.

To put this all in perspective, suppose now we want to compute $\int_{\mathcal{C}}\alpha$ for $[\mathcal{C}] \in H_{n-1}(X,\mathbb{Z})$, $[\alpha] \in F^pH^{n-1}_{pr}(X,\mathbb{C})$. Then up to coboundary on X, $\alpha \equiv \frac{1}{2\pi i}\mathrm{Res}(\beta)$ for $\beta \in Z^0(F^{p+1}\Omega^n_{(\mathbb{P}^n)^\infty}(\mathrm{dlog}X))$, and up to coboundary on $\mathbb{P}^n - X$, $\beta \equiv \omega \in H^0(\Omega^n_{\mathbb{P}^n}(kX)) = A^n_k$. Consequently by Stokes' theorem

$$\int_{\mathcal{C}}\alpha = \frac{1}{2\pi i}\int_{\mathrm{Tube}(\mathcal{C})}\beta = \frac{1}{2\pi i}\int_{\mathrm{Tube}(\mathcal{C})}\frac{P\Omega}{F^k}$$

for some $P \in S^{kD-n-1}$.

4.2.4. Computation of $\bar{\nabla}$. Now take a local 1-parameter family of smooth hypersurfaces

$$\mathcal{X} = \mathcal{X}_D \subset \mathbb{P}^n \times D \quad , \quad \mathcal{X} = \bigcup_{t\in D}\{X_t\}$$

and consider the sheaves on D

$$\mathcal{F}^{n-k}\mathcal{H}^{n-1}_{X_t,pr} \hookrightarrow \mathbb{R}^{n-1}_{\pi_*}\Omega^{\bullet\geq n-k}_{\mathcal{X}/D} \twoheadrightarrow \mathcal{F}^{n-k}\mathcal{H}^{n-1}_{X_t,var}$$

(this is *not* an exact sequence!); we would like to "differentiate" sections of the "var" sheaf. The Gauss-Manin connection $\nabla_{\partial/\partial t}$ on the middle sheaf induces such a derivative; the periods of the derivative in $\mathbb{R}^{n-1}_{\pi_*}\Omega^{\bullet\geq n-k-1}_{\mathcal{X}/D}$ give derivatives of the periods. We can think of H_{pr} and H_{var} as functionals

$$H^{n-1}_{pr}(X_t) = \mathrm{Hom}\left(\mathrm{coker}\left\{H_{n+1}(\mathbb{P}^n,\mathbb{Q}) \to H_{n-1}(X_t,\mathbb{Q})\right\},\mathbb{C}\right)$$

$$H^{n-1}_{var}(X_t) = \mathrm{Hom}\left(\ker\left\{H_{n-1}(X_t,\mathbb{Q}) \to H_{n-1}(\mathbb{P}^n,\mathbb{Q})\right\}, \mathbb{C}\right),$$

so that (for n odd) H_{var} ignores periods on cycles outside a (large) subgroup and H_{pr} consists of functionals *zero* on $\mathrm{im}\left\{H_{n+1}(\mathbb{P}^n,\mathbb{Z}) \to H_{n-1}(X_t,\mathbb{Z})\right\}$. The fact that they are isomorphic indicates that we can always lift a section of $\mathcal{H}^{n-1}_{X_t,var}$ to $\mathcal{H}^{n-1}_{X_t,pr}$, by subtracting the pullback of a smooth form on $\mathbb{P}^n \times D$.

So it suffices to differentiate a section α_t of $\mathcal{F}^{n-k}\mathcal{H}^{n-1}_{X_t,pr}$, which begets a family of β_t's [in $Z^0(F^{p+1}\Omega^n_{(\mathbb{P}^n)^\infty}(\mathrm{dlog}X_t))$, satisfying $\frac{1}{2\pi i}\mathrm{Res}_{X_t}\beta_t = \alpha_t$] and ω_t's [$\mathrm{d}\xi_t + \beta_t = \omega_t = \frac{P_t\Omega}{F_t^k} \in H^0(\Omega^n_{\mathbb{P}^n}(kX))$]. Consider a vertically closed lift $\tilde{\omega} \in H^0(\Omega^n_{\mathbb{P}^n\times D}(k\mathcal{X}))$ and differentiate periods using Stokes' theorem

$$2\pi i\frac{\partial}{\partial t}\int_{\mathcal{C}_t}\alpha_t = \frac{\partial}{\partial t}\int_{\mathrm{Tube}(\mathcal{C}_t)}\omega_t \;=\; \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{\left(\bigcup_{t\in[0,\epsilon]}\mathrm{Tube}(\mathcal{C}_t)\right)}\tilde{\omega} \;=\; \int_{\mathrm{Tube}(\mathcal{C}_0)}\langle\partial/\partial t, \mathrm{d}\tilde{\omega}\rangle_{t=0}$$

where $\mathcal{C}_t$ and $\mathrm{Tube}(\mathcal{C}_t)$ are continuous families of $(n-1)$- and n-cycles, respectively. Suppose

$$\tilde{\omega} = \frac{P(t)\Omega}{\left(F - \frac{t}{k}G\right)^k},\;\; P(t) \in S^{kD-n-1}[t],\;\; G \in S^D$$

where $\left\{F_t = F - \frac{t}{k}G = 0\right\}$ cuts out X_t; this is completely general since F_t is varying in $H^0(\mathbb{P}^n, \mathcal{O}(D))$, and

$$\theta^1_D \cong TH^0(\mathbb{P}^n, \mathcal{O}(D)) \cong S^D.$$

Then

$$\mathrm{d}\tilde{\omega} = \frac{(F - \frac{t}{k}G)\frac{\partial P}{\partial t}\mathrm{d}t\wedge\Omega + P(t)G\mathrm{d}t\wedge\Omega}{\left(F - \frac{t}{k}G\right)^{k+1}} \implies \langle\partial/\partial t, \mathrm{d}\tilde{\omega}\rangle_{t=0} = \frac{P(0)G\Omega}{F^{k+1}} + \frac{\frac{\partial P}{\partial t}(0)\Omega}{F^k},$$

and

$$2\pi i\int_{\mathcal{C}_0}\left(\nabla_{\partial/\partial t}\alpha_t\right) = \int_{\mathrm{Tube}(\mathcal{C}_0)}\left(\frac{P(0)G\Omega}{F^{k+1}} + \frac{\frac{\partial P}{\partial t}(0)\Omega}{F^k}\right).$$

This is useful if we work modulo periods of $F^{n-k\,(=p)}$ forms on X_t, or equivalently modulo periods of forms in $A^n_k(X_t)$ on $\mathbb{P}^n - X_t$. For we have proved the following statement about equality of periods vectors in the corresponding quotient spaces of functionals: namely, if $\{\mathcal{C}^i_0\}$ are a basis for $\ker\left\{H_{n-1}(X) \to H_{n-1}(\mathbb{P}^n)\right\}$,

$$\begin{array}{ccc} \left\langle \int_{\mathcal{C}^i_0}\left(\nabla_{\partial/\partial t}\alpha_t\right)_{t=0}\right\rangle & \equiv & \left\langle \int_{\mathrm{Tube}(\mathcal{C}^i_0)}\left(\frac{P(0)G\Omega}{F^{k+1}} = \frac{G}{F}\cdot\frac{P(0)\Omega}{F^k}\right)\right\rangle \\ \text{in } Gr^{n-k+1}_F H^{n-1}_{pr}(X) & & \text{in } Gr^{k+1}_{\mathcal{G}}\mathfrak{H}^n(X) \end{array}$$

In a commutative diagram,

$$\begin{array}{ccccc}
P & \mapsto & \dfrac{P\Omega}{F^k} & & \\
R_F^{kD-n-1} & \xrightarrow{\cong} & Gr_{\mathcal{G}}^k \mathfrak{H}^n(X) & \xrightarrow{\cong} & Gr_F^{n-k} H_{pr}^{n-1}(X) \\
\Big\downarrow {\scriptstyle \times G} & & \Big\downarrow {\scriptstyle (\bar{\nabla}_{\partial/\partial t})} & & \Big\downarrow {\scriptstyle \bar{\nabla}_{\partial/\partial t}} \\
R_F^{(k+1)D-n-1} & \xrightarrow{\cong} & Gr_{\mathcal{G}}^{k+1} \mathfrak{H}^n(X) & \xrightarrow{\cong} & Gr_F^{n-(k+1)} H_{pr}^{n-1}(X). \\
Q & \mapsto & \dfrac{Q\Omega}{F^{k+1}} & &
\end{array}$$

4.2.5. First application. We refine this a bit for a family

$$\mathcal{X} \longrightarrow \mathcal{S} \underset{\text{Zar. op.}}{\subset} \mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}(D))$$

of smooth $\{X_t\}$. First, note that for each $t \in \mathcal{S}$

$$\theta^1_{\mathcal{S},t} \cong S^D/(F_t) \quad \text{where} \quad (F_t) = \left(\sum z_j \frac{\partial F}{\partial z_j} \right) \subseteq J_F^{(\geq D)},$$

and so we have the slightly more informative commuting diagram (with μ =multiplication)

$$\begin{array}{ccc}
S^D/(F_t) \otimes R_{F_t}^{kD-n-1} & \xrightarrow{\mu} & R_{F_t}^{(k+1)D-n-1} \\
\Big\downarrow \cong & & \Big\downarrow \cong \\
\theta^1_{\mathcal{S},t} \otimes Gr_F^{n-k} H_{pr}^{n-1}(X_t) & \longrightarrow & Gr_F^{n-k-1} H_{pr}^{n-1}(X_t).
\end{array}$$

One easily sheafifies primitive and variable cohomology (and the graded pieces thereof); e.g. we could define $\mathcal{H}^{n-1}_{X_t,var}$ via

$$0 \to \mathbb{R}^{n-1}_{\pi_*} \Omega^\bullet_{\mathbb{P}^n \times \mathcal{S}/\mathcal{S}} \to \mathbb{R}^{n-1}_{\pi_*} \Omega^\bullet_{\mathcal{X}/\mathcal{S}} \to \mathcal{H}^{n-1}_{X_t,var} \to 0.$$

With this is mind the bottom map becomes a map of sheaves on $\mathcal{S}$ which dualizes to[19] the Gauss-Manin connection

$$Gr^k_{\mathcal{F}} \mathcal{H}^{n-1}_{X_t,var} \overset{\bar{\nabla}_{(k)}}{\longrightarrow} \Omega^1_{\mathcal{S}} \otimes Gr^{k-1}_{\mathcal{F}} \mathcal{H}^{n-1}_{X_t,var}.$$

[19] As the reader can check, this assertion reduces to the fact that, for α_t and β_t sections of $\mathcal{F}^k \mathcal{H}^{n-1}_{X_t}$ and $\mathcal{F}^{n-k} \mathcal{H}^{n-1}_{X_t}$ (so that $\alpha_t \wedge \beta_t = 0$),

$$0 = \int_{X_0} \nabla_{\partial/\partial t} (\alpha_t \wedge \beta_t)_{t=0} = \int_{X_0} (\nabla_{\partial/\partial t} \alpha_t)_{t=0} \wedge \beta_0 + \int_{X_0} \alpha_t \wedge (\nabla_{\partial/\partial t} \beta_t)_{t=0}$$

Noting that μ is obviously surjective for $kD-n-1 \geq 0$ (so that S^{kD-n-1} is not zero), we conclude that $\bar{\nabla}_{(k)}$ is *injective for* $1 \leq k \leq n-1$ provided $D \geq n+1$, or equivalently $K_{X_t} \geq 0$ (or $H^0(\Omega^{n-1}_{X_t}) \neq 0$). This result will be applied in the next section to show that for $t \in \mathcal{S}$ "very general" the image of $\mathrm{Gy}: H^{n-3}(\hat{V}_t) \to H^{n-1}(X_t)$ is just [the span of] the hyperplane class.

4.2.6. Second application and Donagi-Green. Similarly the reader may easily verify that the map

$$R^{kD-n-1}_{F_t} \otimes \bigwedge^2 S^D/(F_t) \xrightarrow{\mu^2_{(k)}} R^{(k+1)D-n-1}_{F_t} \otimes S^D/(F_t)$$

$$\bar{P} \otimes (\bar{G}_1 \wedge \bar{G}_2) \longmapsto \overline{PG_1} \otimes \bar{G}_2 - \overline{PG_2} \otimes \bar{G}_1$$

is isomorphic to the dual of the G-M connection

$$\Omega^1_{\mathcal{S}} \otimes Gr^k_{\mathcal{F}} \mathcal{H}^{n-1}_{X_t,var} \xrightarrow{\bar{\nabla}^2_{(k)}} \Omega^2_{\mathcal{S}} \otimes Gr^{k-1}_{\mathcal{F}} \mathcal{H}^{n-1}_{X_t,var}$$

induced locally by

$$\sum_i \mathrm{d}t_i \otimes \alpha_i \longmapsto \sum_{i,j} (\mathrm{d}t_i \wedge \mathrm{d}t_j) \otimes \bar{\nabla}_{\partial/\partial t_j} \alpha_i.$$

In the next section we will want to use injectivity of $\bar{\nabla}^2_{(n-1)}$ (it's the crux of the argument), so we compute here the range of D for which $\mu^2_{(n-1)}$ is surjective. We include this in a more general problem by observing that, for $k = n-1$, $R^{(k+2)D-n-1}_{F_0} = R^{(n+1)D-n-1}_{F_0} = Gr^{n+1}_{\mathcal{G}} \mathfrak{H}^n(X) = 0$, so that it is enough to prove exactness at the middle term of the bottom (trivially $\Longrightarrow$ top) row of

$$\begin{array}{ccccc}
R^{b-D}_F \otimes \bigwedge^2 S^D/(F) & \xrightarrow{\mu^2} & R^b_F \otimes S^D/(F) & \xrightarrow{\mu} & R^{b+D}_F \\
\uparrow & & \uparrow & & \parallel \\
R^{b-D}_F \otimes \bigwedge^2 S^D & \xrightarrow{\mu^2} & R^b_F \otimes S^D & \xrightarrow{\mu} & R^{b+D}_F
\end{array}$$

for $b = (k+1)D - n - 1 = nD - n - 1$. (That $R^{b+D}_F = 0$ for this b in no way simplifies the proof, so we will ignore it.) That is, we prove vanishing of a certain "Koszul cohomology" group (for certain b and D).

For $b \geq D-1$, J^b_F contains all the generators $\partial F/\partial z_i$ and so $\mu: J^b_F \otimes S^D \to J^{b+D}_F$ is surjective. Considering the diagram with exact columns

$$\begin{tikzcd}
J_F^{b-D} \otimes \bigwedge^2 S^D \arrow[r] \arrow[d, hook] & J_F^b \otimes S^D \arrow[r, two heads] \arrow[d, hook] & J_F^{b+D} \arrow[d, hook] \\
S^{b-D} \otimes \bigwedge^2 S^D \arrow[r] \arrow[d, two heads] & S^b \otimes S^D \arrow[r, two heads] \arrow[d, two heads] & S^{b+D} \arrow[d, two heads] \\
R_F^{b-D} \otimes \bigwedge^2 S^D \arrow[r] & R_F^b \otimes S^D \arrow[r, two heads] & R_F^{b+D},
\end{tikzcd}$$

it is enough to prove exactness of the middle row (at its middle term). More generally by a lemma of Donagi and Green,

$$\bigwedge^2 S^a \otimes S^{b-a} \overset{\mu^2}{\longrightarrow} S^a \otimes S^b \overset{\mu}{\longrightarrow} S^{b+a}$$

is exact at the middle term for $b > a$, and so our result (that $\bar{\nabla}^2_{(n-1)}$ is injective) is proved for $b[= nD - n - 1] \geq D + 1$, or $(n-1)(D-1) \geq 3$.

For completeness we include a proof of Donagi-Green, namely that $\ker(\mu) \subseteq \mathrm{im}(\mu^2)$. Following [**DG**], let I and J be multi-indices and

$$\sum_{\substack{|I| = a \\ |J| = b}} c_{I,J} z_I \otimes z_J \in \ker(\mu)$$

be an element of the kernel; then $\sum_{I+J=K} c_{I,J} = 0$ for all fixed multi-indices K (with $|K| = a + b$). Now by definition

$$\mathrm{im}(\mu^2) = \left\{ z_I \otimes z_{I'+L} - z_{I'} \otimes z_{I+L} \,\middle|\, |I| = |I'| = a,\ |L| = b - a \boxed{> 0} \right\};$$

if we can show the following

CLAIM. $\mathrm{im}(\mu^2)$ contains all elements of the form

$$\left\{ z_I \otimes z_{K-I} - z_{I'} \otimes z_{K-I'} \,\middle|\, |I| = |I'| = a,\ |K| = a + b,\ K \geq I, I' \right\},$$

then all $z_I \otimes z_J$ with $I + J = K$ are equivalent modulo $\mathrm{im}(\mu^2)$ and so from

$$\sum_{I+J=K} c_{I,J} = 0$$

follows

$$\sum_{I+J=K} c_{I,J} z_I \otimes z_J \equiv 0 \qquad \mathrm{mod}\ \mathrm{im}(\mu^2)$$

follows

$$\sum_{\substack{|I|=a\\|J|=b}} c_{I,J} z_I \otimes z_J = \sum_{|K|=a+b} \sum_{I+J=K} c_{I,J} z_I \otimes z_J \equiv 0 \qquad \operatorname{mod} \operatorname{im}(\mu^2).$$

But this is the same as saying our (arbitrary) element of $\ker(\mu)$ is in the image of μ^2.

To prove the claim it is enough to show that

$$\operatorname{im}(\mu^2) \ni z_I \otimes z_{K-I} - z_{I'} \otimes z_{K-I'} \text{ for } I \text{ and } I' \text{ differing by one index}$$

(and the same conditions on I, I', K as in the claim). So for

$$I - I' = (0, \ldots, 0, \underset{i}{1}, 0, \ldots, 0, \underset{j}{-1}, 0, \ldots, 0) =: \delta_i - \delta_j$$

we will show, working modulo $\operatorname{im}(\mu^2)$, that $z_I \otimes z_{K-I} \equiv z_{I'} \otimes z_{K-I'}$.

More generally for any I, I', K (satisfying the claim's conditions) for which there are multi-indices J and I'' with

$$|J| = b, \; J - I \geq 0, \; J - I' \geq 0, \; I'' + J = K,$$

we have

$$z_I \otimes z_{K-I} \equiv z_{I''} \otimes z_J \equiv z_{I'} \otimes z_{K-I'}$$

(using the definition of $\operatorname{im}(\mu^2)$, e.g. for the first $\equiv$ replacing the L and I' of the definition by $J - I$ and I''). Since we only care about the end terms we can forget I'' and replace the condition $I'' + J = K$ by $J \leq K$.

What's left is to show we can choose such a J for $I - I' = \delta_i - \delta_j$. $\boxed{\text{Since } b > a,}$ there exists $J \geq \delta_j + I$ with $|J| = b$; we than have immediately $J - I \geq \delta_j \geq 0$ and $J - I' = (J - I) + (I - I') \geq \delta_j + \delta_i - \delta_j = \delta_i \geq 0$. Now $K \geq I, I' \implies K \geq \delta_j + I = \delta_i + I'$, and so it is possible to choose this J both $\geq \delta_j + I$ and $\leq K$.

4.3. The Vanishing Theorem in Codimension 1

4.3.1. Spreading an element of the Tame kernel. We would now like to study the Milnor regulator on a smooth degree D hypersurface $X \subseteq \mathbb{P}^n$ ($d = \dim X = n - 1$); the family of all such X is written

$$\mathcal{X} \xrightarrow{\pi} \mathcal{S} := \mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}(D)) \setminus \Delta$$

where Δ starts out as the discriminant locus (of singular fibers) and grows by a finite number of Zariski-closed subsets during the course of the discussion below. In fact we shall begin by throwing into it the hyperplane at ∞ (in $\mathcal{S}$) so that we may parametrize $\mathcal{S}$ by the coordinates $t_1, \ldots, t_N$ which generate its meromorphic function field. If $\{\sigma_j(\mathbf{z}) = \sigma_j(z_0, \ldots, z_n)\}_{j=0}^N \in S^D$ is a monomial basis then $\mathcal{X} \subseteq \mathbb{P}^n \times \mathcal{S}$ is cut out by

$$0 = \sigma_0(\mathbf{z}) + \sum_{j=1}^N t_j \sigma_j(\mathbf{z})\,;$$

if we write $(\beta_1^s, \ldots, \beta_N^s)$ for the coordinates of $s \in \mathcal{S}$, then

$$X_s = \left\{ 0 = \sigma_0(\mathbf{z}) + \sum_{j=1}^N \beta_j^s \sigma_j(\mathbf{z}) \right\}.$$

Since the only interesting cohomology of X is in dimension $(n-1)$, we will look at the regulator (mod torsion)

$$R:\ K_n^M(\mathbb{C}(X)) \to$$
$$H^{n-1}(\eta_X, \mathbb{C}/\mathbb{Q}(n)) \ := \ \varinjlim_{V \subset X} \ \mathrm{Hom}\,\{H_{n-1}(X - V, \mathbb{Q}), \mathbb{C}/\mathbb{Q}(n)\}$$

in the version given by integrating $R_{\mathbf{f}}$ over closed cycles.[20] In particular, we will study how this behaves in the family, if the "rigidity condition" that $\{\mathbf{f}_s\} \in \ker(\mathrm{Tame})$ or $K_n^M(X_s)$ is imposed. Except when X_s is an elliptic curve (the case $n = 2$, $D = 3$), we will show that for the weaker ($= \ker(\mathrm{Tame})$) rigidity condition the infinitesimal invariant (="$\nabla[R_{\mathbf{f}_s}]$") is zero, while the stronger criterion[21] $\implies$ the regulator image is actually zero.

We can use this latter result about the family to show that the image of the regulator[22]

$$R:\ K_n^M(X_0) \to \mathrm{im}\left\{H^{n-1}(X_0, \mathbb{C}/\mathbb{Q}(n)) \to H^{n-1}(\eta_X, \mathbb{C}/\mathbb{Q}(n))\right\}$$

is zero for a very general X_0 (again excepting elliptic curves). That is, the base point[23] $0 \in \mathcal{S}$ must be chosen in the compliment of *countably* many Zariski-closed (proper) subsets.[24] This essentially means that X_0 is of no arithmetic interest; indeed to prove our result it is sufficient to assume that $\beta_1^0, \ldots, \beta_N^0$ are algebraically independent (over $\mathbb{Q}$).

Consider on a very general X_0 a fixed element

$$\{\mathbf{f}_0\} \in \prod_\alpha \{f_{1\alpha}^0, \ldots, f_{n\alpha}^0\}^{m_\alpha} \in \ker(\mathrm{Tame}^{[1]}) \subseteq K_n^M(\mathbb{C}(X_0)).$$

Passing if necessary to a finite cover

[20] which gives the same result as integrating $R'_{\mathbf{f}}$ (mod $\mathbb{Q}(n)$).

[21] For a definition of $K_n^M(X)$ see §1.2.3.

[22] see §2.4.2; note that it makes *sense* to want to look at this part of the Milnor regulator – it is the part that has nothing to do with residues (and so "more" to do with the cohomology of X). While it's not obvious, we will show below that $\mathrm{im}\left\{H^{n-1}(X) \to H^{n-1}(\eta_X)\right\}$ is generally large enough for the map to be interesting.

[23] 0 does *not* mean $t_1 = \ldots = t_N = 0$.

[24] "general" means $0 \in$the compliment of *finitely* many Zariski closed subsets (which is just a Zariski *open* set). Though we start with 0 very general, our "spread" shall be defined on a general member of the family.

$$\begin{tikzcd} \mathcal{X} \times_{\mathcal{S}} \tilde{\mathcal{S}} \arrow[r, equal] & \tilde{\mathcal{X}} \arrow[r, "\pi"] \arrow[d, "\rho"] & \tilde{\mathcal{S}} \arrow[d, "\rho"] \\ & \mathcal{X} \arrow[r, "\pi"] & \mathcal{S} \end{tikzcd}$$

after removing (finitely many) more Zariski-closed subsets of $\mathcal{S}$ into Δ (including the branch locus of this cover), we may "spread" $\{\mathbf{f}_0\}$ to an element

$$\{\mathbf{F}\} = \prod_\alpha \{F_{1\alpha}, \dots, F_{n\alpha}\}^{m_\alpha} \in \ker(\mathrm{Tame}) \subseteq K_n^M(\mathbb{C}(\tilde{\mathcal{X}})).$$

This engenders a whole family $\{\mathbf{f}_s\} \in \ker(\mathrm{Tame})$, for all $s \in \tilde{\mathcal{S}}$, by restriction; and there is an $s_0 \in \rho^{-1}(0)$ (which we shall henceforth call 0) such that $\{\mathbf{f}_{s_0}\} = \{\mathbf{f}_0\}$ on[25] X_{s_0}. We caution that the extension of $\{\mathbf{F}\}$ to the entire branched cover $\bar{\tilde{\mathcal{X}}}$ would not be in ker(Tame); the functions $F_{i\alpha}$ may be badly behaved over certain points of $\bar{\tilde{\mathcal{S}}}$ (including, but not limited to, the discriminant and branch loci).

Here is how the spread, and cover, arise if $\beta_1^0, \dots, \beta_N^0$ are algebraically independent. First lift $\{f_{i\alpha}^0\} \in \mathbb{C}(X_0)$ to the rational functions $\{\tilde{f}_{i\alpha}^0\}$ in $\mathbb{C}(\mathbb{P}^n) = \mathbb{C}(Z_1, \dots, Z_n)$ [here $Z_i := z_i/z_0$] they come from, and collect together all the coefficients involved in these functions into the finite set $\mathfrak{C}_0 \subset \mathbb{C}$. Pick out a maximal subset $\mathfrak{C}^{tr} = \{\gamma_1, \dots, \gamma_M\}$ algebraically independent over $\mathbb{Q}(\beta_1^0, \dots, \beta_N^0)$ and name the remainder $\mathfrak{C}^{alg}$, so that (writing $\boldsymbol{\beta}^0$ for $\{\beta_1^0, \dots, \beta_N^0\}$)

$$\mathbb{Q}(\boldsymbol{\beta}^0; \mathfrak{C}_0) = \mathbb{Q}(\boldsymbol{\beta}^0, \mathfrak{C}^{tr})(\mathfrak{C}_0^{alg}) \,/\mathbb{Q}(\boldsymbol{\beta}^0;\ \mathfrak{C}^{tr})$$

is an algebraic (finite) field extension, and the functions we started with live in $\mathbb{Q}(\boldsymbol{\beta}^0; \mathfrak{C}_0)(X_0)$, with corresponding rational functions

$$\{\tilde{f}_{i\alpha}^0\} \in \mathbb{Q}(\boldsymbol{\beta}^0; \mathfrak{C}_0)(Z_1, \dots, Z_n).$$

Now write

$$\mathbb{Q}(\boldsymbol{\beta}^0; \mathfrak{C}_0) \subseteq \overline{\mathbb{Q}(\mathfrak{C}^{tr})}(\boldsymbol{\beta}^0)(\mathfrak{C}_0^{alg}) =: k(\boldsymbol{\beta}^0)(\mathfrak{C}_0^{alg}) =: \ell_0,$$

so that now $[\ell_0 : k(\boldsymbol{\beta}^0)]$ is finite and will in fact be the order of our cover. In particular, there is a cover $\tilde{\mathcal{S}} \overset{\rho}{\longmapsto} \mathcal{S}$ completing the diagram

[25] $\rho^{-1}(0)$ is just several copies of X_0.

$$\begin{tikzcd} k(\mathcal{S}) \arrow[d, hook, "\rho^*"] & k(t_1, \dots, t_N) \arrow[l, "\cong"'] \arrow[r, "t_j \mapsto \beta_j^0 \atop \text{evaluation}", "\cong"'] & k(\beta^0) \arrow[d, hook] \\ k(\tilde{\mathcal{S}}) \arrow[rr, "e_0", "\cong"'] & & \ell_0 \end{tikzcd}$$

where $k(\tilde{\mathcal{S}})$ is just $k(t_1, \dots, t_N)$ with some algebraic functions of the $\{t_j\}$ (e.g., $\sqrt[3]{t_1 - t_3^2}$) adjoined. (This also explains the necessity of the cover, to make these functions well-defined.) In fact evaluation maps $k(\tilde{\mathcal{S}}) \xrightarrow{e_s} \mathbb{C}$ are possible for every $s \in \tilde{\mathcal{S}}$, simply by taking $t_j \mapsto \beta_j^s$ and choosing appropriate branches for the algebraic functions. Denote the image by ℓ_s; it is isomorphic for s very general (and obviously for $s = 0$), as is e_s in the following diagram:

$$\begin{tikzcd} \mathbb{C}(\tilde{\mathcal{S}})(Z_1, \dots, Z_n) \arrow[d] & k(\tilde{\mathcal{S}})(Z_1, \dots, Z_n) \arrow[l, hook'] \arrow[r, "e_s"] \arrow[d] & \ell_s(Z_1, \dots, Z_n) \arrow[r, hook] \arrow[d] & \mathbb{C}(\mathbb{P}^n) \arrow[d] \\ \mathbb{C}(\tilde{\mathcal{X}}) & k(\tilde{\mathcal{X}}) \arrow[l, hook'] \arrow[r, "e_s"] & \ell_s(X_s) \arrow[r, hook] & \mathbb{C}(X_s). \end{tikzcd}$$

Given the original rational functions $\{\tilde{f}_{i\alpha}^0\} \in \ell_0(Z_1, \dots, Z_n)$ it is now clear that the isomorphism e_0 yields $\{F_{i\alpha}\} \in k(\tilde{\mathcal{X}}) \hookrightarrow \mathbb{C}(\tilde{\mathcal{X}})$ as desired. However these functions may have "vertical" zeroes and poles due to the behavior of "coefficients"$\in k(\tilde{\mathcal{S}})$ (including the above functions of $\{t_j\}$), and so the bottom e_0 is not functorial as far as the tame symbol is concerned, until we omit the (finitely many) Zariski-closed subsets of $\tilde{\mathcal{S}}$ over which this happens. The result is still denoted by $\tilde{\mathcal{X}} \to \tilde{\mathcal{S}}$.

4.3.2. Development of the infinitesimal invariant $\nabla R_{\mathbf{f}_s}$. We now consider established the existence of $\{\mathbf{F}\} \in \ker(\text{Tame})$ on this new $\tilde{\mathcal{X}}$, giving by fiberwise restriction $\{\mathbf{f}_s\} \in \ker(\text{Tame}) \subseteq K_n^M(\mathbb{C}(X_s))$. There are corresponding regulator currents $R_{\mathbf{F}} \in \Gamma({}'\mathcal{D}^{n-1}_{\tilde{\mathcal{X}}})$ generating by fiberwise pullback $\iota^*_{X_s} R_{\mathbf{F}} = R_{\mathbf{f}_s} \in \Gamma({}'\mathcal{D}^{n-1}_{X_s})$. Writing $\tilde{V} = V_{\mathbf{F}} \subset \tilde{\mathcal{X}}$ and $V_s = V_{\mathbf{f}_s} = \tilde{V} \cap X_s$, we want to differentiate the periods of $R_{\mathbf{f}_s}$ on cycles $\mathcal{C}_s \subset X_s \setminus V_s$. Note that $R_{\mathbf{f}_s}$ is not even d-closed on $X_s \setminus V_s$ (i.e. as a current in the complex ${}'\mathcal{D}^{\bullet}_{(X_s \setminus V_s)^\infty}$), but can be made so without chaging the periods $\in \mathbb{C}/\mathbb{Q}(n)$ as follows: trivially by dimension $\Omega_{\mathbf{f}_s} = \bigwedge^n \mathrm{dlog} \mathbf{f}_s = 0$, so there exists (as in §2.2.1) $R'_{\mathbf{f}_s} = R_{\mathbf{f}_s} + (2\pi i)^n \partial^{-1}_{(X_s, V_s)} T_{\mathbf{f}_s}$. This is what we must lift to differentiate periods, to some kind of $R'_{\mathbf{F}}$. While this is impossible globally on $\tilde{\mathcal{X}}$

(see below), it is possible to construct $R'_{\mathbf{F}}$ on $\pi^{-1}(\mathcal{U}) =: \mathcal{X}_{\mathcal{U}} \subset \tilde{\mathcal{X}}$ for $\mathcal{U} \subset \tilde{\mathcal{S}}$ a small acyclic neighborhood of 0.

Indeed for $\mathcal{U}$ sufficiently small there is a relative homeomorphism $(\mathcal{X}_{\mathcal{U}}, V_{\mathcal{U}}) \simeq (X_0 \times \mathcal{U}, V_0 \times \mathcal{U})$ under which the image of $T_{\mathbf{F}} \cap \mathcal{X}_{\mathcal{U}}$ is homologous to $T_{\mathbf{f}_0} \times \mathcal{U}$ (which is homologous to zero) $\mod (X_0 \times \partial\mathcal{U}) \cup (V_0 \times \mathcal{U})$; and so $T_{\mathbf{F}} \cap \mathcal{X}_{\mathcal{U}} \sim 0 \mod \mathcal{X}_{\partial\mathcal{U}} \cup V_{\mathcal{U}}$. Alternatively one could consider $\Omega_{\mathbf{F}} = \bigwedge^n \mathrm{dlog}\mathbf{F}$ on $\mathcal{X}_{\mathcal{U}} \setminus V_{\mathcal{U}}$ as a closed (holomorphic) n-form with[26] $\iota_0^* \Omega_{\mathbf{F}} = \Omega_{\mathbf{f}_0} = 0$ and note that $[\iota_0^*]$ factors (ignoring Hodge type)

$$H^n(\mathcal{X}_{\mathcal{U}} \setminus V_{\mathcal{U}}) \cong H^n((X_0 \setminus V_0) \times \mathcal{U}) \cong H^n(X_0 \setminus V_0) \times H^0(\mathcal{U}) \xrightarrow{\cong} H^n(X_0 \setminus V_0);$$

therefore $[\Omega_{\mathbf{F}}] = 0$ as a *class* in $H^n(\mathcal{X}_{\mathcal{U}} \setminus V_{\mathcal{U}})$. (This is a naive use of the rigidity of H^*_{DR}.) Since on $\mathcal{X}_{\mathcal{U}} \setminus V_{\mathcal{U}}$ (in fact on $\tilde{\mathcal{X}} \setminus \tilde{V}$) $\mathrm{d}[R_{\mathbf{F}}] = \Omega_{\mathbf{F}} - (2\pi i)^n T_{\mathbf{F}}$, we see that $[T_{\mathbf{F}}] = 0$ in $H_*(\mathcal{X}_{\mathcal{U}}, V_{\mathcal{U}} \cup \mathcal{X}_{\partial\mathcal{U}})$.

Either way we have a chain $\partial^{-1} T_{\mathbf{F}}$ whose boundary is $T_{\mathbf{F}}$, plus stuff on $V_{\mathcal{U}} \cup \mathcal{X}_{\partial\mathcal{U}}$; and writing $R'_{\mathbf{F}} := R_{\mathbf{F}} + (2\pi i)^n \partial^{-1} T_{\mathbf{F}}$ we have an $(n-1)$-current on $\mathcal{X}_{\mathcal{U}}$ which *on* $\mathcal{X}_{\mathcal{U}} \setminus V_{\mathcal{U}}$ has coboundary $\mathrm{d}[R'_{\mathbf{F}}] = \Omega_{\mathbf{F}}$.

Now let $\{\mathcal{C}_s\}_{s \in \mathcal{U}}$ be a continuous family of $(n-1)$-cycles on $X_s \setminus V_s$ and

$$[0 \in] \mathcal{U}_i := \mathcal{U} \cap \left\{ t_j = \beta_j^0 \,|\, \forall j \neq i \right\} \subseteq \mathcal{U}$$

be "disks" with holomorphic tangent vectors $\{\partial/\partial t_i\}$. Noting that for $s \in \mathcal{U}$

$$\int_{\mathcal{C}_s} R_{\mathbf{f}_s} \equiv \int_{\mathcal{C}_s} R'_{\mathbf{f}_s} = \int_{\mathcal{C}_s} [\iota_s^*] R'_{\mathbf{F}} \qquad \mod \mathbb{Z}(n)$$

and denoting by $[0, \epsilon]_i$ (with endpoint ϵ_i) a simple path in $\mathcal{U}_i$ from $t_i = \beta_i^0$ to $\beta_i^0 + \epsilon$, we may differentiate periods

$$\left(\nabla_{\partial/\partial t_i} R_{\mathbf{f}_s} \right)_{s=0} (\mathcal{C}_0) := \left(\frac{\partial}{\partial t_i} \int_{\mathcal{C}_s} R_{\mathbf{f}_s} \right)_{s=0} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(\int_{\mathcal{C}_{\epsilon_i}} R'_{\mathbf{F}} - \int_{\mathcal{C}_0} R'_{\mathbf{F}} \right)$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\mathcal{C}_{[0,\epsilon]_i}} \mathrm{d}[R'_{\mathbf{F}}] = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\mathcal{C}_{[0,\epsilon]_i}} \Omega_{\mathbf{F}} = \int_{\mathcal{C}_0} \iota_{X_0}^* \left\langle \widetilde{\partial/\partial t_i}, \bigwedge^n \mathrm{dlog}\mathbf{F} \right\rangle,$$

where the interior product with $\widetilde{\partial/\partial t_i}$ is being taken away from $V_{\mathcal{U}}$. We ask: in what sense is it valid to write

$$\nabla R_{\mathbf{f}_s} = \sum \mathrm{d}t_i \otimes \iota^*_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}} \left\langle \widetilde{\partial/\partial t_i}, \bigwedge^n \mathrm{dlog}\mathbf{F} \right\rangle$$

– in particular, where does the $\iota^*_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}}$-part live? We can do better than the cohomology of $X_s \setminus V_s$.

Up to this point we have not used the fact that $\{\mathbf{F} \in \ker(\mathrm{Tame})$ on $\tilde{\mathcal{X}}$. Recall that the map $K_n^M(\mathbb{C}(\tilde{\mathcal{X}})) \to \Gamma(F^{n\prime}\mathcal{D}^n_{\tilde{\mathcal{X}}})$ given by sending $\{\mathbf{F}\} \mapsto$

[26] It's worth emphsizing the difference between the pullback $\iota_s^* \Omega_{\mathbf{F}} \in H^0(\Omega^n_{X_s})$ (which must be zero as $\dim X_s = n-1$) and the restriction $\Omega_{\mathbf{F}}\big|_{X_s} \in H^0(\Omega^n_{\mathcal{X}} \otimes \mathcal{O}_{X_s})$ which is *not* zero.

$\Omega_{\{\mathbf{F}\}}(:= \Omega_{\mathbf{F}}$ for any representative $\mathbf{F}$) is well-defined, and also that $\mathrm{d}[\Omega_{\{\mathbf{F}\}}] = \Omega_{\mathrm{Tame}\{\mathbf{F}\}}$ which in this case is zero on $\tilde{\mathcal{X}}$. So $\Omega_{\mathbf{F}} = \bigwedge^n \mathrm{dlog}\mathbf{F}$ is a d-closed current on $\tilde{\mathcal{X}}$, and in fact using an argument like that in §1.3.3 one can show it is a holomorphic n-form there. However, the extension of $\{\mathbf{F}\}$ to $\bar{\tilde{\mathcal{X}}}$ is *not* in ker(Tame) and so the current $\Omega_{\mathbf{F}}$ is not closed on the compact $\bar{\tilde{\mathcal{X}}}$; therefore the corresponding holomorphic form on $\tilde{\mathcal{X}}$ may have residues along the "bad" fibers and the line of reasoning in §1.3.3 does *not* show $\bigwedge^n \mathrm{dlog}\mathbf{F} = 0$. The reason $\ker(\mathrm{Tame}^{[1]})$ is enough to work with here, is that $\Omega_{\mathbf{F}}$ has no higher codimension residues (unlike $R_{\mathbf{F}}$).

Now $\iota^*_{X_s}\left\langle \widetilde{\partial/\partial t_i}, \bigwedge^n \mathrm{dlog}\mathbf{F}\right\rangle$ gives a section of $R^0_{\pi_*}\Omega^{n-1}_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}}$ if $\bigwedge^n \mathrm{dlog}\mathbf{F}$ is holomorphic on $\tilde{\mathcal{X}}$; alternately one could use closedness of $\Omega_{\mathbf{F}}$ to prove directly the weaker statement, that $\iota^*_{X_s}\left\langle \widetilde{\partial/\partial t_i}, \Omega_{\mathbf{F}}\right\rangle$ can be replaced by a holomorphic $(n-1)$-form on X_s as far as its periods on $\mathcal{C}_s \subset X_s \setminus V_s$ (indeed on *any* cycle on X_t) are concerned. This is done simply by checking that it is a d-closed element of $\Gamma(F^{n-1\prime}\mathcal{D}^{n-1}_{X_s})$, by integrating against $\mathrm{d}\alpha$ (for $\alpha \in \Gamma(\Omega^{n-1}_{(X_s)^\infty})$) and $\alpha_1 \in Z^0(F^1\Omega^{n-2}_{(X_s)^\infty})$ [i.e., $\mathrm{d}\alpha_1 = 0$]:

$$\int_{X_0} \alpha \wedge \mathrm{d}\left[\left\langle \widetilde{\partial/\partial t_i}, \Omega_{\mathbf{F}}\right\rangle\right] := \int_{X_0} \mathrm{d}\alpha \wedge \left\langle \widetilde{\partial/\partial t_i}, \Omega_{\mathbf{F}}\right\rangle = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{\mathcal{X}_{[0,\epsilon]_i}} \mathrm{d}\tilde{\alpha}\wedge\Omega_{\mathbf{F}}$$

$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\int_{\mathcal{X}_{[0,\epsilon]_i}} \tilde{\alpha}\wedge \mathrm{d}[\Omega_{\mathbf{F}}] + \int_{X_\epsilon} \alpha_\epsilon \wedge \Omega_{\mathbf{f}_s} - \int_{X_0} \alpha_0 \wedge \Omega_{\mathbf{f}_0}\right)$$

which is 0 ($\mathrm{d}[\Omega_{\mathbf{F}}] = 0$, $\Omega_{\mathbf{f}_s} = \Omega_{\mathbf{f}_0} = 0$) and

$$\int_{X_0} \alpha_1 \wedge \left\langle \widetilde{\partial/\partial t_i}, \bigwedge^n \mathrm{dlog}\mathbf{F}\right\rangle = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{\mathcal{X}_{[0,\epsilon]_i}\subseteq \mathcal{X}_{\mathcal{U}_i}} \tilde{\alpha}_1 \wedge \bigwedge^n \mathrm{dlog}\mathbf{F} = 0$$

because $\mathcal{X}_{\mathcal{U}_i}$ can only support n $\mathrm{d}z$'s.

So differentiating the periods $\int_{\mathcal{C}_s} R_{\mathbf{f}_s}$ *gives rise to* a section

$$\sum \mathrm{d}t_i \otimes \iota^*_{X_s}\left\langle \widetilde{\partial/\partial t_i}, \Omega_{\mathbf{F}}\right\rangle \quad \text{of} \quad \Omega^1_{\tilde{\mathcal{S}}} \otimes R^0_{\pi_*}\Omega^{n-1}_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}} \hookrightarrow \Omega^1_{\tilde{\mathcal{S}}} \otimes R^{n-1}_{\pi_*}\mathbb{C},$$

and it is fair to ask whether this goes to zero under the map

$$\Omega^1_{\tilde{\mathcal{S}}} \otimes R^{n-1}_{\pi_*}\mathbb{C} \longrightarrow \Omega^2_{\tilde{\mathcal{S}}} \otimes R^{n-1}_{\pi_*}\mathbb{C}.$$

This is what we will prove; the weaker statement that it goes to zero under the map of graded pieces

$$\Omega^1_{\tilde{\mathcal{S}}} \otimes R^0_{\pi_*}\Omega^{n-1}_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}} \overset{\bar{\nabla}^2_{(n-1)}}{\longrightarrow} \Omega^2_{\tilde{\mathcal{S}}} \otimes R^1_{\pi_*}\Omega^{n-2}_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}}$$

then follows automatically.

4.3.3. Killing the infinitesimal invariant. (Rigidity). The present situation is summarized by the diagram (over $\mathcal{U}$)

$$\begin{array}{ccccc}
 & & \nu_s := \sum \mathrm{d}t_i \otimes \iota^*_{X_s} < \widetilde{\partial/\partial t_i}, \bigwedge^n \mathrm{dlog}\mathbf{F} > & \longmapsto & ? \\
 & & \Omega^1_{\tilde{\mathcal{S}}} \otimes \mathcal{F}^{n-1}\mathcal{H}^{n-1}_{X_s} & \xrightarrow{\nabla} & \Omega^2_{\tilde{\mathcal{S}}} \otimes \mathcal{H}^{n-1}_{X_s} \\
 & & \downarrow & & \downarrow \\
\mathcal{H}^{n-1}_{X_s \setminus V_s} & \longrightarrow & \Omega^1_{\mathcal{S}} \otimes \mathcal{H}^{n-1}_{X_s \setminus V_s} & \longrightarrow & \Omega^2_{\mathcal{S}} \otimes \mathcal{H}^{n-1}_{X_s \setminus V_s} \\
R'_{\mathbf{f}_s} & \longmapsto & "\nabla R'_{\mathbf{f}_s}" & \longmapsto & 0
\end{array}$$

where the question arises from the fact that ν_s (which is mapping vertically to $\nabla R'_{\mathbf{f}_s}$) is not given by differentiating periods of something on X_s. The question would be resolved if we could show that the right-hand vertical map is "almost injective" in the sense that

$$\mathcal{K}_s := \ker\left\{H^{n-1}(X_s, \mathbb{C}) \to H^{n-1}(X_s \setminus V_s, \mathbb{C})\right\} = \mathrm{im}\left\{H^{n-1}_{V_s}(X_s, \mathbb{C}) \to H^{n-1}(X_s, \mathbb{C})\right\}$$

is just

$$\mathrm{im}\left\{H^{n-1}(\mathbb{P}^n, \mathbb{C}) \to H^{n-1}(X_s, \mathbb{C})\right\}.$$

Then since[27] $\nu_s \in \Omega^1_{\tilde{\mathcal{S}}} \otimes \mathcal{H}^{n-1}_{X_s,pr}$, and $\nabla\nu_s \in \Omega^2_{\tilde{\mathcal{S}}} \otimes \mathcal{K}_s$ (by the diagram),

$$\nabla\nu_s \in \Omega^2_{\tilde{\mathcal{S}}} \otimes \mathcal{H}^{n-1}_{X_s,pr} \xrightarrow{\cong} \Omega^2_{\tilde{\mathcal{S}}} \otimes \mathcal{H}^{n-1}_{X_s,var} = \Omega^2_{\tilde{\mathcal{S}}} \otimes \mathcal{H}^{n-1}_{X_s} / \mathcal{K}_s$$

must be zero.[28] Note that the statement about $\mathcal{K}_s$ is equivalent to the statement regarding cycles that the natural injection

$$\mathrm{im}\left\{H_{n-1}(X_s \setminus V_s, \mathbb{Q}) \to H_{n-1}(X_s, \mathbb{Q})\right\} \to \ker\left\{H_{n-1}(X_s, \mathbb{Q}) \to H_{n-1}(\mathbb{P}^n, \mathbb{Q})\right\}$$

[27] Assuming n odd (so there's an issue), a form gives a class in $H^{n-1}_{pr}(X_s, \mathbb{C})$ if it integrates to zero on cycles in the (1-dimensional) image of $H_{n+1}(\mathbb{P}^n, \mathbb{Q}) \to H_{n-1}(X_s, \mathbb{Q})$. Since this image is spanned by the cycle $X_s \cap \mathbb{P}^{\frac{n+1}{2}}$, trivially by Hodge theory a holomorphic form ω_s (like $\iota^*_{X_s}\left\langle \widetilde{\partial/\partial t_i}, \bigwedge^n \mathrm{dlog}\mathbf{F} \right\rangle$) must be "primitive". The new cohomology class $[\nabla_{\partial/\partial t_j}\omega_s] \in F^{n-2}H^{n-1}(X_s, \mathbb{C})$ obtained by differentiating its periods is also primitive, since

$$\int_{X \cap \mathbb{P}^{\frac{n+1}{2}}} \nabla_{\partial/\partial t_j}\omega_s = \frac{\partial}{\partial t_j} \int_{X \cap \mathbb{P}^{\frac{n+1}{2}}} \omega_s = \frac{\partial}{\partial t_j}(0) = 0.$$

[28] This is also clear in principle from the fact that $\bigwedge^n \mathrm{dlog}\mathbf{F}$ is closed (as a current on $\tilde{\mathcal{X}}$). We have chosen this kind of argument because we have to deal with K_s in any case!

is an equality. In words: for a very general hypersurface in $\mathbb{P}^n$ (of sufficiently high degree, as we shall see), one can still "get" all the "interesting" cycles even if they have to avoid an arbitrary configuration of divisors.

So it is left to show that

$$(\mathcal{K}_s =)\mathrm{im}\left\{H_{V_s}(X_s,\mathbb{C}) \to H^{n-1}(X_s,\mathbb{C})\right\} \overset{?}{=} \mathrm{im}\left\{H^{n-1}(\mathbb{P}^n,\mathbb{C}) \to H^{n-1}(X_s,\mathbb{C})\right\}$$

for all s in some Zariski-open subset of $\tilde{\mathcal{S}}$ (which as usual we rename $\tilde{\mathcal{S}}$). For s very general (say, $= 0$) we know that we can choose a small analytic neighborhood (say, $\mathcal{U}$) where for all $s \in \mathcal{U}$ the relative situation (X_s, V_s) is topologically identical to that at $s = 0$. As the above equality is an "algebraic" matter, if it holds on $\mathcal{U}$ then it holds on a Zariski-open set $\supset \mathcal{U}$. So we will prove it on $\mathcal{U}$, assuming that $K_{X_0} \geq 0$ (that is, $D \geq n+1$), in order that $H^0(\Omega^{n-1}_{X_0}) \cong H^0(\Omega^n_{\mathbb{P}^n}(\mathrm{dlog}X)) \cong H^0(K_{\mathbb{P}^n}(D)) \cong S^{D-n-1} \neq \{0\}$. Otherwise already $\nu_s = 0$ ($\Longrightarrow \nabla R'_{\mathbf{f}_s} = 0$) and we don't need this result.[29]

Now we make two key observations about the left-hand image $\mathcal{K}_s$: the first is that it is generated over $\mathbb{Q}$ (one could just change $\mathbb{C}$ to $\mathbb{Q}$ and tensor by $\mathbb{C}$ on the outside), and so one may choose a local *basis* of rational sections

$$\{\sigma_k\} \in \Gamma(\mathcal{U}, R^{n-1}_{\pi_*}\mathbb{Q}) \qquad \text{for} \quad \Gamma(\mathcal{U},\mathcal{K}_s).$$

These are necessarily flat ($\nabla\sigma_k \in \Omega^1_{\mathcal{U}} \otimes R^{n-1}_{\pi_*}\mathbb{C}$ are zero), essentially by local rigidity of periods $\in \mathbb{Q}$. The second observation is that, since $V_s \subset X_s$ is codimension 1, its cohomology has image $\mathcal{K}_s \subseteq F^1H^{n-1}(X_s,\mathbb{C})$. Putting these together, we find that the $\{\sigma_k\}$ are sections over $\mathcal{U}$ of

$$\ker\left\{\mathcal{F}^1\mathcal{H}^{n-1}_{X_s} \overset{\nabla}{\longrightarrow} (\mathcal{F}^0)\mathcal{H}^{n-1}_{X_s} \otimes \Omega^1_{\tilde{\mathcal{S}}}\right\}.$$

Taking the quotient by $\mathrm{im}\left\{H^{n-1}(\mathbb{P}^n,\mathbb{C}) \to H^{n-1}(X_s,\mathbb{C})\right\}$ gives sections $\{\sigma_{k,var}\}$ of the middle sheaf in the short exact sequence

$$0 \to \ker\left\{\mathcal{F}^2\mathcal{H}^{n-1}_{X_s,var} \overset{\nabla}{\longrightarrow} \mathcal{F}^1\mathcal{H}^{n-1}_{X_s,var} \otimes \Omega^1_{\tilde{\mathcal{S}}}\right\} \to \ker\left\{\mathcal{F}^1\mathcal{H}^{n-1}_{X_s,var} \overset{\nabla}{\longrightarrow} (\mathcal{F}^0)\mathcal{H}^{n-1}_{X_s,var} \otimes \Omega^1_{\tilde{\mathcal{S}}}\right\}$$

$$\to \ker\left\{Gr^1_{\mathcal{F}}\mathcal{H}^{n-1}_{X_s,var} \overset{\bar{\nabla}_{(1)}}{\longrightarrow} Gr^0_{\mathcal{F}}\mathcal{H}^{n-1}_{X_s,var} \otimes \Omega^1_{\tilde{\mathcal{S}}}\right\} \to 0.$$

Since $D \geq n+1$, §4.2.5 applies to show that $\bar{\nabla}_{(1)}$ is injective, and the last term is zero (and $\{\sigma_{k,var}\}$ pulls up to the first term). Now one simply increases all the $\mathcal{F}$ and $Gr_{\mathcal{F}}$ superscripts by 1, writes this out again with $\{\sigma_{k,var}\}$ in

[29] which in fact fails without the degree bound ($D \geq n+1$). For instance, a "general" quadric surface $\subset \mathbb{P}^{3[=n]}$ is birational to $\mathbb{P}^1 \times \mathbb{P}^1$, and removing the divisors $\{0\} \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \{0\}$ from this eliminates H_2 completely.

the middle and repeats the bootstrapping procedure. This continues, using injectivity of $\bar{\nabla}_{(2)}$, $\bar{\nabla}_{(3)}$, etc. until $\{\sigma_{k,var}\}$ winds up in $\mathcal{F}^{\frac{n}{2}}\mathcal{H}^{n-1}_{X_s,pr} \cap R^{n-1}_{\pi_*}\mathbb{Q}$ for n even or $\mathcal{F}^{\frac{n+1}{2}}\mathcal{H}^{n-1}_{X_s,var} \cap R^{n-1}_{\pi_*}\mathbb{Q}$ for n odd, which are zero. So there is only one σ_k (as they were a basis) and it was in $\Gamma\left(\mathcal{U}, \mathrm{im}\left\{\mathcal{H}^{n-1}_{\mathbb{P}^n} \to \mathcal{H}^{n-1}_{X_s}\right\}\right)$. This completes the proof that $\nabla\nu_s = 0$ on $\tilde{\mathcal{S}}$ for all degrees D.

So in particular ν_s gives a section of

$$\ker\left\{\bar{\nabla}^2_{(n-1)} : \Omega^1_{\tilde{\mathcal{S}}} \otimes \mathcal{F}^{n-1}\mathcal{H}^{n-1}_{X_s(,var)} \to \Omega^2_{\tilde{\mathcal{S}}} \otimes Gr^{n-2}_{\mathcal{F}}\mathcal{H}^{n-1}_{X_s,var}\right\},$$

which was shown to be zero in the last section for $(n-1)(D-1) \geq 3$. Recalling that this argument was valid for $D \geq n+1$, and that we had *trivially* $\nu_s = 0$ for $D < n+1$, wee see that it is possible to have $\nu_s \neq 0$ only if $D \geq n+1$ and $(n-1)(D-1) \leq 2$. For $n \geq 2$ the only solution to these inequalities is $n = 2$, $D = 3$, the case of a family of elliptic curves. This establishes the first result promised at the outset: that the infinitesimal invariant $\nabla R_{\mathbf{f}_s} = 0$ for any section $\{\mathbf{f}_s\} \in \ker(\mathrm{Tame}) \subseteq K^M_n(\mathbb{C}(X_s))$ over [a cover of a Zariski-open subset of] a complete linear family of hypersurfaces $\subset \mathbb{P}^n$ [elliptic curves in $\mathbb{P}^2$ excepted].[30] So the regulator image $\int_{(\cdot)} R_{\mathbf{f}_s}$ in $H^{n-1}(X_s \setminus V_s, \mathbb{C}/\mathbb{Q}(n))$ is flat (the periods are constant), but that's all;[31] indeed this tells us nothing about the image of the regulator on the $\{\mathbf{f}_0\} \in \ker(\mathrm{Tame}) \subseteq K^M_n(\mathbb{C}(X_0))$ we started from. So we start over with the stronger criterion on $\{\mathbf{f}_0\}$ and give a very brisk, but at this point much easier, argument.

4.3.4. Vanishing argument for $K^M_n(X)$. Let $\mathbf{f}_0$ be a "good" representative of any class in $K^M_n(X_0) \subseteq K^M_n(\mathbb{C}(X_0))$ so that $\overline{\gamma_{\mathbf{f}_0}}$ is the generic (=codimension 0) component of a ($\partial_{\mathcal{B}}$-closed) higher Chow cycle $\Gamma_0 \in Z^n(X_0, n)$. The pair $(\mathbf{f}_0, \Gamma_0)$ then spreads to $(\mathbf{F}, \tilde{\Gamma})$ on $\tilde{\mathcal{X}} \to \tilde{\mathcal{S}}$ as before, where $\tilde{\Gamma} \in Z^n(\tilde{\mathcal{X}}, n)$ is $\partial_{\mathcal{B}}$-closed (but its closure on $\tilde{\bar{\mathcal{X}}}$ is not). Writing γ^i_s for components of codimension ≥ 1, this gives by restriction a family of higher Chow cycles

$$\Gamma_s = \overline{\gamma_{\mathbf{f}_s}} + \sum_{i \geq 1} \gamma^i_s \qquad \left(\sum_i \mathrm{supp}(\pi_{X_s}(\gamma^i_s)) =: V_s\right)$$

giving rise to currents

$$R_{\Gamma_s} = R_{\mathbf{f}_s} + \sum_{i \geq 1} R_{\gamma^i_s}, \qquad \mathrm{d}[R_{\Gamma_s}] = -(2\pi i)^n T_{\Gamma_s}$$

(so that one has R'_{Γ_s} d-closed) since $\Omega_{\Gamma_s} = 0$ by type considerations (on X_s). In fact on $\tilde{\mathcal{X}}$, also by type, the current $\Omega_{\tilde{\Gamma}}$ has no codimension ≥ 1 "components"; so $\Omega_{\tilde{\Gamma}} = \Omega_{\mathbf{F}} = \bigwedge^n \mathrm{dlog}\mathbf{F}$, which is once again holomorphic on

[30] In this case Collino has constructed a family $\mathbf{f}_s$ with infinitesimal invariant he shows to be zero by means of thetanulls.

[31] For a monodromy argument we need $[R_{\mathbf{f}_s}] \in \mathrm{im}\left\{H^{n-1}(X_s) \to H^{n-1}(X_s \setminus V_s)\right\}$ (already true for $n = 2, 3$ but we are working more generally).

$\tilde{\mathcal{X}}$ with possible poles on $\tilde{\bar{\mathcal{X}}}$. It has trivial class locally, e.g. over $\mathcal{U}$, so that on $\mathcal{X}_{\mathcal{U}}$ one can write a $R'_{\tilde{\Gamma}}$ with

$$\iota^*_{X_s} R'_{\tilde{\Gamma}} = R'_{\Gamma_s}, \qquad \mathrm{d}[R'_{\tilde{\Gamma}}] = \bigwedge^n \mathrm{dlog}\mathbf{F}$$

for all $s \in \mathcal{U}$.

Lifting $\mathbf{f}_s$ to Γ_s has saved us from the headache of working away from V_s. There are actually two lifts going on here: since $\{\mathbf{f}_s\} \in K_n^M(X_s) \subseteq K_n^M(\mathbb{C}(X_s))$ the class $[R_{\mathbf{f}_s}] \in \mathrm{im}\left\{H^{n-1}(X_s, \mathbb{C}/\mathbb{Q}(n)) \to H^{n-1}(X_s \setminus V_s, \mathbb{C}/\mathbb{Q}(n))\right\}$; $[R_{\Gamma_s}] \in H^{n-1}(X_s, \mathbb{C}/\mathbb{Q}(n))$ gives a global lift over $\tilde{\mathcal{S}}$, of which $[R'_{\Gamma_s}] \in H^{n-1}(X_s, \mathbb{C})$ is a local lift, e.g. over $\mathcal{U}$. However, it may be analytically continued to a "multivalued section" over all of $\tilde{\mathcal{S}}$, and one can look at its monodromy in $H^{n-1}(X_s, \mathbb{Q}(n))$.

Before doing this we show locally that $[R'_{\Gamma_s}]$ is flat, i.e. $\nabla R'_{\Gamma_s} = 0$. Since $\mathrm{d}[R'_{\tilde{\Gamma}}] = \bigwedge^n \mathrm{dlog}\mathbf{F}$ we have as before[32]

$$\nabla R'_{\Gamma_s} = \sum \mathrm{d}t_i \otimes \left\langle \widetilde{\partial/\partial t_i}, \bigwedge^n \mathrm{dlog}\mathbf{F} \right\rangle \in \ker\left\{\Omega^1_{\tilde{\mathcal{S}}} \otimes \mathcal{F}^{n-1}\mathcal{H}^{n-1}_{X_s} \to \Omega^2_{\tilde{\mathcal{S}}} \otimes Gr^{n-2}_{\mathcal{F}}\mathcal{H}^{n-2}_{X_s,var}\right\},$$

which is zero except for $n = 2$, $D = 3$ by Donagi-Green.

Describing monodromy by the map

$$\rho : \pi_1(\tilde{\mathcal{S}}, 0) \to \mathrm{Aut}\left\{H^{n-1}(X_0, \mathbb{C})\right\},$$

we note the difference of classes

$$\rho(\alpha)[R'_{\Gamma_0}] - [R'_{\Gamma_0}] \in H^{n-1}(X_0, \mathbb{Q}(n))$$

since they both go to the same $[R_{\Gamma_0}] \in H^{n-1}(X_0, \mathbb{C}/\mathbb{Q}(n))$. Now recall that if α goes around a divisor in $\tilde{\bar{\mathcal{S}}} \setminus \tilde{\mathcal{S}}$ over which X_s acquires a node ($[R_{\Gamma_s}]$ need not be defined there), we may speak of the vanishing cycle $\delta \in H_{n-1}(X_0, \mathbb{Q})$ associated to α, whose "flat transport" to the nodal X_s is homologous to zero. It is a fact that such δ span $\ker\{H_{n-1}(X_0, \mathbb{Q}) \to H_{n-1}(\mathbb{P}^n, \mathbb{Q})\}$; let $\{\delta_i\}$ be a basis (with associated loops α_i) and $\{\hat{\delta}_i\}$ a dual basis[33] for $H^{n-1}_{var}(X_0, \mathbb{Q})$, which one easily lifts to $H^{n-1}_{pr}(X_0, \mathbb{Q})$.

Since $[R'_{\Gamma_s}]$ is flat, the Picard-Lefschetz formula (see **[L1]** or **[GH]**) applies to compute

$$\rho(\alpha_i)[R'_{\Gamma_0}] - [R'_{\Gamma_0}] = \pm\left(\int_{\delta_i} R'_{\Gamma_o}\right) \cdot \hat{\delta}_i.$$

Combined with the above this gives *immediately*

$$\int_{\delta} R'_{\Gamma_0} \in \mathbb{Q}(n), \qquad \forall \delta \in \ker\{H_{n-1}(X_0, \mathbb{Q}) \to H_{n-1}(\mathbb{P}^n, \mathbb{Q})\},$$

[32]In fact this is better than before, as we may skip all the $\mathcal{K}_s$ business: ∇R_{Γ_s} *is* ∇ of something in $\mathcal{H}^{n-1}_{X_s}$.

[33]that is, $\int_{\delta_i} \hat{\delta}_j = \delta_{ij}$.

which is to say

$$[R_{\Gamma_0}] \in \mathrm{im}\left\{H^{n-1}(\mathbb{P}^n, \mathbb{C}/\mathbb{Q}(n)) \to H^{n-1}(X_0, \mathbb{C}/\mathbb{Q}(n))\right\};$$

and so (except for $n = 2$, $D = 3$)

$$[R_{\mathbf{f}_0}] = 0 \in H^{n-1}(X_0 \setminus V_0, \mathbb{C}/\mathbb{Q}(n)),$$

since the composition $H^{n-1}(\mathbb{P}^n) \to H^{n-1}(X_s) \to H^{n-1}(X_s \setminus V_s)$ is zero.

This proves the vanishing theorem in codimension 1. One striking consequence is that, if $\{\mathbf{f}_0\}$, $\{\mathbf{f}'_0\} \in K_n^M(\mathbb{C}(X_0))$ have all the same Res^i then they have the same regulator image in $H^{n-1}(X_0 \setminus V_0, \mathbb{C}/\mathbb{Q}(n))$ (because their difference is in $K_n^M(X_0)$); so residues *determine* the image. By the earlier result they are also *rigid* in the family if Res^1 vanishes. This makes one wonder if there are *geometrically* constructible examples (on arithmetically uninteresting X_0 at dimension ≥ 3) of regulator currents with interesting [polylogarithmic] higher residues?

4.4. The Vanishing Theorem in Codimension 2

4.4.1. Extending the Koszul complex to capture $H^0(\Omega^n_Y)$. Let Y be a fixed smooth projective hypersurface (of degree D_Y) in $\mathbb{P}^{n+1}$, and let $L = \mathcal{O}_Y(D_X)$ be very ample.[34] Consider the family[35]

$$\mathcal{X} \to \mathcal{S} \underset{\text{Zar. op.}}{\subseteq} \mathbb{P}H^0(Y, L)$$

of smooth hypersurfaces X_s (with dimension still $= n-1$). We will prove the vanishing of the holomorphic regulator image for a very general member of this family provided $L \gg 0$. Although X_s *is* a complete intersection $\subset \mathbb{P}^{n+1}$ of multidegree (D_Y, D_X), we do not also vary Y in the family. And so this result does not belong to §4.5, where we consider, say, the family of *all* complete intersections of multidegree (D_Y, D_X). Moreover, while we do not produce concrete (lower) degree bounds on D_X (to quantify "$L \gg 0$"), one can obtain such bounds and they are quite high. It may be that the methods employed were not optimal, or it could be a result of examining a subfamily of the complete family of §4.5 (doubtful, but it would interesting from the geometric-constructive perspective mentioned on §4.1.1).

In a picture our situation is

[34]The notation $L \gg 0$ (or "for L sufficiently ample") will always mean simply "for D_X sufficiently large".

[35](we no longer differentiate between $\tilde{\mathcal{X}} \to \tilde{\mathcal{S}}$ and $\mathcal{X} \to \mathcal{S}$.)

$$
\begin{tikzcd}
\mathcal{X} \arrow[r, hook, "\iota_{\mathcal{X}}"] \arrow[rr, bend left, "(\iota_{\mathcal{X}})"] \arrow[dr, "\pi"'] & \mathcal{Y} = Y \times \mathcal{S} \arrow[r, "p_Y"] \arrow[d, "\pi"] & Y \subset \mathbb{P}^{n+1} \\
& \mathcal{S} &
\end{tikzcd}
$$

Unlike $\mathbb{P}^n$, Y may have interesting $H^{n,0}$, and (as we shall see) the right Koszul complex to use for the infinitesimal invariant does not begin with $\mathcal{H}^{n-1,0}_{X_s} \otimes \Omega^1_{\mathcal{S}}$ but with $H^0(\Omega^n_Y) \otimes \mathcal{O}_{\mathcal{S}}$. We begin by introducing the first map $'\bar{\nabla}$ in the complex, which links these two sheaves and "precedes" the Gauss-Manin maps, and which corresponds to cupping with a "gradient" instead of the Kodaira-Spencer class. Define $'\bar{\nabla}$ as the composition

$$
\begin{tikzcd}
R^0_{\pi_*}\Omega^n_{\mathcal{Y}/\mathcal{S}} \arrow[r, equal] \arrow[d, "p_Y^*"] & \mathcal{O}_{\mathcal{S}} \otimes H^0(\Omega^n_Y) \arrow[dd, dashed, "'\bar{\nabla}"] \\
R^0_{\pi_*}\Omega^n_{\mathcal{Y}} \arrow[d, "\iota_{\mathcal{X}}^*"] & \\
R^0_{\pi_*}\Omega^n_{\mathcal{X}} \arrow[r, "\alpha"'] & R^0_{\pi_*}\Omega^{n-1}_{\mathcal{X}/\mathcal{S}} \otimes \Omega^1_{\mathcal{S}}
\end{tikzcd}
$$

where α is given by considering once again the short exact sequences used in defining the spectral sequence for $\bar{\nabla}$

$$
\mathcal{L}^1\Omega^n_{\mathcal{X}} \to \Omega^n_{\mathcal{X}} \to 0 \quad , \qquad \mathcal{L}^2\Omega^n_{\mathcal{X}} \to \mathcal{L}^1\Omega^n_{\mathcal{X}} \to \Omega^{n-1}_{\mathcal{X}/\mathcal{S}} \otimes \pi^*\Omega^1_{\mathcal{S}},
$$

$$
\mathcal{L}^3\Omega^n_{\mathcal{X}} \to \mathcal{L}^2\Omega^n_{\mathcal{X}} \to \Omega^{n-2}_{\mathcal{X}/\mathcal{S}} \otimes \pi^*\Omega^2_{\mathcal{S}},
$$

and building out of them the standard construction for exact $\triangle$'s (or as close as we can get)

$$
\begin{tikzcd}
& R^0_{\pi_*}\mathcal{L}^1\Omega^n_{\mathcal{X}} \arrow[r, equal] \arrow[d, equal] & R^0_{\pi_*}\Omega^n_{\mathcal{X}} \arrow[d, dashed, "\alpha"] & & & \\
{} \arrow[r] & R^0_{\pi_*}\mathcal{L}^1\Omega^n_{\mathcal{X}} \arrow[r] \arrow[rr, squiggly, bend right, "\text{zero}"'] & R^0_{\pi_*}\Omega^{n-1}_{\mathcal{X}/\mathcal{S}} \otimes \Omega^1_{\mathcal{S}} \arrow[r] \arrow[drr, bend right] & R^1_{\pi_*}\mathcal{L}^2\Omega^n_{\mathcal{X}} \arrow[r] \arrow[d, equal, "\bar{\nabla}"] & {} & \\
& & {} \arrow[r] & R^1_{\pi_*}\mathcal{L}^2\Omega^n_{\mathcal{X}} \arrow[r] & R^1_{\pi_*}\Omega^{n-2}_{\mathcal{X}/\mathcal{S}} \otimes \Omega^2_{\mathcal{S}} \arrow[r] & {}
\end{tikzcd}
$$

which both defines α and shows the composition $\bar{\nabla} \circ \alpha = 0$, and therefore $\bar{\nabla} \circ {}'\bar{\nabla} = 0$. We shall often omit p_Y^* and write ${}'\bar{\nabla} = \alpha \circ \iota_{\mathcal{X}}^*$ (i.e. identify a form on Y with its pullback to $Y \times \mathcal{S}$).

Note that it is possible to define a ${}'\bar{\nabla}_{(2)}$ on $\Omega^1_{\mathcal{S}} \otimes H^0(\Omega^n_Y)$ analogously by writing down the long exact sequences derived from

$$\mathcal{L}^2\Omega^{n+1}_{\mathcal{X}} \to \Omega^{n+1}_{\mathcal{X}} \to 0 \ , \qquad \mathcal{L}^3\Omega^{n+1}_{\mathcal{X}} \to \mathcal{L}^2\Omega^{n+1}_{\mathcal{X}} \to \Omega^{n-1}_{\mathcal{X}/\mathcal{S}} \otimes \pi^*\Omega^2_{\mathcal{S}},$$

obtaining

$$\alpha_2 : R^0_{\pi_*}\Omega^{n+1}_{\mathcal{X}} \to R^0_{\pi_*}\Omega^{n-1}_{\mathcal{X}/\mathcal{S}} \otimes \Omega^2_{\mathcal{S}} \, ;$$

and then using

$$\pi^*\Omega^1_{\mathcal{S}} \otimes \Omega^n_{\mathcal{Y}/\mathcal{S}} \xrightarrow{p_Y^*} \Omega^{n+1}_{\mathcal{Y}} \xrightarrow{\iota_{\mathcal{X}}^*} \Omega^{n+1}_{\mathcal{X}}$$

to induce

$$\iota_{\mathcal{X}}^* : \Omega^1_{\mathcal{S}} \otimes R^0_{\pi_*}\Omega^n_{\mathcal{Y}/\mathcal{S}} \to R^0_{\pi_*}\Omega^{n+1}_{\mathcal{X}},$$

we have the composition

$${}'\bar{\nabla}_{(2)} = \alpha_2 \circ \iota_{\mathcal{X}}^* : \Omega^1_{\mathcal{S}} \otimes R^0_{\pi_*}\Omega^n_{\mathcal{Y}/\mathcal{S}} \to \Omega^2_{\mathcal{S}} \otimes R^0_{\pi_*}\Omega^{n-1}_{\mathcal{X}/\mathcal{S}}.$$

To define the gradient, which gives locally a map

$$\theta^1_{\mathcal{S},o} \to H^0(X_0, N_{X_0/Y})$$

describing how X_s moves in Y, consider the "exact square" (where intuitively "$\mathcal{X}/\mathcal{S} = X_s$" and "$\mathcal{Y}/\mathcal{S} = Y$")

$$\begin{tikzcd}
\theta^1_{\mathcal{X}/\mathcal{S}} \arrow[r, "\iota_*^{\mathcal{X}/\mathcal{S}}"] \arrow[d, "i"] & \theta^1_{\mathcal{X}} \arrow[r, "q"] \arrow[d, "{[\iota_*^{\mathcal{X}}]}"] & \pi^*\theta^1_{\mathcal{S}} \arrow[d, "\cong"] \\
\theta^1_{\mathcal{Y}/\mathcal{S}} \arrow[r] \arrow[d, "{[\mathrm{bar}]}"] & \theta^1_{\mathcal{Y}}|_{\mathcal{X}} \arrow[r] \arrow[d] \arrow[l, dashed, shift right=2, "p_*^Y"'] & \pi^*\theta^1_{\mathcal{S}} \arrow[l, dashed, shift right=2] \\
N_{(\mathcal{X}/\mathcal{S})/(\mathcal{Y}/\mathcal{S})} \arrow[r, "\cong"] & N_{\mathcal{X}/\mathcal{Y}} &
\end{tikzcd}$$

of vector fields with support on $\mathcal{X}$, where we write $|_{\mathcal{X}}$ instead of$\otimes\mathcal{O}_{\mathcal{X}}$. Without the backwards dotted arrows (this splitting is given unambiguously by the product structure) this diagram commutes. Some directions commute even then: $i = p_Y \circ [\iota_*^{\mathcal{X}}] \circ \iota_*^{\mathcal{X}/\mathcal{S}}$, so that $[\mathrm{bar}] \circ p_*^Y \circ [\iota_*^{\mathcal{X}}] \circ \iota_*^{\mathcal{X}/\mathcal{S}} = 0$ and

$$[\mathrm{bar}] \circ p_*^Y \circ [\iota_*^{\mathcal{X}}] \circ q^{-1} : \quad \pi^*\theta^1_{\mathcal{S}} \to N_{(\mathcal{X}/\mathcal{S})/(\mathcal{Y}/\mathcal{S})}$$

is a well-defined map of sheaves (even though q^{-1} is ambiguous). Taking $R^0_{\pi_*}$ this induces a map of sheaves on $\mathcal{S}$

$$\theta^1_{\mathcal{S}} \longrightarrow R^0_{\pi_*}N_{(\mathcal{X}/\mathcal{S})/(\mathcal{Y}/\mathcal{S})}$$

given locally by

$$\partial/\partial t_i \longmapsto \overline{p_*^Y(\widetilde{\partial/\partial t_i})} =: \theta_i$$

which gives the desired gradient; this may be thought of as a section

$$\mathrm{grad}(X_s) \in \Omega^1_{\mathcal{S}} \otimes R^0_{\pi_*} N_{(\mathcal{X}/\mathcal{S})/(\mathcal{Y}/\mathcal{S})}$$

which maps by the obvious connecting homomorphism to the Kodaira-Spencer class

$$\eta(X_s) \in \Omega^1_{\mathcal{S}} \otimes R^1_{\pi_*} \theta^1_{\mathcal{X}/\mathcal{S}}.$$

It is easy to see that $'\bar{\nabla}$ is given by cup product with $\mathrm{grad}(X_s)$, so that we may write locally[36]

$$'\bar{\nabla}\left(\sum_i f_i(\mathbf{t}) \otimes \omega^i_Y\right) = \sum_{i,j} f_i(\mathbf{t})\mathrm{d}t_j \otimes \left\langle \theta^{\mathbf{t}}_j \,, \omega^i_Y \Big|_{X_{\mathbf{t}}} \right\rangle_{\mathcal{X}/\mathcal{S}}$$

for $f_i \in H^0(\mathcal{O}_{\mathcal{S}}) \left[\subseteq \mathbb{C}(\bar{\mathcal{S}})\right]$ and (a fixed basis) $\{\omega^i_Y\} \in H^0(\Omega^n_Y)$.

Conclusion. In particular, this $'\bar{\nabla}$ is $\mathcal{O}_{\mathcal{S}}$-linear like $\bar{\nabla}$ (the f_i don't get differentiated).

Remark 4.4.1. This all looks more obvious if we rewrite the $\partial/\partial t_i \to \theta^{\mathbf{t}}_i$ map (using Kodaira vanishing for L sufficiently ample)

$$\left(\theta^1_{\mathcal{S},\mathbf{t}} =\right) T_{\mathbf{t}}\mathbb{P}H^0(Y,L) \overset{(\cong)}{\longrightarrow} H^0(X_{\mathbf{t}}, L\,|_{X_{\mathbf{t}}}) \cong H^0(Y,L)\,/\mathrm{im}_{\mathbf{t}} H^0(\mathcal{O}_Y)$$

which in case $Y = \mathbb{P}^n$, $L = \mathcal{O}_Y(D)$ gives back the familiar isomorphism (observing that $\mathrm{im}_{\mathbf{t}}$ is given by multiplication by $F_{\mathbf{t}}$)

$$T_{\mathbf{t}}\mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}(D)) \overset{\cong}{\longrightarrow} S^D\,/(F_{\mathbf{t}})\,.$$

For reference we remind the reader that Kodaira vanishing says:

$$H^q(\Omega^p_Y(D_X)) = 0 \quad \text{for}$$

$$(a)\ D_X \gg 0 \ \text{ and } \ q > 0 \left[\ \overset{\text{(Serre duality)}}{\Longrightarrow} \quad (a')\ D_X \ll 0 \ \text{ and } \ q < n\right] \quad \text{or}$$

$$(b)\ D_X > 0 \ \text{ and } \ p+q > n\ \left[\quad \Longrightarrow \quad (b')\ D_X < 0 \ \text{ and } \ p+q < n\right].$$

[36] with $f_i(s) = f_i(t_1, \ldots, t_N) = f(\mathbf{t})$; also, we have written $\mathcal{X}/\mathcal{S}$ to emphasize that the interior product takes place *entirely vertically* (on $X_s \subset Y$) and so naturally yields a section of $\mathcal{H}^{n-1,0}_{X_s}$, without taking $\iota^*_{\mathcal{X}/\mathcal{S}}$ afterwards.

4.4.2. Killing the infinitesimal invariant. So far we have shown:

$$\mathcal{O}_{\mathcal{S}} \otimes H^0(\Omega^n_Y) \overset{'\bar{\nabla}}{\longrightarrow} \Omega^1_{\mathcal{S}} \otimes \mathcal{F}^{n-1}\mathcal{H}^{n-1}_{X_s} \overset{\bar{\nabla}}{\longrightarrow} \Omega^2_{\mathcal{S}} \otimes Gr^{n-2}_{\mathcal{F}}\mathcal{H}^{n-1}_{X_s} \to$$

is a *complex* of sheaves on $\mathcal{S}$, where all the maps are $\mathcal{O}_{\mathcal{S}}$-linear (given by cup product with grad and η respectively).

LEMMA 4.4.2. *This is exact at the middle term for $L \gg 0$.*

(We will prove this at the end of the section.) Now if $\{\mathbf{f}_0\} \in K^M_n(X_0)$ for 0 very general, then as in §4.3 we may "spread" to obtain: $\{\mathbf{F}\} \in K^M_n(\mathcal{X})$, $R_\Gamma \in \Gamma('\mathcal{D}^{n-1}_{\mathcal{X}})$ restricting to $R_{\mathbf{F}}$ on $\eta_{\mathcal{X}}$, and to $R_{\mathbf{f}_0}$ on η_{X_0}, and local lifts (say, on a ball in $\mathcal{S}$) R'_Γ having $\mathrm{d}[R'_\Gamma] = \bigwedge^n \mathrm{dlog}\mathbf{F}$ holomorphic (and closed) on $\mathcal{X}$. So

$$\nu_{\mathbf{t}} := "\nabla R'_{\Gamma_{\mathbf{t}}}" = \sum \mathrm{d}t_j \otimes \iota^*_{\mathcal{X}/\mathcal{S}} \left\langle \widetilde{\partial/\partial t_j}, \bigwedge^n \mathrm{dlog}\mathbf{F} \right\rangle$$

gives a section of[37]

$$\ker\left(\Omega^1_{\mathcal{S}} \otimes \mathcal{F}^{n-1}\mathcal{H}^{n-1}_{X_{\mathbf{t}}} \overset{\nabla}{\longrightarrow} \Omega^2_{\mathcal{S}} \otimes \mathcal{F}^{n-2}\mathcal{H}^{n-1}_{X_{\mathbf{t}}} \right) \subseteq$$

$$\ker\left(\Omega^1_{\mathcal{S}} \otimes \mathcal{F}^{n-1}\mathcal{H}^{n-1}_{X_{\mathbf{t}}} \overset{\bar{\nabla}}{\longrightarrow} \Omega^2_{\mathcal{S}} \otimes Gr^{n-2}_{\mathcal{F}}\mathcal{H}^{n-1}_{X_{\mathbf{t}}} \right) = \mathrm{im}\left(\mathcal{O}_{\mathcal{S}} \otimes H^0(\Omega^n_Y) \overset{'\bar{\nabla}}{\longrightarrow} \Omega^1_{\mathcal{S}} \otimes \mathcal{F}^{n-1}\mathcal{H}^{n-1}_{X_{\mathbf{t}}} \right),$$

i.e. we get only that

$$\nu_{\mathbf{t}} = '\bar{\nabla}\left(\sum_i f_i(\mathbf{t}) \otimes \omega^i_Y \right).$$

So we must refine our analysis a bit further to show it vanishes.

Consider the (not necessarily commutative) diagram of sheaves

$$\begin{array}{ccccc} H^0(\Omega^n_Y) \otimes \mathcal{O}_{\mathcal{S}} & \xrightarrow{'\bar{\nabla}} & \mathcal{H}^{n-1,0}_{X_{\mathbf{t}}} \otimes \Omega^1_{\mathcal{S}} & \xrightarrow{\nabla^{(0,1)}} & \mathcal{H}^{n-2,1}_{X_{\mathbf{t}}} \otimes \Omega^2_{\mathcal{S}} \\ \downarrow \mathrm{d} & & \downarrow \nabla^{(1,0)} & & \qquad (*) \\ H^0(\Omega^n_Y) \otimes \Omega^1_{\mathcal{S}} & \xrightarrow{'\bar{\nabla}} & \mathcal{H}^{n-1,0}_{X_{\mathbf{t}}} \otimes \Omega^2_{\mathcal{S}} & & \end{array}$$

where $\nabla : \mathcal{F}^{n-1}\mathcal{H}^{n-1}_{X_{\mathbf{t}}} \otimes \Omega^1_{\mathcal{S}} \to \mathcal{F}^{n-2}\mathcal{H}^{n-1}_{X_{\mathbf{t}}} \otimes \Omega^2_{\mathcal{S}}$ may be split unambiguously by the principle of two types, and $\nabla^{(0,1)}$ is exactly equivalent to $\bar{\nabla}$; moreover d is given by simply

$$\sum_i f_i \otimes \omega^i_Y \longmapsto \sum_i \mathrm{d}f_i \otimes \omega^i_Y.$$

[37] see §4.3.4 (the details are the same).

We want to prove that $(*)$ commutes.

It is clear from the definition of ∇ $(= \nabla^{(0,1)} + \nabla^{(1,0)})$ that

$$\begin{tikzcd}
R^0_{\pi_*}\Omega^n_{\mathcal{X}} \arrow[r, "\mathrm{d}"] \arrow[d, "\alpha"] & R^0_{\pi_*}\Omega^{n+1}_{\mathcal{X}} \arrow[d, "\alpha_2"] \\
\mathcal{F}^{n-1}\mathcal{H}^{n-1}_{X_{\mathbf{t}}} \otimes \Omega^1_{\mathcal{S}} \arrow[r, "\nabla"] & \mathcal{F}^{n-2}\mathcal{H}^{n-1}_{X_{\mathbf{t}}} \otimes \Omega^2_{\mathcal{S}}
\end{tikzcd} \qquad (**)$$

commutes[38] (but note that the image of the composition is in $\mathcal{F}^{n-1}\mathcal{H}^{n-1}_{X_{\mathbf{t}}} \otimes \Omega^2_{\mathcal{S}}$!). Consider $\omega_Y[\otimes 1] \in H^0(\Omega^n_Y) \otimes \mathcal{O}_{\mathcal{S}}$ in $(*)$, where $\omega_Y \in H^0(\Omega^n_Y)$ is a fixed form on Y (or $Y \times \mathcal{S}$, omitting p_Y^*); then by construction of ${}'\bar{\nabla}$, we have ${}'\bar{\nabla}(\omega_Y) = \alpha \circ \iota^*_{\mathcal{X}}(\omega_Y)$ where $\iota^*_{\mathcal{X}}(\omega_Y)$ is a section of $R^0_{\pi_*}\Omega^n_{\mathcal{X}}$ in $(**)$. The top row of $(*)$ being exact,

$$\nabla^{(1,0)} \circ {}'\bar{\nabla}(\omega_Y) = \nabla \circ {}'\bar{\nabla}(\omega_Y) = \nabla \circ \alpha\left(\iota^*_{\mathcal{X}}(\omega_Y)\right) = \alpha_2\left(\mathrm{d}\{\iota^*_{\mathcal{X}}(\omega_Y)\}\right)$$

$$= \alpha_2\left(\iota^*_{\mathcal{X}}(\mathrm{d}\omega_Y)\right) = 0$$

[38] A Note on (**): Although it seems straightforward that ∇ is computed by d, a subtlety worth mention comes up in formally proving this. The composition $\nabla \circ \alpha$ is computed by the composition (where $\mathbb{R}^n_{\pi_*}\Omega^n_{\mathcal{X}}[-n] = R^0_{\pi_*}\Omega^n_{\mathcal{X}}$ and $\mathbb{R}^{n+1}_{\pi_*}\Omega^{\bullet-2\geq n-2}_{\mathcal{X}/\mathcal{S}} = \mathbb{R}^{n-1}_{\pi_*}\Omega^{\bullet\geq n-1}_{\mathcal{X}/\mathcal{S}}[-1]$)

$$\begin{tikzcd}
& \mathbb{R}^n\mathcal{L}^1\Omega^n_{\mathcal{X}}[-n] \arrow[r, equal] \arrow[d, equal] & \mathbb{R}^n\Omega^n_{\mathcal{X}}[-n] \arrow[dddrr, dashed, bend left] & & \\
\mathbb{R}^n\mathcal{L}^2\Omega^n_{\mathcal{X}}[-n] \arrow[r] & \mathbb{R}^n\mathcal{L}^1\Omega^n_{\mathcal{X}}[-n] \arrow[r] & \Omega^1_{\mathcal{S}} \otimes \mathbb{R}^n\Omega^{n-1}_{\mathcal{X}/\mathcal{S}}[-n] \arrow[d, equal] & & \\
\mathbb{R}^n\mathcal{L}^2\Omega^{\bullet\geq n}_{\mathcal{X}} \arrow[r] & \mathbb{R}^n\mathcal{L}^1\Omega^{\bullet\geq n}_{\mathcal{X}} \arrow[r] & \Omega^1_{\mathcal{S}} \otimes \mathbb{R}^n\Omega^{\bullet-1\geq n-1}_{\mathcal{X}/\mathcal{S}} \arrow[r, "\delta"] & \mathbb{R}^{n+1}\mathcal{L}^2\Omega^{\bullet\geq n}_{\mathcal{X}} \arrow[d, equal] & \\
& & & \mathbb{R}^{n+1}\mathcal{L}^2\Omega^{\bullet\geq n}_{\mathcal{X}} \arrow[r] & \Omega^2_{\mathcal{X}} \otimes \mathbb{R}^{n+1}\Omega^{\bullet-2\geq n-2}_{\mathcal{X}/\mathcal{S}}.
\end{tikzcd}$$

In particular, $\Omega^n_{\mathcal{X}}[-n]$ is *not* $\Omega^{\bullet\geq n}_{\mathcal{X}}$, while $\Omega^{n-1}_{\mathcal{X}/\mathcal{S}}[-n]$ *is* $\Omega^{\bullet-1\geq n-1}_{\mathcal{X}/\mathcal{S}}$. A representative of $\mathbb{R}^n\Omega^n_{\mathcal{X}}[-n] = R^0_{\pi_*}\Omega^n_{\mathcal{X}}$ is a $\bar{\partial}$-closed holomorphic n-form on $\mathcal{X}$ (quasi-projective), while a representative of $\mathbb{R}^n_{\pi_*}\Omega^{\bullet\geq n}_{\mathcal{X}}$ would be d-closed (which we do *not* want). The main point here is that since $\Omega^{\bullet\geq n}_{\mathcal{X}}$, $\mathcal{L}^1\Omega^{\bullet\geq n}_{\mathcal{X}}$, etc. are resolved by (p,q)-forms with $p \geq n$, and total differential d (not $\bar{\partial}$ as for $\Omega^n_{\mathcal{X}}$!), δ is computed by taking d. (The remainder *is* straightforward.)

(Y is compact so ω_Y is closed). It is trivial that the other direction ${}'\bar{\nabla} \circ \mathrm{d}$ in $(*)$ on $\omega_Y \otimes 1$, is 0. Now immediately this implies

$$\nabla^{(1,0)} \left(\sum_{i,j} \langle \theta_j^{\mathbf{t}}\,,\, \omega_Y^i \,|_{X_{\mathbf{t}}} \rangle \otimes f_i \right) \;=\; \sum_{i,j} \langle \theta_j^{\mathbf{t}}\,,\, \omega_Y^i \,|_{X_{\mathbf{t}}} \rangle \otimes \mathrm{d} f_i$$

so that the last equality in

$${}'\bar{\nabla} \circ \mathrm{d} \left(\sum \omega_Y^i \otimes f_i \right) \;=\; {}'\bar{\nabla} \left(\sum_i \omega_Y^i \otimes \mathrm{d} f_i \right) \;=\; \sum_{i,j} \langle \theta_j^{\mathbf{t}}\,,\, \omega_Y^i \,|_{X_{\mathbf{t}}} \rangle \otimes \mathrm{d} f_i$$

$$=\; \nabla^{(1,0)} \circ {}'\bar{\nabla} \left(\sum \omega_Y^i \otimes f_i \right)$$

holds, and the diagram $(*)$ commutes.

Now since *both* pieces of ∇ on $\nu_{\mathbf{t}}$ are zero, in particular

$$0 \;=\; \nabla^{(1,0)} \nu_{\mathbf{t}} \;=\; \nabla^{(1,0)} \circ {}'\bar{\nabla} \left(\sum_i f_i(\mathbf{t}) \otimes \omega_Y^i \right) \;=\; {}'\bar{\nabla} \circ \mathrm{d} \left(\sum_i f_i(\mathbf{t}) \otimes \omega_Y^i \right)$$

$$=\; {}'\bar{\nabla} \left(\sum_i \mathrm{d} f_i \otimes \omega_Y^i \right).$$

Assume for now the following

LEMMA 4.4.3. *For $L \gg 0$,*

$$H^0(\Omega_Y^n) \otimes \Omega_{\mathcal{S}}^1 \xrightarrow{\;{}'\bar{\nabla}\;} \mathcal{H}_{X_{\mathbf{t}}}^{n-1,0} \otimes \Omega_{\mathcal{S}}^2 \qquad \textit{is injective.}$$

Then for $L \gg 0$, $\sum_i \mathrm{d} f_i \otimes \omega_Y^i \;=\; 0$; and since $\{\omega_Y^i\}$ was a basis, $\mathrm{d} f_i = 0$ $(\forall i)$ and so $\sum f_i \otimes \omega_Y^i \;=\; 1 \otimes \omega_Y$ for some $\omega_Y \in H^0(\Omega_Y^n)$. We now have

$$\nu_{\mathbf{t}} = {}'\bar{\nabla}(\omega_Y)\,;$$

or alternately, noticing that $\nu_{\mathbf{t}}$ is just $\alpha\,(\bigwedge^n \mathrm{dlog}\mathbf{F})$,

$$\alpha \left(\bigwedge^n \mathrm{dlog}\mathbf{F} - \iota_{\mathcal{X}}^* \omega_Y \right) = 0.$$

Using the diagram with exact rows

$$\begin{tikzcd}
 & & & (0\to)R^0_{\pi_*}\mathcal{L}^1\Omega^n_{\mathcal{X}} \arrow[r, equal] \arrow[d, equal] & R^0_{\pi_*}\Omega^n_{\mathcal{X}} \arrow[d, bend right, "\alpha"'] \\
 & & (0\to)R^0_{\pi_*}\mathcal{L}^2\Omega^n_{\mathcal{X}} \arrow[r] \arrow[d, equal] & R^0_{\pi_*}\mathcal{L}^1\Omega^n_{\mathcal{X}} \arrow[r] & R^0_{\pi_*}\Omega^{n-1}_{\mathcal{X}/\mathcal{S}}\otimes\Omega^1_{\mathcal{S}} \\
 & (0\to)R^0_{\pi_*}\mathcal{L}^3\Omega^n_{\mathcal{X}} \arrow[r] \arrow[d, equal] & R^0_{\pi_*}\mathcal{L}^2\Omega^n_{\mathcal{X}} \arrow[r] & \boxed{R^0_{\pi_*}\Omega^{n-2}_{\mathcal{X}/\mathcal{S}}\otimes\Omega^2_{\mathcal{S}}} & \\
(0\to)R^0_{\pi_*}\mathcal{L}^4\Omega^n_{\mathcal{X}} \arrow[r] & R^0_{\pi_*}\mathcal{L}^3\Omega^n_{\mathcal{X}} \arrow[r] & \boxed{R^0_{\pi_*}\Omega^{n-3}_{\mathcal{X}/\mathcal{S}}\otimes\Omega^3_{\mathcal{S}}} & &
\end{tikzcd}$$

etc.

where the boxed terms are zero (Lefschetz hyperplane $\implies$ $R^0_{\pi_*}\Omega^k_{\mathcal{X}/\mathcal{S}} = 0$ for $0 < k < n-1$) and in which we continue until we reach $\mathcal{L}^n\Omega^n_{\mathcal{X}} = \pi^*\Omega^n_{\mathcal{S}}$, one sees that $\ker(\alpha) = \Omega^n_{\mathcal{S}}$ and so (for some $\zeta_{\mathcal{S}} \in H^0(\Omega^n_{\mathcal{S}})$)

$$\bigwedge^n \mathrm{dlog}\mathbf{F} \;=\; \iota^*_{\mathcal{X}}\omega_Y + \pi^*\zeta_{\mathcal{S}} \qquad \text{in } H^0(\Omega^n_{\mathcal{X}}).$$

Moreover

$$\bigwedge^n \mathrm{dlog}\mathbf{F} \;\in\; F^nH^n(\mathcal{X},\mathbb{C}) \cap H^n(\mathcal{X},\mathbb{Z}(n))$$

and it is easy to show that $[\pi^*]\zeta_{\mathcal{S}}$, and therefore $\iota^*_{\mathcal{X}}\omega_Y$, has $\mathbb{Q}(n)$-periods as well.

To see this, let β be a form representing the hyperplane class $[H] \in H^{1,1}(Y,\mathbb{Z})$ (e.g. the Fubini-Study form); noting that $\int_{X_{\mathbf{t}}} \iota^*_{X_{\mathbf{t}}}\beta^{n-1} = D_XD_Y$ and $\beta^{n-1}\wedge\omega_Y = 0$ (by type), we have for $\mathcal{C} \in Z_n(\mathcal{S})$

$$D_XD_Y\int_{\mathcal{C}}\zeta_{\mathcal{S}} \;=\; \int_{\mathcal{C}}\zeta_{\mathcal{S}}\cdot\pi_*\iota^*_{\mathcal{X}}\beta^{n-1} \;=\; \int_{\pi^{-1}(\mathcal{C})\cap\mathcal{X}} \pi^*\zeta_{\mathcal{S}}\wedge\iota^*_{\mathcal{X}}\beta^{n-1}$$

$$= \int_{\pi^{-1}(\mathcal{C})\cap\mathcal{X}} (\pi^*\zeta_{\mathcal{S}} + \iota^*_{\mathcal{X}}\omega_Y)\wedge\iota^*_{\mathcal{X}}\beta^{n-1} \;=\; \int_{\pi^{-1}(\mathcal{C})\cap\mathcal{X}\cap p_Y^{-1}(H^{n-1})} \bigwedge^n \mathrm{dlog}\mathbf{F} \;\in\; \mathbb{Z}(n)$$

(where $p_Y^{-1}(H^{n-1}) = p_Y^{-1}(Y\cap\mathbb{P}^2) = (Y\cap\mathbb{P}^2)\times\mathcal{S}$ is the pullback of the intersection of $n-1$ hyperplanes in Y) and so $\int_{\mathcal{C}}\zeta_{\mathcal{S}} \in \mathbb{Q}(n)$, and

$$\iota^*_{\mathcal{X}}\omega_Y \in F^nH^n(\mathcal{X},\mathbb{C}) \cap H^n(\mathcal{X},\mathbb{Q}(n)).$$

Now since $\iota_{\mathcal{X}_*} : \theta^1_{\mathcal{X}} \to \theta^1_Y$ is obviously surjective on tangent planes, and by Lemma 1.3.7 (and the remark immediately following it) applied to $\bar{\mathcal{X}}$, we have[39]

$$H^0(\Omega^n_Y) \overset{\iota^*_{\mathcal{X}}}{\hookrightarrow} H^0(\Omega^n_{\bar{\mathcal{X}}}) \hookrightarrow \frac{F^n H^n(\mathcal{X},\mathbb{C})}{\mathrm{num}\cap H^n(\mathcal{X},\mathbb{Q}(n))}$$

and with the above this $\implies \omega_Y = 0$, or

$$\bigwedge^n \mathrm{dlog}\mathbf{F} \;=\; \pi^*\zeta_{\mathcal{S}} \;\in\; \ker(\alpha)$$

and the infinitesimal invariant $\nu_{\mathbf{t}}$ vanishes.

The monodromy argument is the same as in §4.3.4 and the vanishing theorem holds (for $L \gg 0$, i.e. D_X sufficiently large) since the vanishing cycles for the family span[40]

$$\ker\left(H_{n-1}(X_0,\mathbb{Q}) \to H_{n-1}(Y,\mathbb{Q})\right) = \ker\left(H_{n-1}(X_0,\mathbb{Q}) \to H_{n-1}(\mathbb{P}^{n+1},\mathbb{Q})\right).$$

4.4.3. Pseudo-Jacobi ideals and Macaulay duality. In order to control the exactness of the sequences in the two Lemmas above, we need an analogue (for more general Y) of the polynomial representation of the graded pieces of $\mathcal{H}^{n-1}_{X_s,pr}$ for $Y = \mathbb{P}^n$. The Jacobi ideals used in that case involved derivatives of $F_t \in H^0(\mathbb{P}^n, \mathcal{O}(D))$, and so we start by "differentiating" $F_t \in H^0(Y,L)$. In fact, following **[GG4]** let $\Sigma_{Y,L}$ be the sheaf of 1st-order differential operators on (local) sections of L; it fits in an exact sequence

$$0 \to \mathcal{O}_Y \longrightarrow \Sigma_{Y,L} \longrightarrow \theta^1_Y \to 0.$$

If locally on $\mathcal{U} \subseteq Y$ we have $\delta \in \Sigma_{Y,L}(\mathcal{U}) \longmapsto \nu \in \theta^1_Y(\mathcal{U})$, $f \in \mathcal{O}_Y(\mathcal{U})$ and $\lambda \in H^0(\mathcal{U}, L)$ then one has a new section of L by $\delta(\lambda)$ or

$$\delta(f\cdot\lambda) \;=\; \nu(f)\cdot\lambda + f\cdot\delta(\lambda),$$

and δ is $\mathcal{O}_Y$-linear if $\nu = 0$ (and may be represented by multiplication with $g \in \mathcal{O}_Y(\mathcal{U})$). One also encounters in this setting the notation $P^1_{Y,L} =$ 1st jet bundle $:= \Sigma^\vee_{Y,L} \otimes L$.

For $Y = \mathbb{P}^n$, $L = \mathcal{O}_{\mathbb{P}^n}(D)$, one computes

$$\Sigma_{\mathbb{P}^n,\mathcal{O}(D)} \;=\; H^0(\mathbb{P}^n, \mathcal{O}(1))^\vee \;\otimes\; \mathcal{O}(1).$$

Here one should think of $\mathcal{O}(1)$ as the dual of the tautological bundle, so that sections are linear functionals[41] (such as $\sum b_j z_j$), their duals linear differential operators (on these functionals, taking them to $\mathbb{C}$, e.g. $\sum c_i \frac{\partial}{\partial z_i}$). A global section of $\Sigma_{\mathbb{P}^n,\mathcal{O}(D)}$ is thus of the form $\sum a_{ij} z_j \frac{\partial}{\partial z_i}$, which indeed operates on

[39] assuming $\bar{\mathcal{X}}$ is smooth; but even if it's not, one can get around this by a resolution-of-singularities argument.

[40] (Y has only interesting nth homology, by Lefschetz hyperplane)

[41] *a priori* on the fiber, but global sections turn out to be linear functionals *on all of* $\mathbb{C}^{n+1}$ (and this is what we have in mind).

sections $F \in H^0(\mathbb{P}^n, \mathcal{O}(D)) = S^D$ (to give another section $\sum a_{ij} z_j \frac{\partial F}{\partial z_j}$). One may sum up the actions of all such differential operators on F by

$$\mathrm{d}F := \sum_i \frac{\partial F}{\partial z_i} \otimes \mathrm{d}z_i \in H^0\left(\mathcal{O}(D-1) \otimes H^0(\mathbb{P}^n, \mathcal{O}(1))^{[\vee\vee]}\right)$$
$$= H^0(\mathcal{O}(D) \otimes \Sigma^\vee_{\mathbb{P}^n, \mathcal{O}(D)}) = H^0(P^1_{\mathbb{P}^n, \mathcal{O}(D)}).$$

To obtain the Jacobi ideal in S^E one simply lets $\Sigma_{\mathbb{P}^n, \mathcal{O}(L)}$ operate on F to get a map

$$H^0\left(\Sigma_{\mathbb{P}^n, \mathcal{O}(D)} \otimes \mathcal{O}(E-D)\right) \xrightarrow{F \in H^0(\mathbb{P}^n, \mathcal{O}(D))} H^0\left(\mathcal{O}(D) \otimes \mathcal{O}(E-D)\right)$$
$$= H^0(\mathcal{O}_{\mathbb{P}^n}(E)) = S^E;$$

the image is $J^E_F = \left(\frac{\partial F}{\partial z_0}, \dots, \frac{\partial F}{\partial z_n}\right)$. This map may be interpreted as $\cup\, \mathrm{d}F$.

The situation is completely analogous for Y, L, $\mathcal{E}$ ($\mathcal{E}$ any coherent analytic sheaf on Y): $J^{\mathcal{E}}_F$ is just the image of

$$H^0(\Sigma_{Y,L} \otimes \mathcal{E} \otimes L^{-1}) \xrightarrow{F \in H^0(Y, L)} H^0(L \otimes \mathcal{E} \otimes L^{-1}) = H^0(Y, \mathcal{E})$$

and is called the pseudo-Jacobi ideal.[42] One has the following "Macaulay duality"

THEOREM. (Green **[Gr2]**) *If $L \gg 0$ and $a + b = n + 1$,*

$$\frac{H^0(\Omega^n_Y \otimes L^a)}{J_F^{\Omega^n_Y \otimes L^a}} \otimes \frac{H^0(\Omega^n_Y \otimes L^b)}{J_F^{\Omega^n_Y \otimes L^b}} \longrightarrow \frac{H^0((\Omega^n_Y)^{\otimes 2} \otimes L^{n+1})}{J_F^{(\Omega^n_Y)^{\otimes 2} \otimes L^{n+1}}} \cong \mathbb{C}$$

is a perfect pairing (more generally one could pair $\mathcal{E} \otimes L^a$ and $(\Omega^n_Y)^2 \otimes \mathcal{E}^\vee \otimes L^b$), i.e. the tensor factors are dual.

This specializes (for $Y = \mathbb{P}^n$, $L = \mathcal{O}(D)$) to the duality between R_F^{kD-n-1} and $R_F^{(n-k+1)D-n-1}$ induced by Serre duality $H^{n-k,k-1}(X) \cong H^{k-1,n-k}(X)$ and the isomorphisms of §4.2.

4.4.4. Proof of the algebraic lemmas. Now we prove Lemmas (4.4.2) and (4.4.3), remarking that while one may compute lower bounds on the degree D_X (rather than just $L \gg 0$) for which they hold, we do not know whether our methods[43] give optimal lower bounds and we do not produce

[42] This is also written as $\cup$ with $\mathrm{d}F \in H^0(\Sigma^\vee_{Y,L} \otimes L) = H^0(P^1_{Y,L})$.

[43] One needs to use the "hard" Bott vanishing theorem (see **[KS]**):

$$H^q(\Omega^p_{\mathbb{P}^{n+1}}(D_X)) = 0 \quad \text{unless} \quad \begin{cases} q = 0 \text{ and } D_X > p \\ q = n+1 \text{ and } D_X < p - n - 1 \\ p = q \text{ and } D_X = 0 \end{cases}$$

together with Kodaira vanishing and various long exact sequences to establish for exactly which D_X we have $H^q(\Omega^p_Y(D_X)) = 0$. Then this would be used to quantify $L \gg 0$ in the various instances where a step holds for L sufficiently ample.

them here. We shall refer freely to theorems and lemmas in [**Gr2**] (using $G(\#.\#)$) in what follows, with the caveat that our n is his $n+1$.

For lemma (4.4.2), since the maps in the sequences of sheaves are $\mathcal{O}_{\mathcal{S}}$-linear, and we already have $\nabla \circ {}'\bar{\nabla} = 0$, it suffices to show $\ker \bar{\nabla} \subseteq \mathrm{im}' \bar{\nabla}$ at $s = 0$. This is what we shall mean henceforth by "middle-exactness". From Remark (4.4.1) we have $H^0(Y, L) \twoheadrightarrow \theta^1_{\mathcal{S},0}$ or $\Omega^1_{\mathcal{S},0} \hookrightarrow H^0(Y, L)^\vee$; and to get middle-exactness of the top row in

$$\begin{tikzcd}
{[\mathcal{O}_{\mathcal{S},o}\otimes]}H^0(\Omega^n_Y) \arrow[r, "{'\bar{\nabla}}"] \arrow[d, equal] & \Omega^1_{\mathcal{S},o} \otimes H^0(\Omega^{n-1}_{X_0}) \arrow[r, "\bar{\nabla}"] \arrow[d, hook] & \Omega^2_{\mathcal{S},0} \otimes H^1(\Omega^{n-2}_{X_0}) \arrow[d, hook] \\
H^0(\Omega^n_Y) \arrow[r] \arrow[d, equal] & H^0(Y,L)^\vee \otimes H^0(\Omega^{n-1}_{X_0}) \arrow[r] \arrow[d, equal] & \bigwedge^2 H^0(Y,L)^\vee \otimes H^1(\Omega^{n-2}_{X_0}) \arrow[d] \\
H^0(\Omega^n_Y) \arrow[r] & H^0(Y,L)^\vee \otimes \dfrac{H^0(\Omega^{n-1}_{X_0})}{\mathrm{im} H^0(\Omega^{n-1}_Y \otimes \mathcal{O}_{X_0})} \arrow[r] & \bigwedge^2 H^0(Y,L)^\vee \otimes \dfrac{H^1(\Omega^{n-2}_{X_0})}{\mathrm{im} H^1(\Omega^{n-2}_Y \otimes \mathcal{O}_{X_0})}
\end{tikzcd}$$

it suffices to check it at the bottom. In particular $\mathrm{im} H^0(\Omega^{n-1}_Y \otimes \mathcal{O}_{X_0}) = 0$ for $L \gg 0$ by using Kodaira + Serre on the right-hand term of

$$\to H^0(\Omega^{n-1}_Y \otimes L^{-1}) \to H^0(\Omega^{n-1}_Y) \to H^0(\Omega^{n-1}_Y \otimes \mathcal{O}_{X_0}) \to H^1(\Omega^{n-1}_Y \otimes L^{-1}) \to$$

and noting that $H^0(\Omega^{n-1}_Y) = 0$ ($\dim Y = n$).

To dualize the bottom row we use Lemmas $G(1.16)$ and $G(1.8)$, advising the reader again that Green's $\dim X = n$ rather than $n-1$ (so we are using the case $p = n$ there); namely that

$$0 \leftarrow \left(\frac{H^1(\Omega^{n-2}_{X_0})}{\mathrm{im} H^1(\Omega^{n-2}_Y \otimes \mathcal{O}_{X_0})} \right)^\vee \leftarrow \left(\frac{H^0(\Omega^{n-1}_{X_0} \otimes L^{n-2})}{\mathrm{im} H^0(\Omega^{n-1}_{X_0} \otimes (\theta^1_Y \otimes L^{-1}) \otimes L^{n-2})} \right)$$

$$\leftarrow \left(\ker \left\{ H^2(\Omega^{n-2}_Y \otimes \mathcal{O}_{X_0}) \to H^2(\Omega_{X_)})^{n-2} \right\} \right)^\vee \leftarrow 0$$

and

$$0 \leftarrow \left(\frac{H^0(\Omega^{n-1}_{X_0})}{\mathrm{im} H^0(\Omega^{n-1}_Y \otimes \mathcal{O}_{X_0})} \right)^\vee \leftarrow \left(\frac{H^0(\Omega^{n-1}_{X_0} \otimes L^{n-1})}{\mathrm{im} H^0(\Omega^{n-1}_{X_0} \otimes (\theta^1_Y \otimes L^{-1}) \otimes L^{n-1})} \right)$$

$$\leftarrow \left(\ker \left\{ H^1(\Omega^{n-1}_Y \otimes \mathcal{O}_{X_0}) \to H^1(\Omega^{n-1}_{X_0}) \right\} \right)^\vee \leftarrow 0$$

are exact for $L \gg 0$. The desired dualization is then the first row of

$$\begin{tikzcd}
\dfrac{H^0(Y,\Omega_Y^n\otimes L^{n+1})}{J_F^{\Omega_Y^n\otimes L^{n+1}}} \arrow[d, equal] & H^0(Y,L)\otimes\dfrac{H^0(\Omega_{X_0}^{n-1}\otimes L^{n-1})}{\left(\begin{matrix}\mathrm{im}H^0(\Omega_{X_0}^{n-1}\otimes L^{n-1}\otimes\theta_Y^1\otimes L^{-1})\\ +\text{more}\end{matrix}\right)} \arrow[l] & \bigwedge^2 H^0(Y,L)\otimes\dfrac{H^0(\Omega_{X_0}^{n-1}\otimes L^{n-2})}{\left(\begin{matrix}\mathrm{im}H^0(\Omega_{X_0}^{n-1}\otimes L^{n-2}\otimes\theta_Y^1\otimes L^{-1})\\ +\text{more}\end{matrix}\right)} \arrow[l] \\
\dfrac{H^0(\Omega_Y^n\otimes L^{n+1})}{J_F^{\Omega_Y^n\otimes L^{n+1}}} & H^0(Y,L)\otimes\dfrac{H^0(\Omega_Y^n\otimes L^n)}{J_F^{\Omega_Y^n\otimes L^n}} \arrow[l] \arrow[u, two heads] & \bigwedge^2 H^0(Y,L)\otimes\dfrac{H^0(\Omega_Y^n\otimes L^{n-1})}{J_F^{\Omega_Y^n\otimes L^{n-1}}} \arrow[l] \arrow[u, two heads] \\
H^0(\Omega_Y^n\otimes L^{n+1}) \arrow[u, two heads] & H^0(Y,L)\otimes H^0(\Omega_Y^n\otimes L^n) \arrow[l] \arrow[u, two heads] & \bigwedge^2 H^0(Y,L)\otimes H^0(\Omega_Y^n\otimes L^{n-1}) \arrow[l] \arrow[u, two heads] \\
J_F^{\Omega_Y^n\otimes L^{n+1}} \arrow[u, hook] & H^0(Y,L)\otimes J_F^{\Omega_Y^n\otimes L^n} \arrow[l, "\boxed{?}"'] \arrow[u, hook] & \bigwedge^2 H^0(Y,L)\otimes J_F^{\Omega_Y^n\otimes L^{n-1}} \arrow[l] \arrow[u, hook]
\end{tikzcd}$$

where the first row is middle-exact (ker $\supseteq$ im) if (a) the 2nd-last row is and (b) the questioned arrow is surjective. Noting that $\Omega_Y^n\otimes\mathcal{O}_{X_0}\cong\Omega_{X_0}^{n-1}\otimes L^{-1}$ (as $L^{-1}|_X\cong N^*_{X_0/Y}$), to construct e.g. the top middle vertical arrow we have used the commuting square

$$\begin{tikzcd}
H^0(\Omega_Y^n\otimes L^n) \arrow[rr, two heads] & & H^0(\Omega_{X_0}^{n-1}\otimes L^{n-1}) \\
& H^0(\Omega_Y^n\otimes L^n\otimes(\theta_Y^1\otimes L^{-1})) \arrow[dr, "|_{X_0}"] & \\
H^0(\Omega_Y^n\otimes L^n\otimes\Sigma_{Y,L}\otimes L^{-1}) \arrow[uu, "J"] \arrow[ur] \arrow[rr] & & H^0(\Omega_{X_0}^{n-1}\otimes L^{n-1}\otimes\theta_Y^1\otimes L^{-1}); \arrow[uu]
\end{tikzcd}$$

while the top left term dualizes $H^0(\Omega_Y^n)$ by Macaulay's theorem.[44] Now for (a) we refer to $G(2.47)$, while for (b) using $G(1.28)$ we are asking for $\mathrm{im}\alpha\twoheadrightarrow\mathrm{im}\beta$ in

$$\begin{tikzcd}
H^0(\Omega_Y^n\otimes L^{n-1}\otimes\Sigma_{Y,L})\otimes H^0(Y,L) \arrow[r, two heads, "G(1.28)"'] \arrow[d, "\alpha"] & H^0(\Omega_Y^n\otimes L^n\otimes\Sigma_{Y,L}) \arrow[d, "\beta"] \\
H^0(\Omega_Y^n\otimes L^n)\otimes H^0(Y,L) \arrow[r, two heads, "G(1.28)"'] & H^0(\Omega_Y^n\otimes L^{n+1}),
\end{tikzcd}$$

which is clear.

Similarly for Lemma (4.4.3) we arrive at a large "dualized" diagram (analogous to the above), whose last three rows are

[44] since $J_F^{\Omega_Y^n}=0$ for $L\gg 0$ using Kodaira + Serre on $H^0(\Omega_Y^n\otimes\Sigma_{Y,L}\otimes L^{-1})$.

$$
\begin{tikzcd}
\dfrac{H^0(\Omega_Y^n\otimes L^{n+2})}{J_F^{\Omega_Y^n\otimes L^{n+2}}} & H^0(Y,L)\otimes\dfrac{H^0(\Omega_Y^n\otimes L^{n+1})}{J_F^{\Omega_Y^n\otimes L^{n+1}}} \arrow[l] & \bigwedge^2 H^0(Y,L)\otimes\dfrac{H^0(\Omega_Y^n\otimes L^{n})}{J_F^{\Omega_Y^n\otimes L^{n}}} \arrow[l, "\xi"] \\
H^0(\Omega_Y^n\otimes L^{n+2}) \arrow[u] & H^0(Y,L)\otimes H^0(\Omega_Y^n\otimes L^{n+1}) \arrow[l] \arrow[u] & \bigwedge^2 H^0(Y,L)\otimes H^0(\Omega_Y^n\otimes L^{n}) \arrow[l] \arrow[u] \\
J_F^{\Omega_Y^n\otimes L^{n+2}} \arrow[u, hook, "\boxed{?}"'] & H^0(Y,L)\otimes J_F^{O_Y^n\otimes L^{n+1}} \arrow[l, "\boxed{?}"'] \arrow[u, hook] & \bigwedge^2 H^0(Y,L)\otimes J_F^{\Omega_Y^n\otimes L^{n}} \arrow[l] \arrow[u, hook]
\end{tikzcd}
$$

and we want to show ξ surjective. Lemma $G(2.47)$ shows the middle row is middle-exact; it suffices to show (in addition to this) that *both* questioned maps are surjective. The only new phenomenon here is the vertical one (so that the upper left-hand term would be zero).

We must show (for $L \gg 0$) that

$$J_F^{\Omega_Y^n\otimes L^{n+2}} := \operatorname{im}\left\{ H^0(\Omega_Y^n\otimes L^{n+1}\otimes\Sigma_{Y,L}) \overset{F\in H^0(Y,L)}{\longrightarrow} H^0(\Omega_Y^n\otimes L^{n+2}) \right\}$$

is all of $H^0(\Omega_Y^n\otimes L^{n+2})$. Noting that (as a vector bundle) $\Sigma_{Y,L}$ has rank $n+1(=\operatorname{rank}\theta_Y^1+\operatorname{rank}\mathcal{O}_Y)$, so that $\bigwedge^{n+2}\Sigma_{Y,L}=0$, we find that cupping repeatedly with $\mathrm{d}F\in H^0(\Sigma_{Y,L}^\vee\otimes L)$ gives rise to a long-exact sequence

$$0\to\bigwedge^{n+1}\Sigma_{Y,L}\otimes L^{-(n+1)}\to\bigwedge^{n}\Sigma_{Y,L}\otimes L^{-n}\to\ldots\to\Sigma_{Y,L}\otimes L^{-1}\to\mathcal{O}_Y\to 0.$$

Tensoring this by $\Omega_Y^n\otimes L^{n+2}$ gives a sequence with terms

$$\mathsf{F}^a=\Omega_Y^n\otimes L^{a+1}\otimes\bigwedge^{n-a+1}\Sigma_{Y,L},\qquad 0\le a\le n+1\ \ (0\ \text{otherwise}).$$

By the hypercohomology trick (see footnote §4.2.2),

$$E_1^{a,b}=H^b(\mathsf{F}^a)=H^b(\Omega_Y^n\otimes L^{a+1}\otimes\bigwedge^{n-a+1}\Sigma_{Y,L})\quad\Longrightarrow\quad E_\infty^{a,b}=0\,;$$

since $a+1>0$ in every term of $\mathsf{F}^\bullet$, for $L\gg 0$ Kodaira vanishing $\Longrightarrow$ $E_1^{a,b}=0$ for $b\neq 0$. Therefore E_2 must already be 0, and so (as $E_1^{n+2,0}=0$) $d_1: E^{n,0}\to E^{n+1,0}$ must be surjective, which is what we want.

4.5. Complete Intersections (Arbitrary Codimension)

4.5.1. Replacing X by a hypersurface Y. Let X be a smooth $(n-1)$-dimensional subvariety of $\mathbb{P}^{n+r}$, cut out by $r+1$ homogeneous polynomials

$$F_0 \in S^{D_0}\ ,\ \ldots\ ,\ F_r \in S^{D_r}$$

in $\{z_0, \ldots, z_r\}$; we can show $(F_0, \ldots, F_r)$ as a section of

$$E = \mathcal{O}_{\mathbb{P}^{n+r}}(D_0) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^{n+r}}(D_r)$$

with X as its zero set.[45] projectivizing the fibers of E (as a vector bundle) gives the projective bundle[46] $\mathbb{P}(E) \to \mathbb{P}^{n+r}$; it contains a hypersurface $Y \subseteq \mathbb{P}(E)$ with cohomology strongly related to that of X. Moreover, since it is a hypersurface we can give a polynomial representation of its (primitive) cohomology in the same spirit as in §4.2. Our original source for this material was the exposition of Dimca's work in **[N]**.

The key is the following "toric" construction of $\mathbb{P}(E)$: let

$$\pi : U = \underset{(z_0, \ldots, z_{n+r})}{\mathbb{C}^{n+r+1} \setminus \{0\}} \times \underset{(x_0, \ldots, x_r)}{\mathbb{C}^{r+1} \setminus \{0\}} \longrightarrow \mathbb{P}(E)$$

be the quotient by this action of $\mathbb{C}^* \times \mathbb{C}^*$:

$$(t_1\,,\, t_2) \cdot (z_0\,,\, \ldots\,,\, z_{n+r}\,;\, x_0\,,\, \ldots\,,\, x_r) = \left(t_2 z_0\,,\, \ldots\,,\, t_2 z_{n+r}\,;\, \frac{t_1 x_0}{t_2^{D_0}}\,,\, \ldots\,,\, \frac{t_1 x_r}{t_2^{D_r}} \right)$$

which is generated infinitesimally by

$$\vec{e}_1 = \sum_{j=0}^{r} x_j \frac{\partial}{\partial x_j}\ , \qquad \vec{e}_2 = \sum_{i=0}^{n+r} z_i \frac{\partial}{\partial z_i} - \sum_{j=0}^{r} D_j \cdot x_j \frac{\partial}{\partial x_j}.$$

We also note that by the Leray-Hirsch theorem (see **[BoT]**)

$$H^m(\mathbb{P}(E)) \cong \bigoplus_{i+j=m} H^i(\mathbb{P}^{n+r}) \otimes H^j(\mathbb{P}^r)$$

with ring structure

$$H^*(\mathbb{P}(E)) \cong \mathbb{C}[\mathsf{Z}, \mathsf{X}] \Big/ \left(\mathsf{Z}^{n+r+1}, \prod_{j=0}^{r} [\mathsf{X} + D_j \cdot \mathsf{Z}]\right)$$

where Z, X are generators for $H^*(\mathbb{P}^{n+r})$, $H^*(\mathbb{P}^r)$ respectively.

Now take Y to be the zero set of

$$F = F(\mathbf{z}, \mathbf{x}) := x_0 F_0(\mathbf{z}) + \ldots + x_r F_r(\mathbf{z})\,;$$

the zero set is clearly invariant under the $\mathbb{C}^* \times \mathbb{C}^*$-action (although F is not). Y has two kinds of fibers over $\mathbb{P}^{n+r}$. Over $\mathbf{z} \in X$ its fiber is all of $\mathbb{P}^r$, as all the $F_j(\mathbf{z})$ are then zero. Otherwise (for $\mathbf{z} \in \mathbb{P}^{n+r} \setminus X$) $F(\mathbf{z}, \mathbf{x})$ is linear in the $\{x_j\}$ and so the fiber of Y is a hyperplane $\mathbb{P}^{r-1}$. In a diagram

[45] We assume all $D_j \geq 2$ to avoid redundancy.

[46] This has dimension $n + 2r$, which will make our numbers different across the board from those in [Nagel].

$$\begin{tikzcd}
\mathbb{P}(E) & Y \arrow[l, hook'] \arrow[dl, "\varphi"'] & \pi^{-1}(X) = X \times \mathbb{P}^r \arrow[l, hook', "i"'] \arrow[dl, "\rho_X"'] \\
\mathbb{P}^{n+r} \arrow[from=u, "\rho"'] & X \arrow[l, hook'] &
\end{tikzcd}$$

We will show (following [**N**])that $i : X \times \mathbb{P}^r \to Y$ induces

$$i^* : H^{n+2r+1}_{var}(Y, \mathbb{C}) \xrightarrow{\cong} H^{n-1}_{var}(X, \mathbb{C}) \otimes H^{2r}(\mathbb{P}^r, \mathbb{C})$$

where

$$H^q_{var}(Y) := \operatorname{coker}\left(H^q(\mathbb{P}(E)) \to H^q(Y)\right).$$

Consider the local systems $R^q_{\rho_*}\mathbb{C}$, $R^q_{\varphi_*}\mathbb{C}$, and $R^q_{(\rho_X)_*}\mathbb{C} = (R^q_{\rho_*}\mathbb{C})_X$. From the above description of Y, since $H^q(\mathbb{P}^{r-1}) \xleftarrow{\cong} H^q(\mathbb{P}^r)$ for $q \neq 2r$, we have $R^q_{\varphi_*}\mathbb{C} \xleftarrow{\cong} R^q_{\rho_*}\mathbb{C}$; while for $q = 2r$, $R^{2r}_{\varphi_*}\mathbb{C} \xrightarrow{\cong} R^{2r}_{(\rho_X)_*}\mathbb{C} = (R^{2r}_{\rho_*}\mathbb{C})_X$. Now consider the Leray spectral sequences

$$^{\rho}E_2^{p,q} := H^p(\mathbb{P}^{n+r}, R_{\rho_*}\mathbb{C}) \implies H^{p+q}(\mathbb{P}(E), \mathbb{C}$$

and

$$^{\varphi}E_2^{p,q} := H^p(\mathbb{P}^{n+r}, R^q_{\varphi_*}\mathbb{C}) \implies H^{p+q}(Y, \mathbb{C})\,;$$

both degenerate at E_2, so that for instance

$$H^{n+2r-1}(Y, \mathbb{C}) \cong \bigoplus_{p+q=n+2r-1} H^p(\mathbb{P}^{n+r}, R^q_{\varphi_*}\mathbb{C}).$$

The proof that i^* is an isomorphism then goes as follows: for $p + q = n + 2r - 1$, $q \neq 2r$,

$$H^p(\mathbb{P}^{n+r}, R^q_{\rho_*}\mathbb{C}) \xrightarrow{\cong} H^p(\mathbb{P}^{n+r}, R^q_{\varphi_*}\mathbb{C})$$

so that the only interesting $H_{var}(Y)$ is found at $q = 2r$:

$$\begin{aligned}
H^{n+2r-1}_{var}(Y, \mathbb{C}) &\cong \bigoplus_{p+q=n+2r-1} \operatorname{coker}\left\{H^p(\mathbb{P}^{n+r}, R^q_{\rho_*}\mathbb{C}) \to H^p(\mathbb{P}^{n+r}, R^q_{\varphi_*}\mathbb{C})\right\} \\
&\cong \operatorname{coker}\left\{H^{n-1}(\mathbb{P}^{n+r}, R^{2r}_{\rho_*}\mathbb{C}) \to H^{n-1}(\mathbb{P}^{n+r}, R^{2r}_{\varphi_*}\mathbb{C})\right\} \\
&\underset{i^*}{\xrightarrow{\cong}} \operatorname{coker}\left\{H^{n-1}(\mathbb{P}^{n+r}, R^{2r}_{\rho_*}\mathbb{C}) \to H^{n-1}(X, R^{2r}_{\rho_*}\mathbb{C})\right\} \\
&\cong \operatorname{coker}\left\{H^{n-1}(\mathbb{P}^{n+r}) \to H^{n-1}(X)\right\} \otimes H^{2r}(\mathbb{P}^r, \mathbb{C})
\end{aligned}$$

$$\cong H^{n-1}_{var}(X,\mathbb{C}) \otimes H^{2r}(\mathbb{P}^r,\mathbb{C}).$$

Since this isomorphism is induced by restriction it preserves periods of forms. We shall write $\xi \in H^2(\mathbb{P}^r,\mathbb{Z})$ for the generator of $H^*(\mathbb{P}^r)$ so that, for $[\alpha] \in H^{n+2r-1}_{var}(Y)$, $[i^*\alpha] \cap \xi^{2r} \in H^{n-1}_{var}(X)$ also has the same periods as $[\alpha]$.

4.5.2. Polynomial representation of rational forms. By way of analogy with §4.2, set

$$\mathcal{A}^m_0 := H^0(\mathbb{C}^{n+2r+2}, \Omega^m_{\mathbb{C}^{n+2r+2}}), \qquad \mathcal{A}^m := \text{rational } m\text{-forms on } \mathbb{C}^{n+2r+2}$$

$$A^m_k = A^m_k(Y) := \text{rational } m\text{-forms on } \mathbb{P}(E) \text{ with poles of order } \leq k \text{ along } Y \text{ (and no other poles)}$$

$$= H^0(\Omega^m_{\mathbb{P}(E)}(kY))$$

$$A^m = A^m(Y) := \bigcup_k A^m_k = H^0(\Omega^m_{\mathbb{P}(E)}(*Y)).$$

Recall that we used eigenvalues with respect to $[\mathrm{d}, \iota_{\vec{e}}]$ (where $\iota_{\vec{e}}$ was interior product with the euler vector field $\vec{e}$) to extend the notion of degree from polynomials $\mathbb{C}[z_0, \ldots, z_n]$ to $\mathcal{A}^m$. Here we will use eigenvalues with respect to the (anti)commutators $[\mathrm{d}, \iota_{\vec{e}_1}]$ and $[\mathrm{d}, \iota_{\vec{e}_2}]$ to define a *bidegree* on $\mathcal{A}^m$ (including polynomials). To begin with,

$$\deg_2(x_j) \cdot x_j := [\mathrm{d}, \iota_{\vec{e}_2}]x_j = \mathrm{d}\,\langle \vec{e}_2, x_j\rangle + \langle \vec{e}_2, \mathrm{d}x_j\rangle$$

$$= 0 + \left\langle \sum z_i \frac{\partial}{\partial z_i} - \sum D_{j'} \cdot x_{j'} \frac{\partial}{\partial x_{j'}}, \mathrm{d}x_j \right\rangle = -D_j \cdot x_j\,,$$

and similarly $\deg_1(x_j) = 1$, $\deg_1(z_i) = 1$, and $\deg_2(z_i) = 0$; so z_i and x_j have, respectively, bidegrees $(0,1)$ and $(1,-D_j)$.

This gives a bigrading on the ring of polynomials: just as in §4.2 we had

$$\mathbb{C}[z_0\,, \ldots\,, z_n] = \oplus S^a\,, \qquad \text{now}$$

$$\mathbb{C}[z_0\,, \ldots\,, z_{n+r}\,;\, x_0\,, \ldots\,, x_r] = \oplus S^{a,b}\,;$$

the bigrading corresponds to elements of $Pic(\mathbb{P}(E)) \cong \mathbb{Z}^2$ (while $Pic(\mathbb{P}^n) \cong \mathbb{Z}$) via the divisors cut out by the bihomogeneous polynomials. For instance, F has bidegree $(1,0)$, i.e. $F \in S^{1,0}$ (and $F^k \in S^{k,0}$), and cuts out Y.

Now we show how to evaluate the bidegrees of some forms we will use: defining

$$\mathrm{d}\mathbf{z} \wedge \mathrm{d}\mathbf{x} := \mathrm{d}z_0 \wedge \ldots \mathrm{d}z_{n+r} \wedge \mathrm{d}x_0 \wedge \ldots \wedge \mathrm{d}x_r \in \mathcal{A}^{n+2r+2}_0,$$

we have

$$\deg_2(\mathrm{d}\mathbf{z}) \cdot \mathrm{d}\mathbf{z} = \mathrm{d}\left\langle \sum z_i \frac{\partial}{\partial z_i} - \sum D_j \cdot x_j \frac{\partial}{\partial x_j}\,,\, \mathrm{d}z_0 \wedge \ldots \wedge \mathrm{d}z_{n+r} \right\rangle + \langle \vec{e}_2\,, \mathrm{d}\mathrm{d}\mathbf{z}(=0)\rangle$$

$$= \mathrm{d}\sum (-1)^i z_i \mathrm{d}z_0 \wedge \ldots \wedge \widehat{\mathrm{d}z_i} \wedge \ldots \wedge \mathrm{d}z_{n+r} = (n+r+1) \cdot \mathrm{d}\mathbf{z}$$

and similarly

$$\deg_1(\mathrm{d}\mathbf{z}) = 0\,, \quad \deg_1(\mathrm{d}\mathbf{x}) = r+1\,, \quad \deg_2(\mathrm{d}\mathbf{x}) = -\sum_{j=0}^{r} D_j\,,$$

so that the bidegree of $\mathrm{d}\mathbf{z} \wedge \mathrm{d}\mathbf{x}$ is $(r+1\,,\, n+r+1-\sum D_j)$. We will need the following two abstract computations for rational forms $\varphi \in \mathcal{A}^m$ with well-defined bidegree (i.e. "bihomogeneous").

LEMMA 4.5.1. *If $\langle \vec{e}_2, \langle \vec{e}_1, \varphi \rangle\rangle \neq 0$ then it has the same bidegree as φ.*

PROOF. $(\deg_1 \langle \vec{e}_2, \langle \vec{e}_1, \varphi \rangle\rangle) \cdot \langle \vec{e}_2, \langle \vec{e}_1, \varphi \rangle\rangle$

$$\begin{aligned}
&:= \mathrm{d}\,\langle \vec{e}_1, \langle \vec{e}_2, \langle \vec{e}_1, \varphi \rangle\rangle\rangle\,(=0) \;+\; \langle \vec{e}_1, \mathrm{d}\,\langle \vec{e}_2, \langle \vec{e}_1, \varphi \rangle\rangle\rangle \\
&\;= \;-\langle \vec{e}_1, \langle \vec{e}_2, \mathrm{d}\,\langle \vec{e}_1, \varphi \rangle\rangle\rangle \;+\; (\deg_2 \langle \vec{e}_1, \varphi \rangle) \cdot \langle \vec{e}_1, \langle \vec{e}_1, \varphi \rangle\rangle\,(=0) \\
&\;= \;\langle \vec{e}_1, \langle \vec{e}_2, \langle \vec{e}_1, \mathrm{d}\varphi \rangle\rangle\rangle\,(=0) \;-\; (\deg_1 \varphi)\,\langle \vec{e}_1, \langle \vec{e}_2, \varphi \rangle\rangle \\
&\;= \;(\deg_1 \varphi) \cdot \langle \vec{e}_2, \langle \vec{e}_1, \varphi \rangle\rangle\,.
\end{aligned}$$

The only difference with $\deg_2$ is that one should switch $\vec{e}_1$ and $\vec{e}_2$ at the beginning (using $\langle \vec{e}_2, \langle \vec{e}_1, \varphi \rangle\rangle = -\langle \vec{e}_1, \langle \vec{e}_2, \varphi \rangle\rangle$). $\square$

LEMMA 4.5.2. *For $i = 1, 2$, if $\langle \vec{e}_i, \varphi \rangle \neq 0$ then it has the same bidegree as φ.*

PROOF. The case of $\deg_j \langle \vec{e}_i, \varphi \rangle$, $i = j$, is easy and left to the reader. On the other hand, $i \neq j$ is harder than Lemma 4.5.1. From

$$\mathrm{d}\,\langle \vec{e}_2, \langle \vec{e}_1, \varphi \rangle\rangle \;=\; -\mathrm{d}\,\langle \vec{e}_1, \langle \vec{e}_2, \varphi \rangle\rangle$$

we get

$$-\langle \vec{e}_2, \mathrm{d}\,\langle \vec{e}_1, \varphi \rangle\rangle + (\deg_2 \langle \vec{e}_1, \varphi \rangle)\,\langle \vec{e}_1, \varphi \rangle \;=\; \langle \vec{e}_1, \mathrm{d}\,\langle \vec{e}_2, \varphi \rangle\rangle - (\deg_1 \langle \vec{e}_2, \varphi \rangle)\,\langle \vec{e}_2, \varphi \rangle$$

and then

$$\begin{aligned}
&\langle \vec{e}_2, \langle \vec{e}_1, \mathrm{d}\varphi \rangle\rangle \;-\; (\deg_1 \varphi)\,\langle \vec{e}_2, \varphi \rangle \;+\; (\deg_2 \langle \vec{e}_1, \varphi \rangle)\,\langle \vec{e}_1, \varphi \rangle \;= \\
&\qquad -\langle \vec{e}_1, \langle \vec{e}_2, \varphi \rangle\rangle \;+\; (\deg_2 \varphi)\,\langle \vec{e}_1, \varphi \rangle \;-\; (\deg_1 \langle \vec{e}_2, \varphi \rangle)\,\langle \vec{e}_2, \varphi \rangle\;;
\end{aligned}$$

the first terms cancel and

$$(\deg_2 \langle \vec{e}_1, \varphi \rangle - \deg_2 \varphi)\,\langle \vec{e}_1, \varphi \rangle \;+\; (\deg_1 \langle \vec{e}_2, \varphi \rangle - \deg_1 \varphi)\,\langle \vec{e}_2, \varphi \rangle \;=\; 0.$$

Taking interior product with $\vec{e}_2$ gives

$$\deg_2 \langle \vec{e}_1, \varphi \rangle \;=\; \deg_2 \varphi \qquad \text{if } \;\langle \vec{e}_1, \varphi \rangle \neq 0$$

and with $\vec{e}_1$

$$\deg_1 \langle \vec{e}_2, \varphi \rangle \;=\; \deg_1 \varphi \qquad \text{if } \;\langle \vec{e}_2, \varphi \rangle \neq 0$$

$\square$

An easy corollary of Lemma 4.5.1 is that the important (nozero) form

$$\Omega \;:=\; \langle \vec{e}_2, \langle \vec{e}_1, \mathrm{d}\mathbf{z} \wedge \mathrm{d}\mathbf{x} \rangle \rangle \;\in\; \mathcal{A}_0^{n+2r}$$

has the same bidegree $(r+1\,,\, n+r+1-\sum D_j)$ as $\mathrm{d}\mathbf{z} \wedge \mathrm{d}\mathbf{x}$.

Now $\Psi \in \mathcal{A}^m$ is *invariant* if $\deg_1 \Psi = \deg_2 \Psi = 0$ and an invariant Ψ "descends" ($\Psi = \pi^*\omega$) to $\mathbb{P}(E)$ iff $\langle \vec{e}_1, \Psi \rangle = \langle \vec{e}_2, \Psi \rangle = 0$ (i.e. if Ψ is also horizontal). Clearly if $P \in S^{a,b}$, $\Psi = (P\Omega/F^{k+r+1}) \in \mathcal{A}^{n+2r}$ has bidegree

$$\Big(a + (r+1) - (k+r+1)\,,\, b + (n+r+1 - \sum D_j) - 0\Big)$$

$$= \Big(a - k\,,\, b - \{\sum D_j - (n+r+1)\}\Big)\,.$$

So a Ψ of this form (which is automatically horizontal as $\langle \vec{e}_1, \Omega \rangle = \langle \vec{e}_2, \Omega \rangle = 0$) is invariant, and thus satisfies $\Psi = \pi^*\omega$, iff

$$a = k \quad \text{and} \qquad b = \sum D_j - (n+r+1) \;=\; \deg(K_X) \;=:\; D(X).$$

So $P\Omega/F^{k+r+1}$ "descends" to A^{n+2r}_{k+r+1} if $P \in S^{k,D(X)}$.

Conversely, if $\Psi = \pi^*\omega\,(\in \mathcal{A}^{n+2r})$ for $\omega \in A^{n+2r}_{k+r+1}$, Ψ is of the above form by an argument somewhat more complicated than the corresponding one in §4.2. Namely, set $\Psi_0 = F^{k+r+1}\Psi \in \mathcal{A}_0^{n+2r}$; since$\Psi$ is invariant (and must have bidegree $(0,0)$), Ψ_0 has bidegree $(k+r+1, 0)$; and since Ψ is horizontal, $\langle \vec{e}_1, \Psi_0 \rangle = \langle \vec{e}_2, \Psi_0 \rangle = 0$. We need to define another "degree" $\deg_{2,z}$ as eigenvalue of $[\mathrm{d}_z, \iota_{\vec{e}_2}]$; this is not necessarily well-defined on Ψ_0, which we may have to break into "eigenpolynomials" $\Psi_{0,\gamma}$ (with the same bidegree $(k+r+1, 0)$ and horizontality). Now $\deg_{2,z} z_i = \deg_{2,z} \mathrm{d}z_i = 1$, $\deg_{2,z} x_j = \deg_{2,z} \mathrm{d}x_j = 0$; since $\Psi_{0,\gamma}$ has no denominator it is made up of these, and in view of its bidegree must have at least one z_i or $\mathrm{d}z_i$, so that $\deg_{2,z} \Psi_{0,\gamma} > 0$. Using the fact that $[\mathrm{d}_z, \iota_{\vec{e}_1}] = 0$,

$$(k+r+1)\left(\deg_{2,z}(\Psi_{0,\gamma})\right)\Psi_{0,\gamma} \;=\; \deg_1(\Psi_{0,\gamma})\,[\mathrm{d}_z\,\langle \vec{e}_2, \Psi_{0,\gamma} \rangle + \langle \vec{e}_2, \mathrm{d}_z \Psi_{0,\gamma} \rangle]$$

$$=\; \deg_1(\Psi_{0,\gamma})\,\langle \vec{e}_2, \mathrm{d}_z \Psi_{0,\gamma} \rangle \;=\; \langle \vec{e}_2, \mathrm{d}_z[\deg_1(\Psi_{0,\gamma})\Psi_{0,\gamma}] \rangle$$

$$=\; \langle \vec{e}_2, \mathrm{d}_z\,[\mathrm{d}\,\langle \vec{e}_1, \Psi_{0,\gamma} \rangle + \langle \vec{e}_1, \mathrm{d}\Psi_{0,\gamma} \rangle] \rangle$$

$$= \langle \vec{e}_2, \mathrm{d}_z\,\langle \vec{e}_1, \mathrm{d}\Psi_{0,\gamma} \rangle \rangle = \langle \vec{e}_2, \langle \vec{e}_1, \mathrm{d}_z \mathrm{d}_x \Psi_{0,\gamma} \rangle \rangle$$

and so, setting

$$\varphi \;=\; \frac{1}{k+r+1} \sum_\gamma \frac{\mathrm{d}_z \mathrm{d}_x \Psi_{0,\gamma}}{\deg_{2,z}(\Psi_{0,\gamma})} \;\in \mathcal{A}_0^{n+2r+2},$$

we can write

$$\Psi \;=\; \frac{\Psi_0}{F^{k+r+1}} \;=\; \frac{\langle \vec{e}_2, \langle \vec{e}_1, \varphi \rangle \rangle}{F^{k+r+1}}.$$

Obviously the only possibility for $\varphi \in \mathbf{a}_0^{n+2r+2}$ is $P\mathrm{d}\mathbf{z} \wedge \mathrm{d}\mathbf{x}$, and so $\Psi = P\Omega/F^{k+r+1}$. In summary, we have worked out a polynomial representation

$$S^{k,D(X)} \xrightarrow{\cong} A^{n+2r}_{k+r+1}(Y)$$

$$P \longmapsto \frac{P\Omega}{F^{k+r+1}}.$$

Next we give a representation for $\mathrm{d}A^{n+2r-1}_{k+r}(Y) + A^{n+2r}_{k+r}(Y) \subseteq A^{n+2r}_{k+r+1}(Y)$. Namely, let[47]

$$J_F^{a,b} := \left(\frac{\partial F}{\partial z_0}, \ldots, \frac{\partial F}{\partial z_{n+r}}, \frac{\partial F}{\partial x_0}, \ldots, \frac{\partial F}{\partial x_r}\right) \subseteq S^{a,b}$$

and $R_F^{a,b} := S^{a,b} \big/ J_F^{a,b}$; then the claim is that under the above map,

$$J_F^{k,D(X)} \xrightarrow{\cong} \mathrm{d}A^{n+2r-1}_{k+r}(Y) + A^{n+2r}_{k+r}(Y).$$

If $\eta \in A^{n+2r-1}_{k+r}$, $\pi^*\eta =: \Psi = \Psi_0/F^{k+r}$, $\Psi_0 \in \mathcal{A}_0^{n+2r-1}$, the exactly the same argument as above expresses Ψ_0 as $\langle \vec{e}_2, \langle \vec{e}_1, \varphi\rangle\rangle$ where $\varphi \in \mathcal{A}_0^{n+2r+1}$ has bidegree $(k+r, 0)$ by Lemma 4.5.1. Such a φ is necessarily of the form

$$\varphi = \sum_{i=0}^{n+r} (-1)^i Q_i(\mathbf{z}, \mathbf{x}) \mathrm{d}\mathbf{z}^{(i)} \wedge \mathrm{d}\mathbf{x} + \sum_{j=0}^{r} (-1)^j R_j(\mathbf{z}, \mathbf{x}) \mathrm{d}\mathbf{z} \wedge \mathrm{d}\mathbf{x}^{(j)}$$

where $\mathrm{d}\mathbf{z}^{(i)} = \mathrm{d}z_0 \wedge \ldots \wedge \widehat{\mathrm{d}z_i} \wedge \ldots \wedge \mathrm{d}z_{n+r}$, $\mathrm{d}\mathbf{x}^{(j)} = \mathrm{d}x_0 \wedge \ldots \wedge \widehat{\mathrm{d}x_j} \wedge \ldots \wedge \mathrm{d}x_r$.

Now we shall differentiate Ψ in two steps. First, since φ has bidegree $(k+r, 0)$, φ/F^{k+r} has bidegree $(0,0)$ and so does[48] $\langle \vec{e}_1, \frac{\varphi}{F^{k+r}}\rangle$ by Lemma 4.5.2. So

$$\begin{aligned}
\mathrm{d}\Psi &= \mathrm{d}\frac{\langle \vec{e}_2, \langle \vec{e}_1, \varphi\rangle\rangle}{F^{k+r}} = \mathrm{d}\left\langle \vec{e}_2 \left\langle \vec{e}_1, \frac{\varphi}{F^{k+r}}\right\rangle\right\rangle \\
&= \overbrace{\left(\deg_2 \left\langle \vec{e}_1, \frac{\varphi}{F^{k+r}}\right\rangle\right)}^{=0} \left\langle \vec{e}_1, \frac{\varphi}{F^{k+r}}\right\rangle - \left\langle \vec{e}_2, \mathrm{d}\left\langle \vec{e}_1, \frac{\varphi}{F^{k+r}}\right\rangle\right\rangle \\
&= \left\langle \vec{e}_2, \left\langle \vec{e}_1, \mathrm{d}\left(\frac{\varphi}{F^{k+r}}\right)\right\rangle\right\rangle - \overbrace{\left(\deg_1 \frac{\varphi}{F^{k+r}}\right)}^{=0} \left\langle \vec{e}_2, \frac{\varphi}{F^{k+r}}\right\rangle \\
&= \left\langle \vec{e}_2, \left\langle \vec{e}_1, \frac{F\mathrm{d}\varphi - (k+r)\mathrm{d}F \wedge \varphi}{F^{k+r+1}}\right\rangle\right\rangle = \frac{\langle \vec{e}_2, \langle \vec{e}_1, F\mathrm{d}\varphi - (k+r)\mathrm{d}F \wedge \varphi\rangle\rangle}{F^{k+r+1}}.
\end{aligned}$$

[47] note $\partial F/\partial x_j = F_j(\mathbf{z})$!

[48] of course $\langle \vec{e}_1, \varphi\rangle \neq 0$ because $\langle \vec{e}_2, \langle \vec{e}_1, \varphi\rangle\rangle = \Psi_0 (\neq 0)$.

Next we plug in the above formula for φ and

$$\mathrm{d}\varphi = \sum_{i=0}^{n+r} \frac{\partial Q_i}{\partial z_i} \mathrm{d}\mathbf{z} \wedge \mathrm{d}\mathbf{x} \pm \sum_{j=0}^{r} \frac{\partial R_j}{\partial x_j} \mathrm{d}\mathbf{z} \wedge \mathrm{d}\mathbf{x} = \left(\sum \frac{\partial Q_i}{\partial z_i} \pm \sum \frac{\partial R_j}{\partial x_j} \right) \mathrm{d}\mathbf{z} \wedge \mathrm{d}\mathbf{x}.$$

We get

$$\mathrm{d}\Psi = \frac{\left\langle \vec{e}_2, \left\langle \vec{e}_1, F\left(\sum \frac{\partial Q_i}{\partial z_i} \pm \sum \frac{\partial R_j}{\partial x_j}\right) \mathrm{d}\mathbf{z} \wedge \mathrm{d}\mathbf{x} - (k+r)\mathrm{d}F \wedge \left(\sum (-1)^i Q_i \mathrm{d}\mathbf{z}^{(i)} \wedge \mathrm{d}\mathbf{x} + \sum (-1)^j \mathrm{d}\mathbf{z} \wedge \mathrm{d}\mathbf{x}^{(j)}\right) \right\rangle \right\rangle}{F^{k+r+1}}$$

$$= \frac{\left\langle \vec{e}_2, \left\langle \vec{e}_1, \left\{ F\left(\sum \frac{\partial Q_i}{\partial z_i} \pm \sum \frac{\partial R_j}{\partial z_j}\right) - \left(\sum (-1)^i \frac{\partial F}{\partial z_i} Q_i + \sum (-1)^j \frac{\partial F}{\partial x_j} R_j\right) \right\} \mathrm{d}\mathbf{z} \wedge \mathrm{d}\mathbf{x} \right\rangle \right\rangle}{F^{k+r+1}}$$

$$= \frac{(FP_1 - P_2)\,\Omega}{F^{k+r+1}},$$

where we have set $P_1 = \sum \frac{\partial Q_i}{\partial z_i} \pm \sum \frac{\partial R_j}{\partial x_j}$, $P_2 = \sum (-1)^i \frac{\partial F}{\partial z_i} Q_i + \sum (-1)^j \frac{\partial F}{\partial x_j} R_j \in J_F$ (and note that $P_1 F \in J_F$ trivially[49]). So $\mathrm{d}\eta \in \mathrm{d}A^{n+2r-1}_{k+r}(Y)$ is of the form $P\Omega/F^{k+r+1}$ for $P \in J_F$, and clearly any $P_2\Omega/F^{k+r+1}$ (with P_2 of the right bidegree) may, modulo $(P_1\Omega/F^{k+r}) \in A^{n+2r}_{k+r}(Y)$, be expressed as such a $\mathrm{d}\eta$.

We have shown that the map $P \mapsto P\Omega/F^{k+r+1}$ induces an isomorphism

$$R_F^{k,D(X)} \xrightarrow{\cong} \frac{A^{n+2r}_{k+r+1}(Y)}{\mathrm{d}A^{n+2r-1}_{k+r}(Y) + A^{n+2r}_{k+r}(Y)}$$

$$= \frac{H^0\left(\Omega^{n+2r}_{\mathbb{P}(E)}((k+r+1)Y)\right)}{\mathrm{d}H^0\left(\Omega^{n+2r-1}_{\mathbb{P}(E)}((k+r)Y)\right) + H^0\left(\Omega^{n+2r}_{\mathbb{P}(E)}((k+r)Y)\right)}.$$

4.5.3. Rational forms and C^∞ dlog forms on $\mathbb{P}(E) \setminus Y$. The missing link between Jacobi rings $R_F^{*,*}$ and cohomology of X, is an argument relating the right-hand side of this isomorphism to the cohomology of Y. We want more than just "an" isomorphism for this missing link. We should be able to take a representative form $\omega \in H^0\left(\Omega^{n+2r}_{\mathbb{P}(E)}((k+r+1)Y)\right)$, add some coboundaries (of C^∞ polar forms), and end up with[50] $\beta \in Z^0_{\mathrm{d}}\left(\Omega^{n+r-k,k+r}_{(\mathbb{P}(E))^\infty}(\mathrm{dlog}Y)\right)$ so that $\alpha := \frac{1}{2\pi i}\mathrm{Res}_Y \beta \in Z^0_{\mathrm{d}}\left(\Omega^{n+r-k-1,k+r}_{Y^\infty}\right)$, and the corresponding class $[i^*\alpha] \cap \xi^{2r} \in H^{n-k-1,k}(X))$, have the same periods as ω. The spectral sequence approach that got pushed into a footnote in §4.2.2, if the isomorphism it provides is interpreted correctly (on the right representative forms), will give such a procedure. So we say some words to resurrect this in our present context.

---

[49]Consider $\sum x_j \frac{\partial F}{\partial x_j} = \sum x_j \frac{\partial}{\partial x_j}(x_0 F_0(\mathbf{z}) + \ldots + x_r F_r(\mathbf{z})) = \sum x_j F_j(\mathbf{z}) = F$, which $\implies F \in J_F$.

[50]Z^0_{d} just means d-closed sections; we use this and Γ for C^∞ sheaves.

We use the following notation, for F a holomorphic (analytic) sheaf on a variety V. First, we write $H^0(V,\mathsf{F})$ for sections as usual (and $H^i(\mathsf{F})$ for ordinary Čech cohomology), but wish to point out that while a section of such a sheaf is necessarily $\bar{\partial}$-closed, it is not always ∂-closed – unless one is in a special case like $\mathsf{F}=\Omega^{n+2r}_{\mathbb{P}(E)}(kY)$, where there is nowhere to go (since $\dim\mathbb{P}(E)=n+2r$). Next, let

$$\mathsf{A}^{0,j}(V,\mathsf{F})=\Gamma(\Omega^{0,j}_{V^\infty}\otimes\mathsf{F})\,;$$

these form a complex with $\bar{\partial}$ as differential, and we denote the $\bar{\partial}$-closed elements as $Z^{0,j}_{\bar{\partial}}(V,\mathsf{F})$ and cohomology by $H^j_{\bar{\partial}}(V,\mathsf{F})$. Note that $H^j_{\bar{\partial}}(\mathsf{F})\cong H^j(\mathsf{F})$ and $Z^{0,0}_{\bar{\partial}}(\mathsf{F})=H^0(\mathsf{F})$. If $\mathsf{F}=\Omega^m_{\mathbb{P}(E)}(kY)$ then $\mathsf{A}^{0,j}(\mathsf{F})=\Gamma\left(\Omega^{m,j}_{(\mathbb{P}(E))^\infty}(kY)\right)$.

Suppose we have a long exact sequence

$$0\to\mathsf{F}_0\xrightarrow{\delta}\dots\xrightarrow{\delta}\mathsf{F}_m\to 0$$

of holomorphic sheaves on V; then as a complex $\mathsf{F}_\bullet\simeq 0$ and the hypercohomology spectral sequence converges to zero:

$$E_1^{i,j}\;=\;H^j(V,\mathsf{F}_i)\qquad\Longrightarrow\qquad\mathbb{H}^*(V,\mathsf{F}_\bullet)[=0].$$

There are different ways of setting up the E_0-term: $E_0^{i,j}=\check{C}^j(V,\mathsf{F}_i)$ will work, but for our purposes

$$E_0^{i,j}\;=\;\mathsf{A}^{0,j}(V,\mathsf{F}_i)\qquad(\Longrightarrow\;0)$$

is preferable. In fact since *both* differentials (δ and $\bar{\partial}$) are defined on $E_0^{i,j}$, one can view this as a double complex, with associated simple complex (under the total differential $\delta+\bar{\partial}$) computing hypercohomology. Then we have that this simple complex[51] is exact. We also point out that the *rows* of the double complex $E_0^{i,j}$ are exact.

There is a version of ("weak") Bott vanishing for ample divisors in toric varieties, which says for $Y\subset\mathbb{P}(E)$

$$H^j_{(\bar{\partial})}\left(\mathbb{P}(E),\Omega^i_{\mathbb{P}(E)}(kY)\right)=0\qquad\text{for } j,\,k>0\quad(\text{and } i\geq 0)\;;$$

in particular this means for us that

$$Z^{0,j}_{\bar{\partial}}\left(\Omega^i_{\mathbb{P}(E)}(kY)\right)\;=\;\bar{\partial}\mathsf{A}^{0,j-1}\left(\Omega^i_{\mathbb{P}(E)}(kY)\right),\quad j,\,k>0.$$

Let $\mathsf{F}_\bullet$ $(0\leq\bullet\leq k+r+1)$ be the exact sequence of holomorphic sheaves (again see [**L1**])

$$(0\to)\Omega^{n+r-k}_{\mathbb{P}(E)}(\mathrm{dlog}Y)\hookrightarrow\Omega^{n+r-k}_{\mathbb{P}(E)}(Y)\xrightarrow{\partial}\frac{\Omega^{n+r-k+1}_{\mathbb{P}(E)}(2Y)}{\Omega^{n+r-k+1}_{\mathbb{P}(E)}(Y)}\xrightarrow{\partial}\dots\xrightarrow{\partial}\frac{\Omega^{n+2r}_{\mathbb{P}(E)}((k+r+1)Y)}{\Omega^{n+2r}_{\mathbb{P}(E)}((k+r)Y)}(\to 0)$$

[51]the terms are just the sums $\oplus_{i+j=\bullet}E_0^{i,j}$ of diagonal elements.

viewed as a complex, so that[52]

$$E_0^{i,j} = \mathsf{A}^{0,j}\left(\mathbb{P}(E), \mathsf{F}_i\right) = \begin{cases} \Gamma\left(\Omega^{n+r-k,\,j}_{\mathbb{P}(E)^\infty}(\mathrm{dlog}Y)\right), & i=0 \\ \Gamma\left(\Omega^{n+r-k,\,j}_{\mathbb{P}(E)^\infty}(Y)\right), & i=1 \\ \dfrac{\Gamma\left(\Omega^{n+r+i-k-1,\,j}_{\mathbb{P}(E)^\infty}(iY)\right)}{\Gamma\left(\Omega^{n+r+i-k-1,\,j}_{\mathbb{P}(E)^\infty}((i-1)Y)\right)}, & i \geq 2 \end{cases}$$

(I) converges to 0 (taking $d_0 = \bar\partial,\ d_1 = \partial$, etc.) as a spectral sequence,

(II) is exact as a simple complex under the total differential $\mathrm{d} = \partial + \bar\partial$;

and as a double complex:

(III) has, by Bott vanishing, exact columns (under $\bar\partial$) except at $i=0$ or $j=0$,

(IV) has exact rows.

Using (I) and (III) we obtain an isomorphism

$$d_{k+r+1} : \left(E^{0,k+r}_{k+r+1} = E_1^{0,k+r}\right) \longrightarrow \left(E^{k+r+1,0}_{k+r+1} = E_2^{k+r+1,0}\right)$$

where

$$E_1^{0,k+r} = \frac{Z_{\bar\partial}\left(\Omega^{n+r-k,\,k+r}_{\mathbb{P}(E)^\infty}(\mathrm{dlog}Y)\right)}{\bar\partial\Gamma\left(\Omega^{n+r-k,\,k+r-1}_{\mathbb{P}(E)^\infty}(\mathrm{dlog}Y)\right)} \qquad \text{and}$$

$$E_2^{k+r+1,0} = \frac{\ker\bar\partial \subseteq E_0^{k+r+1,0}}{\partial\left\{\ker\bar\partial \subseteq E_0^{k+r,0}\right\}} = \frac{Z_{\bar\partial}\left(\Omega^{n+2r}_{\mathbb{P}(E)^\infty}((k+r+1)Y)\right)}{\partial Z_{\bar\partial}\left(\Omega^{n+2r-1}_{\mathbb{P}(E)^\infty}((k+r)Y)\right) + Z_{\bar\partial}\left(\Omega^{n+2r}_{\mathbb{P}(E)^\infty}((k+r)Y)\right)}$$

$$= \frac{H^0\left(\Omega^{n+2r}_{\mathbb{P}(E)}((k+r+1)Y)\right)}{\mathrm{d}H^0\left(\Omega^{n+2r-1}_{\mathbb{P}(E)}((k+r)Y)\right) + H^0\left(\Omega^{n+2r}_{\mathbb{P}(E)}((k+r)Y)\right)} = \frac{A^{n+2r}_{k+r+1}}{\mathrm{d}A^{n+2r-1}_{k+r} + A^{n+2r}_{k+r}}.$$

Perhaps it makes more sense to regard $E_1^{0,k+r}$ as the $(n+r-k)$th Hodge-graded piece of

$$\frac{Z_{\mathrm{d}}\left(\Omega^{n+2r}_{\mathbb{P}(E)^\infty}(\mathrm{dlog}Y)\right)}{\mathrm{d}\Gamma\left(\Omega^{n+2r-1}_{\mathbb{P}(E)}(\mathrm{dlog}Y)\right)} \cong H^{n+2r}\left(\mathbb{P}(E)\right) \setminus Y, \mathbb{C});$$

one gets better-behaved representatives by observing that

$$\frac{Z_d\left(\Omega^{n+r-k,\,k+r}_{\mathbb{P}(E)^\infty}(\mathrm{dlog}Y)\right)}{\partial\bar\partial\Gamma\left(\Omega^{n+r-k-1,\,k+r-1}_{\mathbb{P}(E)^\infty}(\mathrm{dlog}Y)\right)} \xrightarrow{\cong} E_1^{0,\,k+r}.$$

[52]We remind the reader that for sections of C^∞ sheaves $\Gamma(\mathsf{E}/\mathsf{F}) = \Gamma(\mathsf{E})/\Gamma(\mathsf{F})$ (think partitions of unity), in contrast to the holomorphic case (where partitions of unity are not available).

To trace through this let

$$\beta \in Z_{\bar\partial}\left(\Omega^{n+r-k,\,k+r}_{\mathbb{P}(E)^\infty}(\mathrm{dlog}Y)\right) \subseteq E_0^{0,\,k+r}\left(\hookrightarrow E_0^{1,k+r}\right)$$

be any representative (of a class in $E_1^{0,k+r}$). Then there are forms

$$\zeta_\ell \in \Gamma\left(\Omega^{n+r+\ell-k-1,\,k+r-\ell}_{\mathbb{P}(E)^\infty}(\ell Y)\right), \qquad \ell = 1, \dots, k+r$$

$$\sigma_\ell \in \Gamma\left(\Omega^{n+r+\ell-k,\,k+r-\ell}_{\mathbb{P}(E)}(\ell Y)\right), \qquad \ell = 1, \dots, k+r$$

(here $\{\zeta_\ell\}$ live in the *numerators* of $E_0^{\ell,\,k+r-\ell}$, $\{\sigma_\ell\}$ in the *denominators* of $E_0^{\ell+1,\,k+r-\ell}$) satisfying

$$\begin{array}{ccc} \bar\partial\sigma_1 = \partial\beta & \qquad & \bar\partial\zeta_1 = \beta \\ \bar\partial\sigma_2 = \partial\sigma_1 & & \bar\partial\zeta_2 = \partial\zeta_1 - \sigma_1 \\ \vdots & & \vdots \\ \bar\partial\sigma_{k+r} = \partial\sigma_{k+r-1} & & \bar\partial\zeta_{k+r} = \partial\zeta_{k+r-1} \pm \sigma_{k+r-1} \\ 0 = \partial\sigma_{k+r} & & \end{array}$$

and

$$\partial\zeta_{k+r} - \sigma_{k+r} \quad \in \quad Z_{\bar\partial}\left(\Omega^{n+2r,\,0}_{\mathbb{P}(E)^\infty}((k+r+1)Y)\right) = H^0\left(\Omega^{n+2r}_{\mathbb{P}(E)}((k+r+1)Y)\right).$$

In a picture,

$$\begin{array}{|llllll} 0 & & & & & \\ \beta & \beta & & & & \\ & \zeta_1 & (\sigma_1) & & & \\ & & \zeta_2 & (\sigma_2) & & \\ & & & \ddots & \ddots & \\ & & & & \ddots & (\sigma_{k+r-1}) \\ & & & & & \zeta_{k+r} \quad (\sigma_{k+r}) \\ \hline \end{array}$$

where the lower diagonal (or rather the images of those forms in the $E_0^{*,*}$ quotient-terms) is an (E_0^*, d)-cycle – this gives d_{k+r+1} – while the sum of all the upper diagonal terms is d-closed without taking quotients. Putting everything together gives the lousy relation

$$\beta - (\mathrm{d}\zeta_1 - \mathrm{d}\zeta_2 + \ldots \pm \mathrm{d}\zeta_{k+r}) + (\sigma_1 + \ldots + \sigma_{k+r}) \;=\; \pm(\partial\zeta_{k+r} \pm \sigma_{k+r}).$$

But if it is observed that taking β to be a d-closed representative (see above) *makes all* $\sigma_\ell = 0$, this relation becomes

$$\beta \mp \partial\zeta_{k+r} \;=\; \mathrm{d}\,(\zeta_1 - \zeta_2 + \ldots \pm \zeta_{k+r})$$

which is not lousy at all. It says that the isomorphism d_{k+r+1} may be interpreted as sending d-closed representatives $\beta \in Z_{\mathrm{d}}\left(\Omega^{n+r-k,\,k+r}_{\mathbb{P}(E)^\infty}(\mathrm{dlog}Y)\right)$ of $E_1^{0,\,k+r}$ to representatives $\partial\zeta_{k+r} \in \partial Z_{\partial\bar\partial}\left(\Omega^{n+2r-1,\,0}_{\mathbb{P}(E)^\infty}((k+r)Y)\right)$ of $E_2^{k+r+1,\,0}$, to which they are related by a d-coboundary which does not change the periods.

4.5.4. Putting everything together; representation of $\bar{\nabla}$. Using Leray-Hirsch to count rank, one finds that for $2r-1 \leq * \leq 2n+2r-1$,

$$H^*(\mathbb{P}(E)) \underset{\cong}{\overset{\cup[H]}{\longrightarrow}} H^{*+2}(\mathbb{P}(E)) \,,$$

where the hyperplane class is given by a tacit embedding (of $\mathbb{P}(E)$ in some $\mathbb{P}^N$) which realizes Y as a hypersurface section (possible since Y is very ample – see [**N**]), say of degree D_Y. If $n \geq 2$ ($\dim X \geq 1$) then $n+2r-3 \leq * \leq n+2r+1$ will do. On the one hand this shows

$$H^{n+2r-1}_{pr}(Y) \overset{\cong}{\longrightarrow} H^{n+2r-1}_{var}(Y)$$

via the diagram (with coefficients $\in \mathbb{Q}$)

$$\begin{tikzcd}
H^{n+2r-1}(Y) \arrow[dd, "\cup([H]\cdot D_Y)"'] \arrow[dr] & H^{n+2r-1}(\mathbb{P}(E)) \arrow[l] \arrow[d, "\cup([H]\cdot D_Y)", "\cong"'] \\
 & H^{n+2r+1}(\mathbb{P}(E)) \arrow[dl] \arrow[d, "\cup([H]\cdot D_Y)", "\cong"'] \\
H^{n+2r+1}(Y) \arrow[r] & H^{n+2r+3}(\mathbb{P}(E)).
\end{tikzcd}$$

On the other hand we get

$$H^{n+2r}(\mathbb{P}(E)\setminus Y) \underset{\cong}{\overset{\mathrm{Res}}{\longrightarrow}} H^{n+2r-1}_{pr}(Y)$$

by considering

$$\begin{tikzcd}
{} \arrow[r] & H^{n+2r-2}(Y) \arrow[r, "\mathrm{Gy}"] & H^{n+2r}(\mathbb{P}(E)) \arrow[r] & H^{n+2r}(\mathbb{P}(E)\setminus Y) \arrow[r] & H^{n+2r-1}(Y) \arrow[r, "\mathrm{Gy}"] \arrow[d, "\cup[H]"] & H^{n+2r+1}(\mathbb{P}(E)) \arrow[r] \arrow[d, "\cup[H]", "\cong"'] & {} \\
 & H^{n+2r-2}(\mathbb{P}(E)) \arrow[u] \arrow[ur, "\cong", "\cup([H]\cdot D_Y)"'] & & & H^{n+2r+1}(Y) \arrow[r] & H^{n+2r+3}(\mathbb{P}(E)). &
\end{tikzcd}$$

Writing

$$Gr_F^a H^b_{[\mathrm{dlog}Y]}(\mathbb{P}(E)\setminus Y,\mathbb{C}) \;:=\; \frac{Z_{\mathrm{d}}\left(F^a\Omega^b_{\mathbb{P}(E)^\infty}(\mathrm{dlog}Y)\right)}{\mathrm{d}\Gamma\left(F^a\Omega^{b-1}_{\mathbb{P}(E)^\infty}(\mathrm{dlog}Y)\right)+Z_{\mathrm{d}}\left(F^{a+1}\Omega^b_{\mathbb{P}(E)^\infty}(\mathrm{dlog}Y)\right)}$$

we now have the whole composition

$$\begin{tikzcd}
R_F^{k,D(X)} \arrow[r, "\cong"] \arrow[d, "\cong"', "P\mapsto \frac{P\Omega}{F^{k+r+1}}"] & Gr_F^{n-k-1}H^{n-1}_{pr}(X,\mathbb{C}) & Gr_F^{n-k-1}H^{n-1}_{var}(X,\mathbb{C})\left[\otimes H^{2r}(\mathbb{P}^r)\right] \arrow[l, "\cong"'] \\
\frac{A^{n+2r}_{k+r+1}(Y)}{\mathrm{d}A^{n+2r-1}_{k+r}+A^{n+2r}_{k+r}} & & Gr_F^{n+r-k-1}H^{n+2r-1}_{var}(Y,\mathbb{C}) \arrow[u, "\cong", "\iota^*"'] \\
\frac{Z_{\mathrm{d}}\left(\Omega^{n+r-k,k+r}_{\mathbb{P}(E)^\infty}(\mathrm{dlog}Y)\right)}{\partial\bar{\partial}\Gamma\left(\Omega^{n+r-k-1,k+r-1}_{\mathbb{P}(E)^\infty}(\mathrm{dlog}Y)\right)} \arrow[u, "\cong"', "d_{k+r}"] \arrow[r, "\cong"] & Gr_F^{n+r-k}H^{n+2r}_{[\mathrm{dlog}Y]}(\mathbb{P}(E)\setminus Y,\mathbb{C}) \arrow[r, "\cong", "\mathrm{Res}"'] & Gr_F^{n+r-k-1}H^{n+2r-1}_{pr}(Y,\mathbb{C}) \arrow[u, "\cong"]
\end{tikzcd}$$

All the maps of forms/cohomology classes respect periods (at least in the graded sense[53]), so that if

$$X_t = \mathcal{V}\left(F_0(t),\dots,F_r(t)\right)\,,\quad F(t) \;=\; F+tG\,,\quad G\in S^{1,0}$$

is a 1-parameter variation of X (and Y), differentiating $\omega_t \in \frac{A^{n+2r}_{k+r+1}(Y_t)}{\mathrm{d}A^{n+2r+1}_{k+r}(Y_t)+A^{n+2r}_{k+r}(Y_t)}$ at $t=0$ leads (by exactly the same argument as in §2.2) to a formula for $\bar{\nabla}_{\partial/\partial t}$:

$$\begin{tikzcd}
R_F^{k,D(X)} \arrow[r, "\cong"] \arrow[d, "\times G"] & \frac{A^{n+2r}_{k+r+1}(Y)}{\mathrm{d}A^{n+2r-1}_{k+r}+A^{n+2r}_{k+r}} \arrow[r, "\cong"] \arrow[d, "(\bar{\nabla}_{\partial/\partial t})"] & Gr_F^{n-k-1}H^{n-1}_{pr}(X) \arrow[d, "\bar{\nabla}_{\partial/\partial t}"] \\
R_F^{k+1,D(X)} \arrow[r, "\cong"] & \frac{A^{n+2r}_{k+r+2}(Y)}{\mathrm{d}A^{n+2r-1}_{k+r+1}+A^{n+2r}_{k+r+1}} \arrow[r, "\cong"] & Gr_F^{n-k}H^{n-1}_{pr}(X).
\end{tikzcd}$$

For $\mathcal{S} \underset{\text{Zar. op.}}{\subset} \mathbb{P}H^0(\mathbb{P}^{n+r},E)$ we have that

$$\theta^1_{\mathcal{S},t}\cong S^{1,0}/(F_t)\;,$$

[53]The maps themselves respect periods on the nose, including d_{k+r} (or d_{k+r}^{-1}) provided the special representatives (see above) are used; the problem is that, even if $\frac{P(t)\Omega}{F(t)^{k+r+1}}$ is such a representative for each t, $\partial/\partial t$ of it is *not* (it may be of the form $\partial\zeta-\sigma$ rather than $\partial\zeta$); the way one remedies this situation involves ignoring periods of the next higher filtration. Details are left to the reader.

so for a "full" family $\mathcal{X} \to \mathcal{S}$ one gets at $0 \in \mathcal{S}$

$$\begin{tikzcd}
\bigwedge^2 S^{1,0}/(F) \otimes R_F^{k-1,D(X)} \arrow[r, "\mu^2_{(k)}"] \arrow[d, "\cong"] & S^{1,0}/(F) \otimes R_F^{k,D(X)} \arrow[r, "\mu^1_{(k+1)}"] \arrow[d, "\cong"] & R_F^{k+1,D(X)} \arrow[d, "\cong"] \\
\bigwedge^2 \theta^1_{\mathcal{S},o} \otimes Gr_F^{n-k} H^{n-1}_{pr}(X) \arrow[r] & \theta^1_{\mathcal{S},o} \otimes Gr_F^{n-k-1} H^{n-1}_{pr}(X) \arrow[r] & Gr_F^{n-k-2} H^{n-1}_{pr}(X)
\end{tikzcd}$$

where the bottom row dualizes to the $\mathcal{O}_{\mathcal{S}}$-linear part of ∇

$$Gr_{\mathcal{F}}^{k+1} \mathcal{H}^{n-1}_{X_s,var} \overset{\bar{\nabla}^1_{(k+1)}}{\longrightarrow} \Omega^1_{\mathcal{S}} \otimes Gr_{\mathcal{F}}^{k} \mathcal{H}^{n-1}_{X_s,var} \overset{\bar{\nabla}^2_{(k)}}{\longrightarrow} \Omega^2_{\mathcal{S}} \otimes Gr_{\mathcal{F}}^{k-1} \mathcal{H}^{n-1}_{X_s,var}$$

at 0. So $\mu^2_{(k)}$ and $\nabla^2_{(k)}$, $\mu^1_{(k+1)}$ and $\nabla^1_{(k+1)}$, are dual.

4.5.5. Algebraic lemmas and multidegree bounds. Now we have set everything up so that the Proposition and Theorem below will go through (for complete intersections $\subset \mathbb{P}^{n+r}$), by exactly the same arguments as in §4.3 (for hypersurfaces $\subset \mathbb{P}^n$), if the following two algebraic lemmas are proved.

LEMMA 4.5.3. $\mu^1_{(k+1)}$ *is surjective for* $1 \le k+1 \le n-1$, *if* $\deg(K_X) \ge 0$.

(The resulting injectivity of $\bar{\nabla}^1_{(k+1)}$ then yields the proof of Proposition.)

LEMMA 4.5.4. $\mu^2_{(n-1)}$ *is surjective for* $\begin{cases} \deg(K_X) > 0\,, & n = 2 \\ \deg(K_X) \ge 0\,, & n \ge 3 \end{cases}$.

(Injectivity of $\bar{\nabla}^2_{(n-1)}$ then is used to show the infinitesimal invariant of the regulator image is zero. Recall that for $\deg(K_X) < 0$ this is automatically zero and no lemma is needed.)

For Lemma 4.5.3 it is enough to show

$$S^{1,0} \otimes S^{k,D(X)} \overset{\text{mult.}}{\twoheadrightarrow} S^{k+1,\,D(X)}.$$

Referring to [**N**] Lemma 3.4, we need to check that

$$\sum_{j=0}^{r} D_j I_j \,+\, D(X) \ge 0$$

for all "multi-indices" I with $|I|(:= \sum I_j) = k$. If D_ℓ is the smallest of the D_j then setting $I = (0, \ldots, \underset{\ell}{k}, \ldots, 0)$ minimizes the left-hand side; using $D(X) = \deg K_X = \sum D_j - (n+r+1)$ we have

$$\sum D_j \,+\, kD_\ell \ge n+r+1 \qquad \text{for} \quad 0 \le k \le n-2$$

if $\deg(K_X) \geq 0$. (So this boils down to k, $D(X) \geq 0$, which is what one would expect.)

For Lemma 4.5.4 consider the diagram

$$\begin{tikzcd}
\bigwedge^2 S^{1,0} \otimes J_F^{n-2,D(X)} \arrow[r] \arrow[d, hook] & S^{1,0} \otimes J_F^{n-1,D(X)} \arrow[r, "m^1_{(n)}"] \arrow[d, hook] & J_F^{n,D(X)} \arrow[d, hook] \\
\bigwedge^2 S^{1,0} \otimes S^{n-2,D(X)} \arrow[r, "M^2_{(n-1)}"] \arrow[d, two heads] & S^{1,0} \otimes S^{n-1,D(X)} \arrow[r, "M^1_{(n)}"] \arrow[d, two heads] & S^{n,D(X)} \arrow[d, two heads] \\
\bigwedge^2 S^{1,0} \otimes R_F^{n-2,D(X)} \arrow[r] \arrow[d, two heads] & S^{1,0} \otimes R_F^{n-1,D(X)} \arrow[r] \arrow[d, two heads] & R_F^{n,D(X)} \arrow[d, equal] \\
\bigwedge^2 S^{1,0}/(F) \otimes R_F^{n-2,D(X)} \arrow[r, "\mu^2_{(n-1)}"] & S^{1,0}/(F) \otimes R_F^{n-1,D(X)} \arrow[r, "\mu^1_{(n)}"] & R_F^{n,D(X)}
\end{tikzcd}$$

We will show (for certain multi-degrees)

(a) $\ker \mu^1_{(n)} \subseteq \mathrm{im}\mu^2_{(n-1)}$

(b) $R_F^{n,\,D(X)} = 0$.

Together these $\implies$ $\mu^2_{(n-1)}$ surjective as desired. By a diagram chase (a) reduces to

($a1$) surjectivity of $m^1_{(n)}$

($a2$) $\ker M^1_{(n)} \subseteq \mathrm{im} M^2_{(n-1)}$.

Now (b) follows from our residue representation of $H^{n-1}_{var}(X)$, since $R_F^{n,\,D(X)} \cong Gr_F^{-1} H^{n-1}_{pr}(X) = 0$; while ($a1$) and ($a2$) are covered respectively by [Nagel] Lemmas 3.6 and 3.8 (where we must substitute n for p and $n+r$ for n).

We work out the multi-degree requirements for these, if necessary re-indexing so that $D_0 \geq \ldots \geq D_r \geq 2$. Nagel's assumption "$r \leq n-3$" corresponds for us to $n \geq 3$; this is a superfluous assumption and plays no role in the proofs, so we assume $n \geq 2$ and $r \geq 1$ (codimension ≥ 2; no need to repeat §$4.2-3$). Since there is no work to do in case $K_X < 0$ we assume also $\sum_{j=0}^r D_j \geq n+r+1$ ($K_X \geq 0$).

Nagel's conditions are (for us)

$$(*) \qquad \sum_{j=0}^{r} D_j \;+\; (n-2)D_r \;\geq\; n+r+2$$

$$(**) \qquad \sum_{j=1}^{r} D_j \ + \ (n-1)D_r \ \geq \ n+r+1.$$

The left-hand side of $(**)$ is $\geq 2r + (n-1)\cdot 2 = 2n+2r-2 \geq n+r+1$ since $n+r \geq 3$; so the second condition offers no resistance. We break $(*)$ into cases:

($X =$ curve) for $n = 2$, we have $\sum_{j=0}^{r} D_j \geq n+r+2$ which demands $K_X > 0$ (no elliptic curves, please!)

($\dim X \geq 2$) for $n \geq 3$, $K_X \geq 0$ cinches it: since $D_r \geq 2$, $(n-2)D_r \geq 1$.

This proves Lemma 4.5.4 in the cases/degrees claimed.

4.5.6. Statement of the Main Theorem. In order to state the final result, we define two types of complete intersections $\subset \mathbb{P}^{n+r}$. Either type requires X to be smooth, of multidegree $(D_0, \ldots, D_r)$ with all $D_j \geq 2$ (to avoid redundancy), and that $\dim X = n-1 \geq 1$ (i.e. $n \geq 2$). Furthermore, type [B] disallows the case

$$\left\{ n = 2 \ (\dim X = 1), \ \deg(K_X)\,(= \sum D_j - (n+r+1)\,) = 0 \right\}$$

of elliptic curves. On the other hand, type [A] requires a non-negative canonical bundle ($\sum D_j \geq n+r+1$).

PROPOSITION 4.5.5. *Let $X \subset \mathbb{P}^{n+r}$ be a very general type* [A] *complete intersection. Then*

$$im\left\{ H^{n-1}(X, \mathbb{C}/\mathbb{Q}(n)) \to H^{n-1}(\eta_X, \mathbb{C}/\mathbb{Q}(n)) \right\} \cong_{n.c.}$$

$$\varinjlim_{V \subset X} \left(coker\left\{ H_V^{n-1}(X, \mathbb{C}/\mathbb{Q}(n)) \to H^{n-1}(X, \mathbb{C}/\mathbb{Q}(n)) \right\} \right) \cong$$

$$H_{var}^{n-1}(X, \mathbb{C}/\mathbb{Q}(n)) \ [\neq 0],$$

where the limit is over all (arbitrary unions of) divisors $V \subset X$.

THEOREM 4.5.6. (Main Theorem)

(I) [Vanishing] *Let $X \subset \mathbb{P}^{n+r}$ be a very general type* [B] *complete intersection. Then the image of the "holomorphic regulator"*

$$R : \ K_n^M(X) \longrightarrow im\left\{ H^{n-1}(X, \mathbb{C}/\mathbb{Q}(n)) \to H^{n-1}(\eta_X, \mathbb{C}/\mathbb{Q}(n)) \right\}$$

is zero. [Collino has proved the counterexample for elliptic curves.]

(II) [Rigidity] *Let $\mathcal{X} \to \mathcal{S} \underset{Zar.\ op.}{\subset} \mathbb{P}H^0(\mathbb{P}^{n+r}, \mathcal{O}(D_0) \oplus \ldots \oplus \mathcal{O}(D_r))$ be a family of type* [B] *complete intersections. Then if $\{\boldsymbol{f}_s\} \in \ker(Tame) \subseteq K_n^M(\mathbb{C}(X))$ is a local section (on an open ball), its regulator image*

$$[R_{\boldsymbol{f}_s}] \in H^{n-1}(\eta_{X_s}, \mathbb{C}/\mathbb{Q}(n))$$

is flat.

CHAPTER 5

Applications to Algebraic Cycles

5.1. Higher Abel-Jacobi Maps

5.1.1. Relative Chow groups revisited. Our work on regulator formulas in Chapters 1 and 2 turns out to be useful in the study of rational equivalence of cycles, in particular the detection of rational *in*equivalences not picked up by $c_{\mathcal{D}}$ (an amalgam of the cycle-class and AJ maps). The most immediate application is in fact to relative cycles (and relative rational equivalences) on $(\square^n, \partial\square^n) = (\mathbb{P}^1 - \{1\}, \{0, \infty\})^n$; this is what we start with.

Let $k \subset \mathbb{C}$ be any subfield with $\operatorname{trdeg}(k/\mathbb{Q}) =: t < \infty$, and consider a relative cycle $\mathcal{Z} \in CH^p\left((\square^n, \partial\square^n)(k)\right)$, $p \leq n$. We may exchange the extension $k/\mathbb{Q}$ for geometry by spreading[1] over a projective variety $S/\mathbb{Q}$, $k \cong \mathbb{Q}(S)$, to a relative cycle

$$\zeta \in Z^p\left(\left\{\eta_S^{\mathbb{Q}} \times (\square^n, \partial\square^n)\right\}(\mathbb{Q})\right) := \varinjlim_{D/\mathbb{Q} \subset S} Z^p\left(\{(S \setminus D) \times \square^n,\, (S \setminus D) \times \partial\square^n\}(\mathbb{Q})\right).$$

As suggested in §1.3.4, we may then consider the "Abel-Jacobi" $R(\zeta)$ of the spread; the resulting composite

$$CH^p\left((\square^n, \partial\square^n)(k)\right) \xrightarrow{\cong} CH^p\left(\left\{\eta_S^{\mathbb{Q}} \times (\square^n, \partial\square^n)\right\}(\mathbb{Q})\right)$$

$$\xrightarrow[\text{see §1.3.4}]{\cong} CH^p(\eta_S^{\mathbb{Q}}, n) \xrightarrow[\text{see §2.4}]{\mathcal{R}} H_{\mathcal{D}}^{2p-n}(\eta_S^{\mathbb{Q}}, \mathbb{Q})$$

may be used to detect $\underset{\text{rat}}{\not\equiv} 0$ (and is well-defined by construction). Provided an extension of the Bloch-Beilinson conjecture to relative quasi-projective varieties holds, it is injective (modulo torsion) and detects rational $\equiv$ or $\not\equiv$ completely. Our point of view is that we are using analysis (really calculus) to detect complex geometry.

In §1.4 we performed a calculation for $p = n$ which in hindsight amounts to saying that the Milnor regulator

$$\mathcal{R}: CH^n(S \setminus D, n) \longrightarrow H_{\mathcal{D}}^n(S \setminus D, \mathbb{Q}(n)) \qquad (\text{under } \varinjlim_{D \subset S})$$

[1] see §4.3 or also **[GG5]**

is "in the same spirit" as the classical Abel-Jacobi map. That is, performing the integrals

$$\int_S \Omega_{\mathbf{f}} \wedge \omega \quad \text{and} \quad \int_S R'_{\mathbf{f}} \wedge \alpha\,,$$

where $\omega \in \Gamma\left(\frac{F^0}{F^{t-n}}\Omega^{2t-n}_{(S\setminus D)^\infty}\right)$ and $\alpha \in \Gamma\left(\Omega^{2t-n+1}_{(S\setminus D)^\infty}\right)$, is the same as finding the bounding chain $\partial_\epsilon^{-1}\gamma_{\mathbf{f}}$ for $\gamma_{\mathbf{f}}^\epsilon$ (if possible) and computing $[\lim_{\epsilon\to 0}]$

$$\int_{\gamma_{\mathbf{f}}} \wedge^n \mathrm{dlog} z_i \wedge \pi_S^*\omega \qquad \text{and} \qquad \int_{\partial_\epsilon^{-1}\gamma_{\mathbf{f}}} \wedge^n \mathrm{dlog} z_i \wedge \pi_S^*\alpha\,,$$

where the latter is the intuitively "obvious" extension of the classical AJ map.

In §5.1.2 we will look at the classical AJ on spreads of cycles $\mathcal{Z} \in Z^p(X(k))$ for X smooth projective $/\mathbb{Q}$. To see that we are doing essentially the same thing here (for $(\square^n, \partial\square^n)$ instead of X) with the above composite, it suffices to check that $\mathcal{R}$ identifies with the "logical" extension of the AJ map. But we never did this for $p < n$, so we sketch the argument now.

If $\zeta \in Z^p\left((S \setminus D) \times \square^n, (S \setminus D) \times \partial\square^n\right)$ is a relative cycle then (for $p < n$) one can show that the *topological* cycle ζ_ϵ on $(S, D) \times (\hat{\square}^n \setminus \partial\hat{\square}^n)$ [derived from Lemma 1.3.3], is always a boundary. This takes the form $\partial(\partial_\epsilon^{-1}\zeta)$, where $\partial_\epsilon^{-1}\zeta \approx \theta(\tilde{\zeta}_\epsilon) + (S^1)^n \times \partial^{-1}_{(S,D)}(T_\zeta)$; since this avoids $S\setminus D \times N_\epsilon(\partial\hat{\square}^n)$, the integrals

$$\int_{\partial_\epsilon^{-1}\zeta} \wedge^n \mathrm{dlog} z_i \wedge \pi_S^*\alpha\,, \qquad \alpha \in \Gamma\left(\Omega^{2t-(2p-n-1)}_{(S\setminus D)^\infty}\right)$$

make sense. Taking their limit defnes by duality the natural extension

$$AJ(\zeta) \in H^{2p-n-1}(S \setminus D, \mathbb{C}/\mathbb{Q}(p)) =: H^{2p}_{\mathcal{D}}(\{(S \setminus D) \times \square^n,\ (S \setminus D) \times \partial\square^n\}, \mathbb{Q}(p))$$

of the Abel-Jacobi class (to this particular relative case).

We can both show the map is well-defined (respects relative rational equivalence) and accomplish our objective (relating AJ to $\mathcal{R}$) in one blow, by showing commutativity of

$$\begin{tikzcd}
CH^p(\eta_S \times \square^n, \eta_S \times \partial\square^n) \arrow[r, "\cong"'] & CH^p(\eta_S, n) \arrow[dd, "\mathcal{R}"] \\
Z^p(\eta_S \times \square^n, \eta_S \times \partial\square^n) \arrow[u, two heads] \arrow[d, "AJ"] & \\
H^{2p}_{\mathcal{D}}(\eta_S \times \square^n, \eta_S \times \partial\square^n, \mathbb{Q}(p)) \arrow[r, "\cong"] & H^{2p-n}_{\mathcal{D}}(\eta_S, \mathbb{Q}(p)).
\end{tikzcd}$$

This reduces to the equality

$$\lim_{\epsilon\to 0}\int_{\partial_\epsilon^{-1}\zeta} \wedge^n \mathrm{dlog} z_i \wedge \pi_S^* \alpha \;=\; \int_S R'_\zeta \wedge \alpha\,,$$

which one proves essentially by "integrating" $\wedge^n \mathrm{dlog} z_i$ along the fibers of $\theta(\tilde{\zeta}) \to \zeta$ (which gives $\iota_\zeta^* R_\square^n$) and pushing down to S.

We remind the reader of the complete absence of Hodge-theoretic constrants on α: for AJ of a codimension-p cycle on the $(t+n)$-dimensional variety $(S\setminus D)\times(\square^n,\partial\square^n)$ one expects to integrate formsrepresenting "sufficiently holomorphic" classes, namely: in

$$F^{t+n-p+1}H^{2t+2n-2p+1}((S,D)\times(\hat{\square}^n\setminus\partial\hat{\square}^n)).$$

But the only class[2] in $H^*(\hat{\square}^n\setminus\partial\hat{\square}^n)$ is $[\wedge^n \mathrm{dlog} z_i]\in F^nH^n(\hat{\square}^n\setminus\partial\hat{\square}^n)$, and so the forms reduce to

$$[\wedge^n \mathrm{dlog} z_i \wedge \alpha] \in\; F^nH^n(\hat{\square}^n\setminus\partial\hat{\square}^n)\otimes F^{t-p+1}H^{2t-(2p-n-1)}(S,D) \;=$$

$$F^nH^n(\hat{\square}^n\setminus\partial\hat{\square}^n)\otimes H^{2t-(2p-n-1)}(S,D).$$

Since $n>p$, $\wedge^n \mathrm{dlog} z_i$ already makes the form sufficiently holomorphic, and so we have total freedom with α. For this reason, completely special to these relative AJ maps, we can always replace integration against α by integration over topological cycles $\mathcal{C}$ on $S\setminus D$.

We give a few examples of the computation of AJ on the spread of $\mathcal{Z}\in Z^p((\square^n,\partial\square^n)(k))$ by a "regulator current" on S, starting with the case $n=p$ of relative 0-cycles.

EXAMPLE 5.1.1. Assuming for simplicity $t<n$, one has a composite

$$\begin{tikzcd}
K_n^M(k) \arrow[r, "\cong"] \arrow[d, "\cong"] & K_n^M(\mathbb{Q}(S)) \arrow[d, "\cong"] \arrow[dr] & \\
CH^n((\square^n,\partial\square^n)(k)) \arrow[r, "\cong"] & CH^n(\{\eta_S^{\mathbb{Q}}\times(\square^n,\partial\square^n)\}(\mathbb{Q})) \arrow[r, "AJ"] & H^{n-1}(\eta_S,\mathbb{C}/\mathbb{Q}(n))
\end{tikzcd}$$

computed by $\mathbf{f}\mapsto R_{\mathbf{f}}$, which one conjectures is injective.

EXAMPLE 5.1.2. More interesting are 1-cycles $(p=n-1)$. The case $n=2$ is trivial as $CH^1((\square^2,\partial\square^2)(k)) \xrightarrow{\cong} CH^1(k,2)=0$ (for algebraically closed k); in fact one can show by geometric arguments $CH^1(\mathbb{P}^1\times\mathbb{P}^1,\#)\cong 0$. For $n=3$ we just get back (from $AJ(\zeta)$) $AJ(\mathcal{Z})$, since

$$CH^2((\square^3,\partial\square^3)(k)) \xrightarrow{\cong} CH^2(\{\eta_S\times(\square^3,\partial\square^3)\}(\mathbb{Q})) \xrightarrow{AJ} H^1_{\mathcal{D}}(\eta_S,\mathbb{Q}(2)) = \mathbb{C}/\mathbb{Q}(2)$$

[2]recall $\hat{\square}^n=(\mathbb{P}^1,\{1\})^n$, $\square^n(\mathbb{P}^1-\{1\})^n$.

is computed by a (constant mod $\mathbb{Q}(2)$) zero-current. (This is nothing more than our AJ on $CH^2(\mathbb{C},3)$. Is this, then, injective – in spite of the codimension 2?) However, to compare with $n=4$ we give the following formula: let $\mathcal{Z}$ be a sum of rational curves of the form

$$\mathcal{Z}_{f,g,h} := \{z \mapsto (f(z), g(z), h(z))\} = \left(a\frac{z-\alpha_0}{z-\alpha_1},\, b\frac{z-\beta_0}{z-\beta_1},\, c\frac{z-\gamma_0}{z-\gamma_1} \right)$$

(so that *e.g.* α_0, α_1 record the values of $z \in \mathbb{P}^1$ where the curve intersects the faces $z_1 = 0, \infty$, and so on). Provided $\mathcal{Z}$ is also a relative cycle we can taje $AJ(\mathcal{Z})$, to which $R_{\mathcal{Z}_{f,g,h}}$ contributes

$$\sum_{i,j=0,1} (-1)^{i+j} \mathrm{Li}_2\left(CR\{\alpha_1, \alpha_0, \gamma_i, \beta_j\}\right) + \log\left(c\frac{\alpha_0-\gamma_0}{\alpha_0-\gamma_1}\right) \log\left(CR\{\alpha_0, \beta_0, \alpha_1, \beta_1\}\right)$$
$$-2\pi i \sum_{T_h \cap T_g} \log f.$$

EXAMPLE 5.1.3. For $n=4$ the composite

$$CH^3((\square^4, \partial\square^4)(k)) \xrightarrow{\cong} CH^3(\{\eta_S \times (\square^4, \partial\square^4)\}(\mathbb{Q})) \xrightarrow{AJ} H^2_{\mathcal{D}}(\eta_S, \mathbb{Q}(2))$$
$$= H^1(\eta_S, \mathbb{C}/\mathbb{Q}(3))$$

is much more interesting, and goes beyond AJ on the original cycle $\mathcal{Z} \in Z^3((\square^4, \partial\square^4)(k))$ (which is always zero!). Again we expect this composite is injective. If $\mathcal{Z}$ is a sum of rational curves as above, so that

$$\left(f(z),\, g(z),\, h(z),\, F(z) = d\frac{z-\Delta_0}{z-\Delta_1} \right) =: \mathcal{Z}_{f,g,h,F}$$

is a component, then ζ has $\zeta_{f,g,h,F}$ (where $\{\alpha_0, \alpha_1, \ldots, \Delta_1\} \in \mathbb{Q}(S)$) as a component. Using the fact that $AJ(\zeta)$ is computed by R_ζ, we see that $R_{\zeta_{f,g,h,F}}$ is $\mathcal{Z}_{f,g,h,F}$'s contribution to the composition. By a computation not unlike that in $[G-Z]$, the interesting terms of $R_{\zeta_{f,g,h,F}}$ are

$$\sum_{\ell,k=0,1} (-1)^{\ell+k} \left\{ \begin{array}{l} \sum_{i,j=0,1} (-1)^{i+j} \mathrm{Li}_2\left(CR\{\alpha_i,\, \gamma_k,\, \Delta_\ell,\, \beta_j\}\right) \\ + \log\left(\frac{\Delta_\ell - \beta_0}{\gamma_k - \beta_\infty}\right) \log\left(CR\{\alpha_0, \alpha_\infty, \Delta_\ell,\, \gamma_k\}\right) \end{array} \right\} \cdot \mathrm{dlog}(\Delta_\ell - \gamma_k).$$

Neither computation is at a point where it can detect $\underset{\mathrm{rat}}{\not\equiv}$ of explicit relative cycles, but there is hope for progress; further comparison with the work in **[GZ]** might be a good place to start.

5.1.2. What comes after the Abel-Jacobi map? Now let $X/\mathbb{Q}$ be a smooth projective variety ($d = \dim X$) and $\mathcal{Z} \in Z^p(X(\mathbb{Q}))$. According to the Bloch-Beilinson conjecture (BBC),

$$CH^p(X(\mathbb{Q})) \otimes \mathbb{Q} \underset{c_{\mathcal{D}}}{\hookrightarrow} H^{2p}_{\mathcal{D}}(X, \mathbb{Q}(p))$$

and so $[\mathcal{Z}]$ and (if $[\mathcal{Z}]=0$) $AJ(\mathcal{Z})$ completely detect the situation $\mathcal{Z} \underset{\text{rat}}{\not\equiv} 0$ (modulo torsion). BBC does not apply to $c_{\mathcal{D}}$ on $CH^p(X(\mathbb{C}))$, however, and so if $\mathcal{Z} \in Z^p(X(\mathbb{C}))$ one expects

$$CH^p(X(\mathbb{C})) \overset{\Psi_0=[\,\cdot\,]}{\longrightarrow} Hg^{p,p}(X) := F^pH^{2p}(X,\mathbb{C}) \cap H^{2p}(X,\mathbb{Z})$$

$$\big(CH^p_{\text{hom}}(X) :=\big) \ker(\Psi_0) \overset{\Psi_1=AJ}{\longrightarrow} J^p(X) := \frac{H^{2p-1}(X,\mathbb{C})}{F^pH^{2p-1}(X,\mathbb{C}) + H^{2p-1}(X,\mathbb{Z})}$$

to be followed by a series of maps

$$\Psi_i : \ker(\Psi_{i-1}) \longrightarrow \left\{ \begin{array}{c} \text{Hodge-theoretically} \\ \text{defined objects} \end{array} \right\}.$$

These target objects cannot be finite-dimensional abelian varieties like $J^p(X)$, even in the case of zero-cycles, by a result of Mumford [**Mu**] (or see [**L1**]). For any natural number N and base point $p_0 \in X$, there is a map

$$S^{(N)}(X(\mathbb{C})) \overset{\pi_N}{\longrightarrow} CH_0^{\text{hom}}(X(\mathbb{C}))$$

$$Sym_N(p_1, \dots, p_N) \longmapsto \sum_{j=1}^{n} [p_j] - N[p_0].$$

Mumford's theorem as generalized by Roitman [**Ro**] (or see [**L1**]), states that [given an appropriate definition of $\dim(\text{im}(\pi_N))$]

$$H^0(\Omega^\ell_X) \neq 0 \text{ for any } \ell \geq 2 \implies (N \cdot d \geq) \dim(\text{im}(\pi_N)) \geq N.$$

This result is clearly different from the case ($d=1$) of X a curve, where for all N, $\dim(\text{im}(\pi_N)) \leq g$(=genus of X). It applies, for example, in case $d \geq 2$ and $\dim(K_X) \geq 0$.

Letting $N \to \infty$, we see that (when this theorem applies) $CH_0^{\text{hom}}(X(\mathbb{C}))$ is "∞-dimensional" in the sense that no finite-dimensional variety parametrizes it (or classifies rational $\equiv$ classes). Also, we see that there are always rational equivalence classes that that cannot be represented by a cycle involving $\leq N$ points: just take a generic element of $\text{im}(\pi_{N \cdot d+1})$. So the best target spaces for $\Psi_{i \geq 2}$ we can hope for are *limits* of finite-dimensional Hodge-theoretically defined objects, and this is exactly what we get below.

Now any $\mathcal{Z} \in Z^p(X(\mathbb{C}))$ is defined over some subfield $k \subset \mathbb{C}$, with $\infty >$ trdeg$(k/\mathbb{Q})$; so once again we may exchange the field for additional geometry, and spread[3] the cycle over $S/\mathbb{Q}$, where $\mathbb{Q}(S) \cong k$ and $\dim_{\mathbb{C}}(S(\mathbb{C})) = t$. There are ambiguities in the "complete" spread $[\tilde{\zeta}] \in CH^p(X \times S(\mathbb{Q}))$ but not its restriction $[\zeta]$ to $X \times \eta_S$. Working modulo torsion and assuming a Bloch-Beilinson conjecture (BBC^q) for quasi-projective varieties, from the diagram

[3] This idea (in this context) has been around for a while; apparently it originated with S. Saito.

$$\begin{tikzcd}
& CH^p(X\times S(\mathbb{Q})) \arrow[dl] \arrow[d] \arrow[r, hook, "(\mathrm{BBC})", "c_{\mathcal{D}}"'] & H^{2p}_{\mathcal{D}}(X\times S,\mathbb{Q}(p)) \arrow[d, "\Phi"] \\
CH^p(X(k)) \arrow[r, "\cong"'] & CH^p(X\times \eta_S(\mathbb{Q})) \arrow[r, hook, "(\mathrm{BBC}^q)", "c_{\mathcal{H}}"'] & H^{2p}_{\mathcal{H}}(X\times \eta_S,\mathbb{Q}(p))
\end{tikzcd}$$

we see (with **[L2]**) that

$$CH^p(X(k)) \overset{\Psi}{\hookrightarrow} \underline{H}^{2p}_{\mathcal{H}}(X\times\eta_S,\mathbb{Q}(p)) := \mathrm{im}(\Phi).$$

Lewis points out that $\mathrm{im}(\Phi)$ is the *lowest weight* part of $H_{\mathcal{H}}$, and constructs on it a Leray filtration $\mathcal{L}^i\underline{H}^{2p}_{\mathcal{H}}$ (with $\mathcal{L}^0=\underline{H}^{2p}_{\mathcal{H}}$) which induces

$$\mathcal{L}^iCH^p(X(k)) := \Psi^{-1}(\mathcal{L}^i\underline{H}^{2p}_{\mathcal{H}}).$$

Notice that one may automatically chop Ψ into pieces[4]

$$\Psi_i:\ \ker(\Psi_{i-1}) = \mathcal{L}^iCH^p(X(k)) \longrightarrow Gr^i_{\mathcal{L}}\underline{H}^{2p}_{\mathcal{H}}(X\times\eta_S,\mathbb{Q}(p)).$$

Writing $\rho:\ X\times\eta_S\to\eta_S$ for the projection, according to **[L2]** these graded pieces sit in a short exact sequence

$$\begin{tikzcd}
Gr^{i-1}_{\mathcal{L}}J^p(X\times\eta_S) \arrow[r] \arrow[d, phantom, "\ddot{\|}"] & Gr^i_{\mathcal{L}}\underline{H}^{2p}_{\mathcal{H}}(X\times\eta_S,\mathbb{Q}(p)) \arrow[r, "\theta"] & Gr^i_{\mathcal{L}}Hg^p(X\times\eta_S) \arrow[d, phantom, "\ddot{\|}"] \\
\frac{\mathrm{Ext}^1_{MH}(\mathbb{Q}(0),W_{-1}H^{i-1}(\eta_S,R^{2p-i}_{\rho*}\mathbb{Q}(p)))}{\hom_{MH}(\mathbb{Q}(0),Gr^0_W H^{2p-1}(X\times\eta_S,\mathbb{Q}(p)))} & & \hom_{MH}(\mathbb{Q}(0),H^i(\eta_S,R^{2p-i}_{\rho*}\mathbb{Q}(p)))
\end{tikzcd}$$

If $[\mathcal{Z}]\in\mathcal{L}^iCH^p(X(k))$, then $\Psi_i(\mathcal{Z}) = Gr^i_{\mathcal{L}}c_{\mathcal{H}}(\zeta)$, and $\therefore\ \theta(\Psi_i(\mathcal{Z})) =: [\zeta]_i$, is defined; if $[\zeta]_i=0$ then $\Psi_i(\mathcal{Z})$ pulls up to $[AJ\zeta]_{i-1}$ in the first term of the above sequence; and if $[AJ\zeta]_{i-1}=0$ then $[\mathcal{Z}]\in\mathcal{L}^{i+1}CH^p(X(k))$, and so on. We will compute (pieces of) the outer terms for 0-cycles in §5.3 (in

[4]This should be viewed as an extension of earlier work chopping up the ordinary cycle-class of the spread, to the entire Deligne (or absolute Hodge) class. Writing

$$Gr^i\Omega^p_{X/\mathbb{Q}} \cong \Omega^i_{S/\mathbb{Q}}\otimes\Omega^{p-i}_{\mathcal{X}/S},$$

Esnault and Paranjape (**[EP]** or **[Gr4]**) filtered the image space of the map:

$$CH^p(X(\mathbb{C}))\otimes\mathbb{C}\to H^p(\mathcal{X},\Omega^p_{\mathcal{X}(\mathbb{Q})/\mathbb{Q}})\otimes\mathbb{C}\cong H^p(\mathcal{X},\Omega^p_{\mathcal{X}(\mathbb{C})/\mathbb{C}})$$

and obtained graded pieces lying in the middle cohomology of the complex

$$\to \Omega^{i-1}_S\otimes R^{p-1}_{\pi_*}\Omega^{p-i+1}_{\mathcal{X}/S} \overset{\nabla}{\longrightarrow} \Omega^i_S\otimes R^p_{\pi_*}\Omega^{p-i}_{\mathcal{X}/S} \overset{\nabla}{\longrightarrow} \Omega^{i+1}_S\otimes R^{p+1}_{\pi_*}\Omega^{p-i+1}_{\mathcal{X}/S}\to$$

which are essentially our $\overline{[\zeta]_i}$ in §5.3 (the infinitesimal invariants of the differential characters we will define there).

particular the image of the denominator in the Ext^1 term will become more clear). We want to emphasize here, that even without BBC^q the Ψ_i and $\mathcal{L}^i$ are still (well-)defined; they just don't exhaust $CH^p(X(k))$.

This computation of the $Gr^i_{\mathcal{L}}\underline{H}^{2p}_{\mathcal{H}}$, combined with hard Lefschetz, shows that (see **[L2]**, Prop. 5.0)

$$\mathcal{L}^{p+1}CH^p(X(k)) = 0\,;$$

this says rational equivalence classes are already captured by $\Psi_0, \Psi_1, \ldots, \Psi_p$. Moreover, $[\zeta]_0$ and $[AJ\zeta]_0$ recover $[\mathcal{Z}]$ and $AJ(\mathcal{Z})$, so that we recover our original Ψ_0 and Ψ_1 (the latter enlarged by $[\zeta]_1$), at least for cycles defined $/k$. To express the Ψ_i as maps on (filtered pieces of) $CH^p(X(\mathbb{C}))$, we first define (with [L])

$$\mathcal{L}^iCH^p(X(\mathbb{C})) := \varinjlim_{k\subset\mathbb{C}} \mathcal{L}^iCH^p(X(\mathbb{C}))$$

(where the limit is over k with finite transcendence degree $/\mathbb{Q}$), and taking a limit over all finite-dimensional varieties $S/\mathbb{Q}$, map

$$\mathcal{L}^iCH^p(X(\mathbb{C})) \xrightarrow{\Psi_i} \varinjlim_{S/\mathbb{Q}} Gr^i_{\mathcal{L}}\underline{H}^{2p}_{\mathcal{H}}(X\times\eta_S, \mathbb{Q}(p)).$$

In the case of 0-cycles, if X is regular ($H^0(\Omega^1_X)=0$) then $Gr^1_{\mathcal{L}}Hg^p(X\times\eta_S) = 0$ for all $S/\mathbb{Q}$, and the target space for Ψ_1 is just $J^p(X(\mathbb{C}))$ (and is finite dimensional). If $H^0(\Omega^\ell_X)\neq 0$ for some $(p\geq)\ell\geq 2$, then the target space for Ψ_ℓ is infinite-dimensional; this gives the connection with Mumford.

We want to point out the equivalence of the above approach with the work **[GG5]** of Griffiths and Green, who look at (graded pieces of) $c_{\mathcal{D}}(\tilde{\zeta})$ instead of $c_{\mathcal{H}}(\zeta)\,[=\Psi(\mathcal{Z})]$. Suppose one has a Leray filtration on $H^{2p}_{\mathcal{D}}(X\times S, \mathbb{Q}(p))$ and a spread $\tilde{\zeta}$ of $\mathcal{Z}$ with $c_{\mathcal{D}}(\tilde{\zeta})\in\mathcal{L}^{p+1}H^{2p}_{\mathcal{D}}$; in order to show $\mathcal{Z}\underset{\text{rat}}{\equiv} 0$ on X, one must produce $\Gamma_D\in Z^{p-1}(X\times D(\mathbb{Q}))$ (for some divisor $D/\mathbb{Q}\subset S$) such that $\tilde{\zeta}\pm\Gamma_D \underset{\text{rat}}{\equiv} 0$ on $X\times S$. The construction of this Γ_D involves the Hodge conjecture (HC), which never explicitly appears in **[L2]**; this is because it is already contained in BBC^q (which essentially $=$BBC$+$HC).

For example, let $\mathcal{Z}$ be a 0-cycle on a 3-fold X, and for simplicity suppose that the fundamental class of $\tilde{\zeta}$ lies entirely in[5]

$$Hg^3(X\times S)\cap\left\{H^2(X)\times H^4(S)\right\}.$$

Then we may annihilate this algebraic obstruction to rational equivalence ($\tilde{\zeta}\underset{\text{rat}}{\equiv} 0$ on $X\times S$) without negotiating $\mathcal{Z}$, by adding cycles with support on $D/\mathbb{Q}\subset S$. The idea is to take a "cross-section" of $\tilde{\zeta}$ by a hyperplane section

[5] or at least (in the sense of [G-G]) $[\zeta]_0=\ldots=[\zeta]_3=0$, $[AJ\zeta]_0=\ldots=[AJ\zeta]_2=0$.

$X_\Lambda/\mathbb{Q}$ of X; $\tilde{\zeta} \cap (X_\Lambda \times S)$ is then suported on $X_{(\Lambda)} \times D$. This "cross-section map" may be formulated as a correspondence

$$[\Lambda \times \Delta_S] \in CH^*((X \times S) \times (X \times S)),$$

so that the isomorphism of hard Lefschetz is induced by its action:

$$H^2(X) \times H^4(S) \underset{\cong}{\overset{[\Lambda\times\Delta_S]_*}{\longrightarrow}} H^4(X) \otimes H^4(S).$$

According to the Lefschetz standard conjecture ($\Leftarrow$ HC + hard Lefschetz), there exists a correspondence $[\Lambda^{-1}]$ (algebraically) inducing its inverse: $[\Lambda^{-1}]_* = ([\Lambda]_*)^{-1}$, or the composition $[\Lambda^{-1} \times \Delta_S]_* \circ [\Lambda \times \Delta_S]_* =$ the identity on cohomology. But because it is "algebraic" (i.e. induced by a correspondence), the composition operates on cycles, and is *not* the identity on $\tilde{\zeta}$. In particular, since already $[\Lambda \times \Delta_S] \cdot \tilde{\zeta}$ is supported on $X \times D$,

$$\Gamma_D := [\Lambda^{-1} \times \Delta_S] \cdot \left([\Lambda \times \Delta_S] \cdot \tilde{\zeta} \right) \subseteq X \times D.$$

As cohomology classes, though,

$$[\Gamma_D] \;=\; [\Lambda^{-1} \times \Delta_S]_* \circ [\Lambda \times \Delta_S]_*[\tilde{\zeta}] \;\; = [\tilde{\zeta}] \in H^2(X) \otimes H^4(X)$$

and so the modification $\tilde{\zeta} - \gamma_D$ kills (the graded piece of) the fundamental class without altering the spread's restriction to $\mathcal{Z}$ on $X \times \eta_S^{\mathbb{Q}}$. Griffiths and Green say that such a class is "in the ambiguities".

5.1.3. Some formal computations for relative varieties. For the rest of Chapter 5 we will be dealing with 0-cycles, continuing the present line of thought with X smooth projective in §5.3. As motivation for this in the meantime we return to relative varieties where we can compute everything, starting (once again) with $X = (\square^n, \partial\square^n)$. There is a perfect pairing

$$H^{2n-i}((\square^n, \partial\square^n), \mathbb{Q}) \otimes H^i(\hat{\square}^n \setminus \partial\hat{\square}^n, \mathbb{Q}) \longrightarrow \mathbb{Q}(-n),$$

while

$$H^i(\hat{\square}^n \setminus \partial\hat{\square}^n, \mathbb{Q}) \cong \begin{cases} \mathbb{Q}(-n) & i = n \\ 0 & \text{otherwise} \end{cases}$$

becayse $[\bigwedge^n \mathrm{dlog} z_i]$ has weight $2n$ (due to the n dlog's; see e.g. [**GS**]). So the $\mathbb{Q}(-n)$ must pair with a $\mathbb{Q}(0)$, and

$$\mathbb{R}^{2n-i}_{\rho_*}\mathbb{Q}(0) \cong \begin{cases} \mathbb{Q}(0) & i = n \\ 0 & \text{otherwise} \end{cases} \qquad \Longrightarrow \qquad R^n_{\rho_*}\mathbb{Q}(n) = \mathbb{Q}(n),$$

which we emphasize is highly peculiar; it has the effect of giving $H^*(\eta_S)$ in the computation below where for X smooth we would have $\operatorname{im}\{H^*(S) \to H^*(\eta_S)\}$. In particular, $H^{n-1}(\eta_S)$ has mixed Hodge structure with weights (in general) between $n-1$ and $2(n-1)$; therefore $H^{n-1}(\eta_S, \mathbb{Q}(n))$ has weights between $-(n+1)$ and -2 so that W_{-1} is everything and $Gr^0_W = 0$. Using this fact and pretending the work described in §5.1.2 applies to $X=(\square^n,\partial\square^n)$, we may compute *formally*

$$Gr^i_{\mathcal{L}}\underline{H}^{2n}_{\mathcal{H}}(X\times\eta_S,\mathbb{Q}(n))=0 \text{ for } i\neq n, \text{ and}$$

$$Gr^n_{\ell}Hg^n(X\times\eta_S)\cong \hom_{MH}\left(\mathbb{Q}(0),H^n\left(\eta_S,R^n_{\rho_*}\mathbb{Q}(n)\right)\right)$$

$$\cong F^nH^n(\eta_S,\mathbb{C})\cap H^n(\eta_S,\mathbb{Q}(n));$$

while

$$Gr^n_{\mathcal{L}}J^p(X\times\eta_S)\cong\frac{\mathrm{Ext}^1_{MH}\left(\mathbb{Q}(0),W_{-1}H^{n-1}(\eta_S,\mathbb{Q}(n))\right)}{\hom_{MH}\left(\mathbb{Q}(0),Gr^0_WH^{2n-1}(X\times\eta_S,\mathbb{Q}(n))\right)}$$

$$\cong\frac{\mathrm{Ext}^1_{MH}\left(\mathbb{Q}(0),H^{n-1}(\eta_S,\mathbb{Q}(n))\right)}{\hom_{MH}\left(\mathbb{Q}(0),\left\{Gr^0_WH^{n-1}(\eta_S,\mathbb{Q}(n))\ =0\right\}\right)}\cong\frac{H^{n-1}(\eta_S,\mathbb{C})}{\{F^nH^{n-1}(\eta_S,\mathbb{C})\ =0\}+H^{n-1}(\eta_S,\mathbb{Q}(n)}$$

$$\cong H^{n-1}(\eta_S,\mathbb{C}/\mathbb{Q}(n)).$$

In other words,

$$Gr^n_{\mathcal{L}}\underline{H}^{2n}_{\mathcal{H}}(X\times\eta_S,\mathbb{Q}(n))\cong H^n_{\mathcal{D}}(\eta_S,\mathbb{Q}(n))$$

is just the target space for the "composite" of §5.1.1 for $n=p$, which simply amounted to (see Example 5.1.1) spreading[6]

$$[\mathcal{Z}]=[\mathbf{a}]=\left[\sum m_j(\alpha_{1j},\ldots,\alpha_{nj})\right]\in CH^n((\square^n,\partial\square^n)(k))\cong\ K^M_n(k)$$
$$\|$$

to $[\zeta]=[\gamma_{\mathbf{f}}]=\left[\sum m_j(f_{1j},\ldots,f_{nj})\right]\in CH^n(\{(\square^n,\partial\square^n)\times\eta_S\}(\mathbb{Q}))\cong K^M_n(\mathbb{Q}(S))$

and applying $\mathcal{R}$. In terms of our formal analogy $\mathcal{R}$ consists of

$$[\zeta]_n=\left([\wedge^n\mathrm{dlog}z_i]^\vee\otimes\right)[\wedge^n\mathrm{dlog}\mathbf{f}]\ \in Gr^n_{\mathcal{L}}Hg^n(X\times\eta_S)$$

and if this vanishes

$$[AJ\zeta]_{n-1}=\left([\wedge^n\mathrm{dlog}z_i]^\vee\otimes\right)[R^{(\prime)}_{\mathbf{f}}]\ \in Gr^{n-1}_{\mathcal{L}}J^n(X\times\eta_S),$$

while automatically (for geometric reasons) all the preceding $[\zeta]_i$, $[AJ\zeta]_{i-1}$ are zero.

REMARK 5.1.4. More generally for $CH^p((\square^n,\partial\square^n)(k))$ $[p\neq n]$ the mas defined in §5.1.1 compute $[\zeta]_{2p-n}$ and $[AJ\zeta]_{2p-n-1}$ (that is, Ψ_{2p-n}) while all the other Ψ_i are 0. So for $p=2$, $n=3$ we had just $[AJ\zeta]_0(=AJ(\mathcal{Z}))$ while for $p=3$, $n=4$ there was the much more interesting invariant $[AJ\zeta]_1$.

[6]Here it is understood that $f_{ij}\longmapsto\alpha_{ij}$ under the embedding $\mathbb{Q}(S)\ \underset{\cong}{\longrightarrow}\ k\subset\mathbb{C}$ (which one should think of as "evaluation at a generic point").

One expects that a similar formal procedure with $X = (\mathbb{P}^1, \{0,\infty\})^n$ – that is, assuming the ideas of §5.1.2 apply – should also predict correct results. We'll do the predictions here, then prove them in §5.2 from a different angle. Understanding the situation for this X is the key for the transition between §5.1.1 and §5.3: $(\mathbb{P}^1, \{0,\infty\})^n$ is at once similar to $(\square^n, \partial\square^n)$ [except that $\mathbb{I}^n$(=all permutations of $\{1\}\times\mathbb{P}^1\times\ldots\times\mathbb{P}^1) \subset (\mathbb{P}^1)^n$ is not thrown away], and to a product of curves [since $(\mathbb{P}^1, \{0,\infty\})$ is essentially a degenerate elliptic curve].

Let σ_i (or σ) denote a choice of i indices

$$1 \leq \sigma_i(1) \leq \ldots \leq \sigma_i(i) \leq n$$

with corresponding projection

$$\pi_i^\sigma : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^i.$$

There are $\binom{n}{i}$ of these, inducing maps on cohomology

$$H^i((\mathbb{P}^1 - \{0,\infty\})^n, \mathbb{Q}) \underset{(\oplus\pi_i^\sigma)^*}{\overset{\cong}{\longleftarrow}} \underset{\left\langle \frac{1}{(2\pi\sqrt{-1})^i}\mathrm{dlog}z_{\sigma_i(1)}\wedge\ldots\wedge\mathrm{dlog}z_{\sigma_i(i)}\right\rangle}{\underset{\|}{\oplus_{\sigma_i} H^i((\mathbb{P}^1 - \{0,\infty\})^i, \mathbb{Q})}} \overset{\cong}{\longleftarrow} \underset{\binom{p}{i}\text{ copies}}{\oplus} H^i(\hat{\square}^i \setminus \partial\hat{\square}^i, \mathbb{Q})$$

which dualize to

$$H^{2n-i}((\mathbb{P}^1, \{0,\infty\})^n) \overset{\cong}{\longrightarrow} \oplus_\sigma H^i((\mathbb{P}^1, \{0,\infty\})^i) \qquad [\text{so } R^{2n-i}_{\rho_*}\mathbb{Q}(n) \cong \oplus_{\sigma_i}\mathbb{Q}(n)].$$

This induces the top $\cong$ in the following diagram, in which we are also (formally) applying functorial aspects of the $\mathcal{L}^i$ (which we have not discussed but are covered in [L]):

$$\begin{array}{ccc}
Gr^i_{\mathcal{L}}\underline{H}^{2n}_{\mathcal{H}}\left((\mathbb{P}^1, \{0,\infty\})^n \times \eta_S, \mathbb{Q}(n)\right) & \overset{\cong}{\longrightarrow} & \oplus_\sigma Gr^i_{\mathcal{L}}\underline{H}^{2i}_{\mathcal{H}}\left((\mathbb{P}^1, \{0,\infty\})^i \times \eta_S, \mathbb{Q}(i)\right) \\
\uparrow \Psi_i(n) & & \uparrow \oplus_\sigma \Psi_i(i) \\
\mathcal{L}^i CH^n\left((\mathbb{P}^1, \{0,\infty\})^n(k)\right) & \xrightarrow{\oplus\pi^\sigma_{i\,*}} & \oplus_\sigma \mathcal{L}^i CH^i\left((\mathbb{P}^1, \{0,\infty\})^i(k)\right).
\end{array}$$

This says that $\Psi_i(n)$ factors through $\oplus\pi^\sigma_{i\,*}$; combining this with injectivity[7] of the $\Psi_i(i)$, we have that

$$\mathcal{L}^{i+1}CH^n((\mathbb{P}^1, \{0,\infty\})^n(k)) = \ker(\Psi_i(n)) = \ker(\oplus\pi^\sigma_{i\,*}) = \{\mathcal{Z} \,|\, \pi_i^\sigma(\mathcal{Z}) \underset{\mathrm{rat}}{\equiv} 0 \ (\forall\sigma)\}$$

consists of 0-cycles whose projections to "i-faces" $(\mathbb{P}^1, \{0,\infty\})^i$ are rationally equivalent to zero. [In §5.2 we rigorously construct such a filtration on CH^n,

[7]of coutse, one has to believe a version of BBC here.

and produce maps from the resulting $F^i_\times CH^n$ to the upper right-hand term $\oplus_\sigma Gr^i_{\mathcal{L}}\underline{H}^{2i}_{\mathcal{H}}$ of the diagram.]

Moreover, the n-boxes $\mathcal{Z}=B_{\mathbf{a}}$ of §1.4.5 lie in $\mathcal{L}^nCH^n$, since $\pi^\sigma_i(B_{\mathbf{a}})$ actually $=0$; and it is in fact easy to see (§5.2.2) that the $B_{\mathbf{a}}$ span $\mathcal{L}^nCH^n$. By the geometric argument in §1.4.5, $[\wedge^n \mathrm{dlog}\mathbf{f}]$, $[R_{\mathbf{f}}]$ on S still give the relative cycle-class and AJ image of $\zeta=B(\gamma_{\mathbf{f}})$. So [according to this formal argument] $\Psi_n(n)$ is given by Milnor-regulator currents on S; moreover, so is $\Psi_i(i)$ for every i. Therefore it is no surprise that a formal computation [exactly like that for $(\square^n,\partial\square^n)$] of the upper-right term in the diagram shows that the target is

$$Gr^i_{\mathcal{L}}\underline{H}^{2n}_{\mathcal{H}}\left((\mathbb{P}^1,\{0,\infty\})^n\times\eta_S,\mathbb{Q}(n)\right)\cong\oplus_\sigma H^i_{\mathcal{D}}(\eta_S,\mathbb{Q}(i)).$$

The geometric argument in §1.4.5 is an important connection and inspired our approach in §5.3.

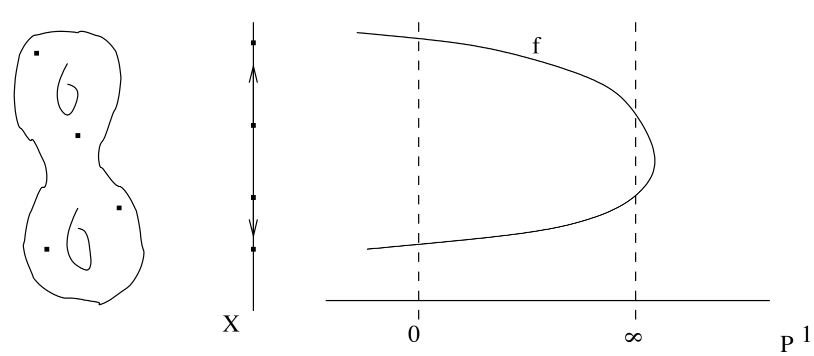
5.2. Zero-cycles on $(\mathbb{P}^1,\{0,\infty\})^n$

5.2.1. Abel for a degenerate elliptic curve. We start by reviewing Abel's theorem, to put the case $n=1$ in context. Let $X/\mathbb{C}$ be a compact Riemann surface, and $\mathcal{Z}=\sum m_j\{p_j\}\in Z^1(X)$ a zero-cycle. We seek a series of Hodge-theoretic invariants $\Psi_i(\mathcal{Z})$ of the cycle-class $[\mathcal{Z}]\in CH^1(X(\mathbb{C}))$, that can tell us when $\mathcal{Z}\underset{\text{rat}}{\equiv}0$. That is, the filtration given by their kernels should "exhaust" CH^1, so that $\mathcal{Z}\in\cap F^iCH^1=:\cap\ker\Psi_i \implies \mathcal{Z}=(f)$ for some $f\in\mathbb{C}(X)$.

So *suppose* $\mathcal{Z}=(f)$: what maps *should* send $\mathcal{Z}\mapsto 0$? The degree $\deg(f)$ of the divisor of a function is always zero so we may take

$$\Psi_0:=\deg:\ CH^1(X)\longrightarrow\mathbb{Z}\cong Hg^1(X)$$
$$\mathcal{Z}\longmapsto\sum m_j$$

as the first map. Now viewing $f:X\to\mathbb{P}^1$ as a "correspondence" in $Z^1(X\times\mathbb{P}^1)$



the 1-chain $\Gamma := \pi_X^f \left\{ (\pi_{\mathbb{P}^1}^f)^{-1} \overleftarrow{[0,\infty]} \right\}$ satisfies $\partial\Gamma = (f)$. If $\omega \in \Omega^1(X)$, then

$$f_*\omega := \pi_{\mathbb{P}^1 *}^f \pi_X^{f\,*}\omega \in \Omega^1(\mathbb{P}^1) = 0 \implies \int_\Gamma \omega = \int_0^\infty f_*\omega = 0.$$

However if we have before us $\mathcal{Z}[= (f)] \in \ker \Psi_0$ but do not know f, the "ideal path" Γ is not available. We can only "connect the dots" with some Γ' for which $\partial\Gamma' = \mathcal{Z}$, and since $[\Gamma' - \Gamma] \in H_1(X, \mathbb{Z})$ one has $\int_{\Gamma'} \omega = \int_{\Gamma'-\Gamma} \omega =$periods $(\neq 0)$.

Accounting for this ambiguity, we have a well-defined map

$$\Psi_1[= AJ] : \ker \Psi_0 \longrightarrow \Omega^1(X)^\vee \big/ \mathrm{im}\{H_1(X,\mathbb{Z})\} \cong J^1(X).$$

Abel's theorem (see **[G2]** for a more complete treatment) says that Ψ_1 is an isomorphism; the resulting exact sequence

$$0 \to J^1(X) \xrightarrow{\Psi_1^{-1}} CH^1(X) \xrightarrow{\Psi_0} \mathbb{Z} \to 0$$

is easily split by a choice of base point $\{0\} \in X$. In particular, the projection

$$CH^1(X) \longrightarrow \ker \Psi_0$$

$$\sum m_j[p_j] \longmapsto \sum m_j[p_j] - (\sum m_j)[0]$$

splits $CH^1(X) \cong \mathbb{Z} \oplus J^1(X) \cong Gr_F^0 CH^1 \oplus Gr_F^1 CH^1$.

Let $X = E_\lambda \subset \mathbb{P}^2$ be the smooth elliptic curve defined by the homogeneous equation

$$F_\lambda(x,y,z) = x^3 + y^3 + z^3 - 3\lambda xyz = 0$$

for $\lambda \in \mathbb{C} \backslash \{\text{cube roots of unity}\}$. Let μ_λ be the meromorphic form on $\mathbb{P}^2$ to which $(\lambda \neq 0)$

$$\lambda \frac{\Omega}{F_\lambda} = \lambda \frac{z\mathrm{d}x \wedge \mathrm{d}y - y\mathrm{d}x \wedge \mathrm{d}z + x\mathrm{d}y \wedge \mathrm{d}z}{x^3 + y^3 + z^3 - 3\lambda xyz}$$

descends, and define $\omega = \mathrm{Res}_E(\mu)$, $u = \int_0 \omega$ (a multivalued coordinate on E_λ), so that $\langle \omega = \mathrm{d}u \rangle = \Omega^1(E)$. The periods of ω (or ambiguities of u) form a lattice $\Lambda \subset \mathbb{C}$ with $\dim_{\mathbb{Z}} \Lambda = 2$, and the conditions for $\sum m_j[p_j] \underset{\text{rat}}{\equiv} 0$ read $\sum m_j = 0$, $\sum m_j \cdot u(p_j) \in \Lambda$. Pulling back along the local parametrization

$$\mathbb{C}^2 \xrightarrow{\sim} U_0 = \mathbb{P}^2 \setminus \{z = 0\}$$

$$(\mathsf{X}, \mathsf{Y}) \longmapsto (\mathsf{X} : \mathsf{Y} : 1),$$

we have

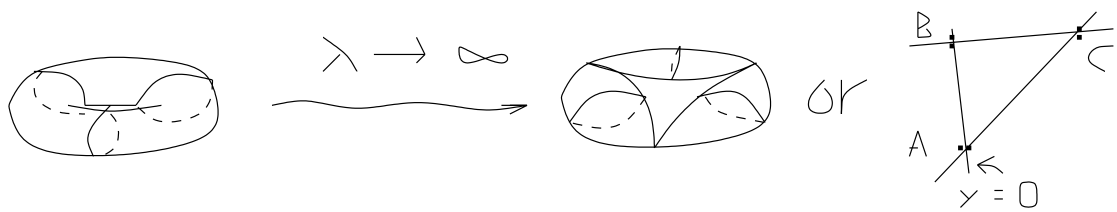
$$F_\lambda = \mathsf{X}^3 + \mathsf{Y}^3 + 1 - 3\lambda\mathsf{XY}, \qquad \mu = \lambda \frac{\mathrm{d}\mathsf{X} \wedge \mathrm{d}\mathsf{Y}}{F_\lambda}$$

so that e.g. where $\partial F_\lambda / \partial \mathsf{Y} \neq 0$,

$$\omega_\lambda = \mathrm{Res}_{\{F_\lambda = 0\}} \left(\frac{\lambda \mathrm{d}\mathsf{X}}{\partial F_\lambda / \partial \mathsf{Y}} \wedge \frac{\mathrm{d}F}{F} \right) = \lambda \frac{\mathrm{d}\mathsf{X}}{3(\mathsf{Y}^2 - \lambda \mathsf{X})}.$$

Now suppose the above conditions for rational equivalence were still a mystery for E_λ; we might degenerate it in $\mathbb{P}^2$ by taking $\lambda \to \infty$ and study rational equivalence on the resulting variety (=union of 3 $\mathbb{P}^1$'s):

$$\{x^3 + y^3 + z^3 - 3\lambda xyz = 0\} = E_\lambda \longrightarrow E_\infty := \{xyz = 0\}$$



$$\frac{\mathrm{d}\mathsf{X}}{\mathsf{X} - \mathsf{Y}^2/\lambda} \longrightarrow \frac{\mathrm{d}\mathsf{X}}{\mathsf{X}} = \mathrm{dlog}\mathsf{X} \text{ (on } y = 0).$$

The study of rational equivalences on E_∞ via functions which agree at A, B, C is equivalent to the study of $\underset{\mathrm{rat}}{\equiv}$ on $\{y = 0\}$ via functions with fixed values, say 1, at A and B. But this is just the study of $\underset{\mathrm{rat}}{\equiv}$ on the *relative* variety $(\{y = 0\}, \{A, B\}) \simeq (\mathbb{P}^1, \{0, \infty\})$. That is,

$$\begin{array}{llr} \mathcal{Z} = \sum m_i \{\alpha_i\} \in Z^1(\mathbb{P}^1, \{0, \infty\}) & & (\alpha_i \in \mathbb{C}) \\ \quad\ \ |||\, \mathrm{rat} & & \\ \quad\ \ 0 \iff \mathcal{Z} = (f),\ f \in \mathbb{C}(\mathbb{P}^1) \text{ and } f = 1 \text{ on } \{0, \infty\}. & & \end{array}$$

Clearly if $\sum m_i = 0$, $\prod \alpha_i^{m_i} = 1$ then $f := \prod (z - \alpha_i)^{m_i}$ does the job. Since ω_λ degenerates essentially to $\mathrm{d}\mathsf{X}/\mathsf{X}$ so that u_λ becomes $\log \mathsf{X}$, rewriting the second condition $\sum m_i \log \alpha_i \in 2\pi i \mathbb{Z} =: \Lambda$ could perhaps lead one towards Abel's theorem were it *not* known. Choosing $\{1\}$ as base point we have the splitting $CH^1(\mathbb{P}^1, \{0, \infty\}) \cong Gr_F^0 CH^1 \oplus Gr_F^1 CH^1 \cong \mathbb{Z} \oplus \mathbb{C}^*$.

For the $K3$ surfaces $X_\lambda \subset \mathbb{P}^3$ cut out by

$$G_\lambda(x, y, z, w) = x^4 + y^4 + z^4 + w^4 - \lambda xyzw$$

($\lambda \in \mathbb{C}$ general), $\Psi_0 = \deg[\to Hg^2(X_\lambda)]$ and $\Psi_1 = AJ(= Alb)[\to J^2(X_\lambda)]$ are well-defined on $CH^2(X(\mathbb{C}))$ by similar arguments; here we say a zero-cycle $\mathcal{Z} \underset{\mathrm{rat}}{\equiv} 0$ if there are (possibly singular) curves $\mathcal{C}_j \subset X$, meromorphic functions $f_j \in \mathbb{C}(\tilde{\mathcal{C}_j})$ such that $\sum \iota_*(f_j) = \mathcal{Z}$. But $\ker \Psi_0 \cap \ker \Psi_1 \subseteq$

$CH^2(X(\mathbb{C}))$ is not zero; in fact according to Mumford it is huge, since we have

$$\Omega^2(X) = \langle \mathrm{Res}(\Omega/G_\lambda)\rangle \neq 0,$$

where

$$\Omega = x\mathrm{d}y\wedge \mathrm{d}z\wedge \mathrm{d}w - y\mathrm{d}x\wedge \mathrm{d}z\wedge \mathrm{d}w + z\mathrm{d}x\wedge \mathrm{d}y\wedge \mathrm{d}w - w\mathrm{d}x\wedge \mathrm{d}y\wedge \mathrm{d}z.$$

On order to understand how this form influences rational (in)equivalence to zero, and find a Ψ_2, we again take $\lambda\to\infty$, degenerating X_λ to a tetrahedron and (locally on the $\mathsf{Z}=0$ plane of the tetrahedron)

$$\Omega_\lambda = \lambda \mathrm{Res}(\Omega/\Gamma_\lambda) = \lambda\frac{\mathrm{d}\mathsf{X}\wedge\mathrm{d}\mathsf{Y}}{\partial G_\lambda/\partial \mathsf{Z}}$$

to $\mathrm{dlog}\mathsf{X}\wedge\mathrm{dlog}\mathsf{Y}$. Once again we exchange the singular X_∞ for a "relative" one, namely $(\mathbb{P}^2,\triangle)$ or – which turns out to be the same as far as zero-cycles are concerned – $(\mathbb{P}^1,\{0,\infty\})^2 = (\mathbb{P}^1\times\mathbb{P}^1,\#)$. The formulation of the solution (the Ψ_2 in §5.2.4) in terms of differential characters produced by membrane integrals (of $\mathrm{dlog}\mathsf{X}\wedge\mathrm{dlog}\mathsf{Y}$) influences heavily the route taken in §5.3 – the first two subsections of which do apply to X_λ (although we can only get a "piece" of Ψ_2 in terms of the membrane integrals).

5.2.2. Computation of $CH^2(\mathbb{P}^1\times\mathbb{P}^1,\#)$. This computation is originally due to Bloch and Suslin.

Recall the Tame symbol map for a smooth curve $\tilde{\mathcal{C}}$

$$\mathrm{Tame}:\ K_2^M(\mathbb{C}(\tilde{\mathcal{C}})) \longrightarrow \coprod_{p\in\tilde{\mathcal{C}}} K_1^M(\mathbb{C}(p)) = \coprod_{p\in\tilde{\mathcal{C}}} \mathbb{C}^*$$

defined on generators by

$$\mathrm{Tame}_p\{f,g\} = \lim_{x\to p}(-1)^{\nu_p(f)\nu_p(g)}\frac{f(x)^{\nu_p(g)}}{g(x)^{\nu_p(f)}} which$$

reduces for $\{f,g\}$ "good" $(|(f)|\cap|(g)|=\emptyset)$ to $f(p)^{\nu_p(g)}/g(p)^{\nu_p(f)}$. *Weil reciprocity* states that for any $f,\,g\in\mathbb{C}(\tilde{\mathcal{C}})$

$$\prod_{p\in\tilde{\mathcal{C}}}\mathrm{Tame}_p\{f,g\} \;=\; 1\in\mathbb{C}^*.$$

We can prove thos using our regulator currents as follows (see [G-H] for the standard proof). It is always possible to "move" $\{f,g\}$ by a Steinberg (which have $\mathrm{Tame}_p=1\ \forall p$) to a product $\prod\{f_\xi,g_\xi\}$ with $|(f_\xi)|\cap|(g_\xi)|=\emptyset$; so it suffices to examine the case where $|(f)|\cap|(g)|=\emptyset$, and prove

$$\sum_{p\in\tilde{\mathcal{C}}}(\nu_p(g)\log f(p) - \nu_p(f)\log g(p)) \equiv 0 \quad \mod 2\pi i\mathbb{Z}.$$

Since $\partial\tilde{\mathcal{C}}=0$

$$0 = \int_{\partial\tilde{\mathcal{C}}}\frac{1}{2\pi i}R_{\{f,g\}} \;=\; \int_{\tilde{\mathcal{C}}}\mathrm{d}\left[\frac{1}{2\pi i}R_{\{f,g\}}\right] \;=\; \int_{\tilde{\mathcal{C}}}\mathrm{d}\left[\frac{1}{2\pi i}\log f\,\mathrm{dlog}g - \log g\cdot\delta_{T_f}\right]$$

$$= \int_{\tilde{\mathcal{C}}} \left\{ \begin{array}{c} \left(\frac{1}{2\pi i}\mathrm{dlog} f \wedge \mathrm{dlog} g + \log f \cdot \delta_{(g)} + \mathrm{dlog} g \cdot \delta_{T_f}\right) \\ -\left(\mathrm{dlog} g \cdot \delta_{T_f} + \log g \cdot \delta_{(f)} + 2\pi i \delta_{T_f \cap \delta_{T_g}}\right) \end{array} \right\}$$

$$= \int_{\tilde{\mathcal{C}}} \left\{ \log f \cdot \delta_{(g)} - \log g \cdot \delta_{(f)} + 2\pi i \delta_{T_f \cap T_g} \right\}$$

$$\underset{(\mathrm{mod}\ 2\pi i \mathbb{Z})}{\equiv} \sum_{p \in |(g)|} \nu_p(g) \log f(p) - \sum_{p \in |(f)|} \nu_p(f) \log g(p)$$

as desired.

Similarly one has a map

$$\mathrm{Tame} : K_3^M(\mathbb{C}(\tilde{\mathcal{C}})) \longrightarrow \coprod_{p \in \tilde{\mathcal{C}}} K_2^M(\mathbb{C})$$

defined on good generators by

$$\mathrm{Tame}_p\{f, g, h\} = \{f(p), g(p)\}^{\nu_p(h)} \{g(p), h(p)\}^{\nu_p(f)} \{h(p), f(p)\}^{\nu_p(g)},$$

and *Suslin reciprocity* states that (for any f, g, $h \in \mathbb{C}(\tilde{\mathcal{C}})$)

$$\prod_{p \in \tilde{\mathcal{C}}} \mathrm{Tame}_p\{f,\, g,\, h\} = 1 \in K_2^{(M)}(\mathbb{C}).$$

The form of this we will use (and which is proved in the Appendix) is that

$$\text{if } h \equiv 1 \text{ on } |(f)| \cup |(g)|, \text{ then } 1 = \prod_{p \in \tilde{\mathcal{C}}} \{f(p), g(p)\}^{\nu_p(h)}$$

(since in that case the other two factors in $\mathrm{Tame}_p\{f,\, g,\, h\}$ vanish). In the additive Milnor K-theory notation, we write $0 = \sum_{p \in \tilde{\mathcal{C}}} \nu_p(h) \cdot \{f(p), g(p)\}$.

A zero-cycle

$$\mathcal{Z} = \sum_j m_j(\alpha_j, \beta_j) \in \mathbb{Z}[\mathbb{C}^* \times \mathbb{C}^*]$$

on $(\mathbb{P}^1 \times \mathbb{P}^1, \#) = X$ is rationally equivalent to 0 if there are curves

$$\tilde{\mathcal{C}}_k \twoheadrightarrow \mathcal{C}_k \subset \mathbb{P}^1 \times \mathbb{P}^1$$

with $f_k \in \mathbb{C}(\tilde{\mathcal{C}}_k)$ such that $\sum_j m_j(\alpha_j, \beta_j) = \sum_k \iota_*(f_k)$ and $f_k \equiv 1$ on $\iota^{-1}(\mathcal{C}_k \cap \#)$. The $\mathcal{C}_k$ may be singular, and *may* intersect the corners of $\#$ since the functions will be $\equiv 1$ there (see Chapter 1).

There are cycle-class (=degree) and AJ (=Albanese) maps

$$\Psi_0 : CH^2(\mathbb{P}^1 \times \mathbb{P}^1, \#) \longrightarrow \mathbb{Z} \cong Hg^2(X)$$

$$\Psi_1 : \ker(\Psi_0) \longrightarrow \mathbb{C}^* \oplus \mathbb{C}^* \cong \frac{\langle \mathrm{dlog}\mathsf{X}, \mathrm{dlog}\mathsf{Y}\rangle^\vee}{\mathrm{im}\{H_1(\mathbb{C}^* \times \mathbb{C}^*, \mathbb{Z})\}} \cong J^2(X)$$

(where $\mathbb{C}^*[= \exp(\mathbb{C}/2\pi i\mathbb{Z})]$ comes from writing $\mathbb{C}/\mathbb{Z}(1)$ multiplicatively), and we write $F^1CH^2 = \ker \Psi_0$, $F^2CH^2 = \ker \Psi_1$, $F^3CH^2 = 0$. Of course

$\Psi_0(\mathcal{Z}) = \sum m_j$, $\Psi_1(\mathcal{Z}) = \left(\prod \alpha_j^{m_j}, \prod \beta_j^{m_j}\right)$ for the above $\mathcal{Z}$, and there are standard projections to the graded pieces induced as follows by maps of cycles ($\in \mathbb{Z}[\mathbb{C}^* \times \mathbb{C}^*]$):

$$\pi_0 : CH^2(X) \twoheadrightarrow Gr_F^0 CH^2(X) \qquad (\alpha, \beta) \mapsto (1,1)$$

$$\pi_1 : CH^2(X) \twoheadrightarrow Gr_F^1 CH^2(X) \qquad (\alpha, \beta) \mapsto (\alpha, 1) - (1,1) + (1, \beta) - (1,1)$$

$$\pi_2 : CH^2(X) \twoheadrightarrow [Gr_F^2 =] F^2 CH^2(X) \qquad \text{by sending}$$

$$\begin{aligned}(\alpha,\beta) \;\mapsto\; B(\alpha,\beta) \;:=\; \mathcal{Z}_{(\alpha,\beta)} \;&=\; (\alpha,\beta) - (1,\beta) - (\alpha,1) + (1,1) \\ &=\; (\alpha,\beta) + \pi_0(\alpha,\beta) - \pi_1(\alpha,\beta).\end{aligned}$$

Here π_2 surjects because it is the identity on $F^2CH^2(X)$:

$$[\mathcal{Z}] \in \ker \Psi_0 \cap \ker \Psi_1 \implies [\pi_1(\mathcal{Z})] = [\pi_0(\mathcal{Z})] = 0 \implies [\pi_2(\mathcal{Z})] = [\mathcal{Z}]$$

since $\pi_2 = \mathrm{id} + \pi_0 - \pi_1$. Now if B (on the level of cycles) sends the three generating Steinberg relations to cycles $\underset{\text{rat}}{\equiv} 0$, we may factor it as follows to produce a well-defined Ξ:

$$\begin{tikzcd}
\mathbb{Z}[\mathbb{C}^* \times \mathbb{C}^*] \arrow[r, two heads, "{[B]}"] \arrow[d, "\text{mod } \langle S_1 := (\alpha,\beta) + (\gamma,\beta) - (\alpha\gamma,\beta)\rangle"'] & F^2CH^2(\mathbb{P}^1 \times \mathbb{P}^1, \#) \\
\mathbb{C}^* \otimes_{\mathbb{Z}} \mathbb{C}^* \arrow[ur] \arrow[d, "\text{mod } \langle S_2 := (\alpha,\beta) + (\beta,\alpha)\rangle"'] & \\
\mathbb{C}^* \bigwedge_{\mathbb{Z}} \mathbb{C}^* \arrow[uur] \arrow[d, "\text{mod } \langle S_3 := (\alpha, 1-\alpha)\rangle"'] & \\
K_2^M(\mathbb{C}) \arrow[uuur, "\Xi"'] &
\end{tikzcd}$$

Assuming this, we can prove injectivity of Ξ using Suslin reciprocity. Writing $K_2(\mathbb{C})$ additively, choose an element $\{\mathbf{a}\} = \sum m_j \{\alpha_j, \beta_j\}$ with $\Xi\{\mathbf{a}\} = 0$, or equivalently a cycle $\mathcal{Z} = \sum m_j(\alpha_j, \beta_j) \in \mathbb{Z}[\mathbb{C}^* \times \mathbb{C}^*]$ with $B(\mathcal{Z}) = \sum m_j T_{(\alpha_j,\beta_j)} \underset{\text{rat}}{\equiv} 0$. That is, there is a collection of curves (and meromorphic functions)

$$\{\tilde{\mathcal{C}}_k \xrightarrow{\iota_k} \mathbb{P}^1 \times \mathbb{P}^1,\, h_k \in \mathbb{C}(\tilde{\mathcal{C}}_k) \;\big|\; h_k \equiv 1 \text{ on } \iota_k^{-1}(\mathcal{C}_k \cap \#)\}$$

such that $B(\mathcal{Z}) = \sum_k \iota_*(h_k)$, i.e. (writing p for $\iota(p)$)

$$\sum_j m_j T_{(\alpha_j,\beta_j)} = \sum_k \sum_{p\in|(h_k)|} \nu_p(h_k)\cdot p = \sum_k \sum_{p\in\tilde{\mathcal{C}}_k} \nu_p(h_k)\cdot(\mathsf{X}(p),\mathsf{Y}(p))$$

on the nose (in $\mathbb{Z}[\mathbb{C}^*\times\mathbb{C}^*]$). Put $f_k = \mathsf{X}\circ\iota_k$, $g_k = \mathsf{Y}\circ\iota_k \in \mathbb{C}(\tilde{\mathcal{C}}_k)$, and note that $h_k \equiv 1$ on $|(f_k)|\cup|(g_k)| = \iota_k^{-1}(\mathcal{C}_k\cap\#)$. Therefore Suslin $\implies$

$$0 = \sum_{p\in\tilde{\mathcal{C}}_k} \nu_p(h_k)\{f_k(p), g_k(p)\} \in K_2^M(\mathbb{C})$$

and summing over k,

$$0 = \sum_k \sum_{p\in\tilde{\mathcal{C}}_k} \nu_p(h_k)\{\mathsf{X}(p),\mathsf{Y}(p)\} = \sum_j m_j\left(\{\alpha_j,\beta_j\} - (\alpha_j, 1\} - \{1,\beta_j\} + \{1,1\}\right)$$

$$\equiv \sum_j m_j\{\alpha_j,\beta_j\}.$$

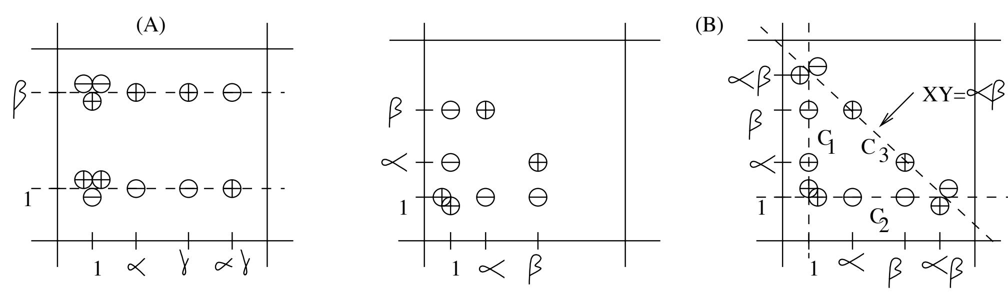
To complete the proof that Ξ is an isomorphism we must show $B(S_i) \underset{\text{rat}}{\equiv} 0$ for $i = 1, 2, 3$, or ($\forall\ \alpha,\ \beta,\ \gamma\in\mathbb{C}^*$)

(A) $\mathcal{Z}_{(\alpha,\beta)} + \mathcal{Z}_{(\gamma,\beta)} - \mathcal{Z}_{(\alpha\gamma,\beta)} \underset{\text{rat}}{\equiv} 0,$

(B) $\mathcal{Z}_{(\alpha,\beta)} + \mathcal{Z}_{(\beta,\alpha)} \underset{\text{rat}}{\equiv} 0,$

(C) $\mathcal{Z}_{(\alpha,1-\alpha)} \underset{\text{rat}}{\equiv} 0.$

(A) and (B) are easily obtained by "plotting" the cycles on $(\mathbb{P}^1\times\mathbb{P}^1, \#)$ and writing down an "explicit rational equivalence" $\{\mathcal{C}_k,\ f_k\}$ with $\sum_k \iota_*(h_k) = \mathcal{Z}$ (and each $h_k = 1$ on $\mathcal{C}_k\cap\#$), to"solve" the diagram. The functions are all variations on the $\frac{(x-a)(x-b)}{(x-ab)(x-1)}$ theme.



We temporarily replace (C) by

(C') $\sum m_j\mathcal{Z}_{(\alpha_j,1-\alpha_j)} \underset{\text{rat}}{\equiv} 0$ for $\sum m_j(\alpha_j)\in\mathbb{Z}[\mathbb{C}^*\setminus\{1\}]$ such that $\prod\alpha_j^{m_j} = \prod(1-\alpha_j)^{m_j}$.

Denote by $\mathsf{A}_2(\mathbb{C})$ the image in $\mathcal{B}_2(\mathbb{C})$ of the set of such $\sum m_j(\alpha_j)$.

If we take $\mathcal{C}$ to be the curve $\subset \mathbb{P}^1 \times \mathbb{P}^1$ parametrized by $t \mapsto (t, 1-t)$ then $f(t) = \prod_j (t-\alpha_j)^{m_j} / (t-(1-\alpha_j))^{m_j}$ is 1 at 0, 1, ∞ and so we have $\mathcal{Z}_0 := \sum m_j[(\alpha_j, 1-\alpha_j) - (1-\alpha_j, \alpha_j)] \underset{\text{rat}}{\equiv} 0$, which $\implies$

$$0 \underset{\text{rat}}{\equiv} B(\mathcal{Z}_0) = \sum m_j \left[T_{(\alpha_j, 1-\alpha_j)} - T_{(1-\alpha_j, \alpha_j)} \right].$$

But (B) $\implies$

$$\sum m_j \left[T_{(\alpha_j, 1-\alpha_j)} + T_{(1-\alpha_j, \alpha_j)} \right] \underset{\text{rat}}{\equiv} 0,$$

and so (modulo 2-torsion) adding these $\implies$ (C').

Now $B_2(\mathbb{C})$ sits in an exact sequence

$$0 \to \ker(\text{st}) \hookrightarrow \mathcal{B}_2(\mathbb{C}) \overset{\text{st}}{\longrightarrow} \wedge^2_{\mathbb{Z}} \mathbb{C}^* \twoheadrightarrow K_2^{(M)}(\mathbb{C}) \to 0,$$

where $\ker(\text{st})$ contains all sums of the form $\sum_{i=0}^4 (-1)^i \{CR(\alpha_0, \ldots, \widehat{\alpha_i}, \ldots, \alpha_4)\}_2$. One can show (see **[GG2]**) that such sums, together with A_2, generate $\mathcal{B}_2(\mathbb{C})$; so $\text{st}(\mathsf{A}_2)$ generates all Steinberg relations $a \wedge (1-a)$ in $\bigwedge^2_{\mathbb{Z}} \mathbb{C}^*$. Therefore 0-cycles of the form (C') (combined with (A) and (B) to give the relations in $\bigwedge^2_{\mathbb{Z}} \mathbb{C}^*$) generate those in (C), which means they are $\underset{\text{rat}}{\equiv} 0$.

We have shown Ξ is an isomorphism and so (mod torsion) $F^2CH^2 \cong K_2^M(\mathbb{C})$; in summary we have

PROPOSITION 5.2.1. $CH^2(\mathbb{P}^1 \times \mathbb{P}^1, \#) \cong_{\otimes\mathbb{Q}} \mathbb{Z} \oplus (\mathbb{C}^* \oplus \mathbb{C}^*) \oplus K_2^M(\mathbb{C})$.

On the one hand this gives a concrete demonstration of the Mumford theorem since $K_2(\mathbb{C})$ is "infinite-dimensional": in particular, the map $\pi_N : S^N(\mathbb{C}^* \times \mathbb{C}^*) \longrightarrow K_2^M(\mathbb{C})$ fails to surject, for all N (there are always elements that cannot be presented as a product of $\leq N$ symbols). On the other hand, the above proof applies just as well to any algebraically closed[8] subfield $k \subseteq \mathbb{C}$, including $\bar{\mathbb{Q}}$. So $F^2CH^2((\mathbb{P}^1 \times \mathbb{P}^1, \#)(\bar{\mathbb{Q}})) \cong K_2^M(\bar{\mathbb{Q}}) \cong 1$ (see **[R]** for a proof) gives a concrete demonstration of Bloch-Beilinson.

5.2.3. Extension of the computation to higher (co)dimension. Here we show how to generalize the above to zero-cycles on $(\mathbb{P}^1, \{0,\infty\})^p$, $p > 2$. The considerations are "combinatorial", and in particular higher Bloch groups do not come in (just as higher logarithms do not enter in §2.1 – 2.2). [The F^iCH^p here are the $F^i_{\times}CH^p$ referred to in §5.1.3.]

We begin with the observation that the result actually applies to any nonsingular complete toric variety T^p, modulo the principal divisors $\{D_\alpha^{p-1}\}$ arising from the codimension 1 faces of its fan (see **[Fu]**). So (working over

[8] or (mod torsion) any $k \subseteq \mathbb{C}$.

$\mathbb{C}$) the generalizations to higher $p > 2$ of $(\mathbb{P}^2, \triangle)$ are also covered by the following

PROPOSITION. *One can put a filtration on* $CH^p(T^p, \cup D_\alpha^{p-1})$ *so that*

$$Gr^i CH^p := \frac{F^i CH^p}{F^{i+1} CH^p} \cong \underbrace{K_i^M(\mathbb{C}) \oplus \ldots \oplus K_i^M(\mathbb{C})}_{\binom{p}{i} \text{ copies}}.$$

This filtration comes from the product structure on $T^p \backslash \cup D_\alpha^{p-1} \cong \underbrace{\mathbb{C}^* \times \ldots \times \mathbb{C}^*}_{p \text{ copies}}.$

Preliminary observation: To prove this we may revert to $T^p = \underbrace{\mathbb{P}^1 \times \ldots \times \mathbb{P}^1}_{p \text{ copies}}$ where the principal divisors are just every permutation of $\{0\} \times \mathbb{P}^1 \times \ldots \times \mathbb{P}^1$ and $\{\infty\} \times \mathbb{P}^1 \times \ldots \times \mathbb{P}^1$. So from now on that's what T^p and D_α^{p-1} mean.

DEFINITION. Let all permutations of $\underbrace{\{1\} \times \ldots \times \{1\}}_{p-i} \times \underbrace{\mathbb{P}^1 \times \ldots \times \mathbb{P}^1}_{i}$ be called *standard i-planes*. We write π_i^σ for the projections to these, where σ (or σ_i) denotes a choice of i indices. An *i-plane* is just any plane parallel to a standard i-plane. By an *i-box* we mean a (codimension p) $0-$cycle consisting of a rectangular configuration of signed points (in an i-plane), such that adjacent points have opposite sign. By a *standard i-box* we mean an i-box with a corner at $\underbrace{\{1\} \times \ldots \times \{1\}}_{p}$ (these are contained in some standard i-plane).

Now we produce (among other things) filtrations on the abelian group of 0-cycles C^p that will induce the desired filtrations on the Chow group $CH^p = C^p / \underset{rat}{\equiv}$. Note that $C^p(T^p, \cup D_\alpha^{p-1})$ and $C_0(T^p \backslash \cup D_\alpha^{p-1}) = C_0(\mathbb{C}^* \times \ldots \times \mathbb{C}^*)$ are just the same thing.

Four conditions on zero-cycles:

(1). $F_{rat}^{i+1} C^p := \left\{ Z \in C^p \mid \pi_i^\sigma(Z) \underset{rat}{\equiv} 0 \ (\forall \sigma) \right\}$, where the rational $\equiv$ is taken inside each relevant i-plane. Write $(\mathbf{rat})^{\mathbf{i}}$ if $Z \in F_{rat}^i C^p(T^p, \cup D_\alpha^{p-1})$.

(2). Writing $F_{\mathbb{P}^1}^0 C^1(\mathbb{P}^1, \{0,\infty\}) = C^1$ and $F_{\mathbb{P}^1}^1 C^1 = \left\{ Z \mid Z \underset{hom}{\equiv} 0 \right\}$, define $F_\times^i C^p := \left\{ Z \mid Z \in \bigoplus_{\sigma \in S^p} (F_{\mathbb{P}^1}^0 \times \ldots \times F_{\mathbb{P}^1}^0) \times (F_{\mathbb{P}^1}^1 \times \ldots \times F_{\mathbb{P}^1}^1) \right\} = \{ Z \mid Z = \Sigma \ i\text{-boxes} \}$ (in particular, these i-boxes are *not* necessarily standard). Write $(\times)^{\mathbf{i}}$ if $Z \in F_\times^i C^p$.

(3). $(\mathbf{st})^{\mathbf{i}}$ means: $\pi_i^\sigma(Z) \underset{rat}{\equiv} \Sigma$ *standard* i-boxes $(\forall \sigma)$.

(4). $(\mathbf{K})^{\mathbf{i}}$ means: $\pi_i^\sigma(Z)$ is "$K_i^M(\mathbb{C})$-equivalent" to 0 ($\forall \sigma$). For this to make sense we must already be thinking of $\pi_i^\sigma(Z)$ as standard i-boxes, and then the elements of $K_i^M(\mathbb{C})$ are the coordinates of the corners *opposite* $\{1, \ldots, 1\}$. So $(K)^i$ implicitly includes $(st)^i$.

Remark on the descent to CH^p: Towards the end of the argument we will use the fact that F_{rat}^i and $F_\times^i$ descend to rational equivalence classes $[Z]$ in a well-defined fashion. This is really easy for F_{rat}^i, since *(A)* below implies trivially that π_i^σ descends to CH^p. For $F_\times^i$ one just defines

$$F_\times^i CH^p := \left\{[Z] \mid \exists\, representative\, Z\, of\, [Z]\, such\, that\, Z \in F_\times^i C^p\right\}.$$

However at present (directly below) we are still working over C^p.

Facts concerning (1)-(4):.

(A). $(\times)^i \implies (\times)^{i-1}$, $(rat)^i \implies (rat)^{i-1}$. To see the second of these (which says that $F_{rat}^{i-1} \supseteq F_{rat}^i$) for $i > 3$, one must note that rational equivalences are given by functions on the *normalization* of an algebraic curve (so the equivalences "project down" just fine); for $i = 3$ use pullbacks of $dlog(z)$ to the relevant curves (which are in this case branched covers of the $\mathbb{P}^1$'s via projection); $i < 3$ is trivial.

The obvious interpretation is that under the descent described above, F_{rat}^* and $F_\times^*$ define *filtrations* on CH^p (and of course also on C^p).

(B). $(\times)^i \implies (rat)^i$. Recall that $(rat)^i$ means that the projection of Z to $(i-1)$-planes is $\underset{rat}{\equiv} 0$. So this merely says that collapsing an i-box to $(i-1)$-space gives 0 on the level of cycles, because the signs cancel.

(C). $(\times)^i \implies (st)^i$. An i-box (in any i-plane) may be made "standard within that i-plane" by adding $\left\{\begin{matrix} + \\ - \end{matrix}\right\}$ of the same points. Projecting this to a standard i-plane gives a standard i-box (or 0).

(D). $(K)^i[+(st)^i] \implies (rat)^{i+1}$ *("Steinberg relations $\rightarrow$ cycles rationally $\equiv$ to 0").* We work in a standard i-plane, say the one corresponding to π_i^σ. The generating relations are "one or two-dimensional": (i)$\{\ldots, [z_k =]a, \ldots, [z_j =]1-a, \ldots\} = 0$; (ii) $\{\ldots, [z_j =]ab, \ldots\} - \{\ldots, a, \ldots\} - \{\ldots, b, \ldots\} = 0$; (iii) $\{\ldots, a, \ldots, b, \ldots\} + \{\ldots, b, \ldots, a, \ldots\} = 0$. Here we are writing Milnor K-theory *additively*. The hypothesis $(K)^i$ says that $\pi_i^\sigma(Z)$ is a formal sum of standard i-boxes; and moreover, that the corresponding formal sum of corners opposite $\{1, \ldots, 1\}$ consists entirely of these Steinberg relations with different k and j. Now consider the 2-plane given by fixing the "$\ldots$"-coordinates and letting z_j and z_k vary. Intersecting it with $\pi_i^\sigma(Z)$ gives a standard 2-box which is $\underset{rat}{\equiv} 0$ by the well-known argument for $p = 2$; e.g., in case (i) we have $Z_{\{a,1-a\}} = (1,1) - (a,1) - (1,1-a) + (a,1-a)$. One can cover the box with planes parallel to this one obtained by changing some of

the "…"-coordinates to 1; these intersections are also $\underset{rat}{\equiv} 0$ by the $p = 2$ case.

REMARK. This is the step where we needed the i-boxes.

(E). $(rat)^{i+1} + (st)^i \implies (K)^i$ ("cycles rationally $\equiv$ to 0 $\rightarrow$ Steinberg relations"). Again we're working in a standard i-plane. Recall that $(K)^i$ is *a priori* a statement about the corners of the standard $i-$boxes opposite $\{1, \ldots, 1\}$. However, it is sufficient to prove that the element of $K_i^M(\mathbb{C})$ generated by *all* the points of Z (not just those corner points) is trivial, since – using $(st)^i$ – all other vertices of a *standard* box are K-theoretically trivial (because each has one coordinate $z_l = 1$). Now the generators for rational equivalence are individual curves $\{\mathcal{C}_n\}$ with meromorphic functions $\{h_n\}$ defined on their normalizations. Since $h = 1$ whenever any coordinate function $\widetilde{z_k}$ blows up (this being the definition of rational equivalence *relative* the faces $\{D_\alpha^{p-1}\}$), Suslin reciprocity on each algebraic curve, summed over n, reduces to

$$0 = \sum_n \sum_{p \in |(h_n)|} \nu_p(h_n).\{\widetilde{z_1}, \ldots, \widetilde{z_i}\} \in K_i^M(\mathbb{C}),$$

where $\sum_n \coprod_{p \in |(h_n)|} \nu_p(h_n) = Z \in C^i(i\text{-plane})$. This is exactly what we wanted.

(F). $(rat)^i \implies (st)^i$. Using *(A)*, we see that $(rat)^i \implies$ all $\pi_j^{\sigma'}(Z) \underset{rat}{\equiv} 0$ for $j < i$, so that for each of the σ_i (there are $\begin{pmatrix} p \\ i \end{pmatrix}$ of them),

$$\pi_i^\sigma(Z) \underset{rat}{\equiv} \pi_i^\sigma(Z) + \sum_{j<i} (-1)^{i-j} \sum_{\sigma' \subset \sigma} \pi_j^{\sigma'}(Z) \;=:\; B_i(\pi_i^\sigma(Z))$$

But the right-hand expression is just a sum of standard i-boxes, as desired. (The point is that this gives, for each σ_i, an element of $K_i^M(\mathbb{C})$.)

(G). Given $(rat)^i$, $(rat)^{i+1} \Leftrightarrow (K)^i$. (This is just the first part of the proposition; see below for more details.)

($\Rightarrow$): by *(F)*, $(rat)^i \implies (st)^i$; this is then just *(E)*: $(rat)^{i+1} + (st)^i \implies (K)^i$.

($\Leftarrow$): this is just *(D)*.

Interpretation: Together with (F), this yields an isomorphism as described in the first part of the proposition. (F) gives a homomorphism $F_{rat}^i C^p \rightarrow \oplus K_i^M(\mathbb{C})$, while (G,$\Rightarrow$) tells us that this is well-defined modulo $F_{rat}^{i+1} C^p$. Surjectivity is trivial: standard i-boxes in standard i-planes certainly surject onto the $K_i^M(\mathbb{C})$ summands, and such boxes are in F_{rat}^i by (B). Injectivity follows from (G,$\Leftarrow$). Finally we have to check that $\frac{F_{rat}^i C^p}{F_{rat}^{i+1} C^p} \cong \frac{F_{rat}^i CH^p}{F_{rat}^{i+1} CH^p}$

"naturally". This is clear from the canonical isomorphisms $CH^p \cong \frac{C^p}{F^{p+1}C^p}$, $F^iCH^p \cong \frac{F^iC^p}{F^{p+1}C^p}$.

To prove the second statement of the proposition we need one more fact:

(H). $(rat)^i \implies (\times)^i$ mod $(rat)^{p+1}$ (converse of (B) in CH^p). This says that if all $\pi^\sigma_{i-1}(Z) \underset{rat}{\equiv} 0$, then Z is rationally equivalent to a sum of i-boxes. For $i=1$ this just says that if the multiplicities in Z of points $\in|Z|$ sum to zero, then one may "connect the dots" using line segments parallel to the coordinate axes, and take their boundaries as the desired 1-boxes. This can (in a slightly altered form) be generalized as follows. Let $\mathbf{a}=(a_1,\ldots,a_p)\in|Z|$ have multiplicity $n_{\mathbf{a}}$ in Z. Define, for an integer k between i and p, an $(i-1)$-index $1\leq \mathbf{l}< k$ (which means: $1\leq l_1<\ldots<l_{i-1}<k$) and an i-plane $P^{\mathbf{a}}_{i,\mathbf{l},k} := (1,\ldots,1,z_{\mathbf{l}},1,\ldots,1,z_k,a_{k+1},\ldots,a_p)$. (The notation means that $z_{l_1},\ldots,z_{l_{i-1}}$ all appear in their respective slots.) There is a natural projection of $\mathbf{a}$ to this P, and a natural way of making $\mathbf{a}$ into an i-box $\subseteq P$, which is like what we did in *(F)*, but there we did it for the projection of *all* of Z to a *standard* i-plane. For example, in the case $p=5$, $i=2$ we have in $P^{\mathbf{a}}_{2,(2),4}=(1,z_2,1,z_4,z_5)$ the 2-box $(1,a_2,1,a_4,a_5)-(1,1,1,a_4,a_5)-(1,a_2,1,1,a_5)+(1,1,1,1,a_5)$. We denote this i-box by $B^{\mathbf{a}}_{i,\mathbf{l},k}$ (so the 2-box in the example would be $B^{\mathbf{a}}_{2,(2),4}$). Then

$$\sum_{\mathbf{a}\in|Z|} n_{\mathbf{a}} \sum_{k=i}^{p}\sum_{1\leq \mathbf{l}<k} B^{\mathbf{a}}_{i,\mathbf{l},k} = Z + \sum\left\{\begin{array}{c}+\\-\end{array}\right\}\text{of the same points} + \sum_{\sigma_i}\sum_{j<i}(-1)^{i-j}\sum_{\sigma'_j\subseteq\sigma_i}\pi^{\sigma'}_j(Z),$$

where the second term *is* 0 and the last is $\underset{rat}{\equiv} 0$, proving that $Z \underset{rat}{\equiv} \sum$ i-boxes. We note, for a given $\mathbf{l}$, that the $\{P^{\mathbf{a}}_{i,\mathbf{l},p}\}_{\mathbf{a}\in|Z|}$ are all the same *standard* i-plane; and the corresponding $\{B^{\mathbf{a}}_{i,\mathbf{l},p}\}$, with the exception of some points that go to the second term, are where the third term comes from [σ_i corresponds to $(\mathbf{l},p)$]. The points for the first term Z come from the $\{B^{\mathbf{a}}_{i,(1,\ldots,i-1),i}\}_{\mathbf{a}\in|Z|}$.

Interpretation: Together with (B), this says that the filtrations $F^i_\times$ and F^i_{rat} on CH^p are the same.. From the definition of $F^i_\times$ it is clear that this filtration respects products, in the sense that we have a (surjective) map

$$F^aCH^p(T^p,\cup D^{p-1}_\alpha)\otimes F^bCH^q(T^q,\cup D^{q-1}_\beta)\to F^{a+b}CH^{p+q}(T^{p+q},\cup D^{p+q-1}_\gamma).$$

REMARK 5.2.2. As in the case of $p=2$, all the results hold if we replace $\mathbb{C}$ by $k\subseteq\mathbb{C}$ algebraically closed, or (modulo torsion) for any $k\subseteq\mathbb{C}$.

5.2.4. Description of the invariants. In §5.1.3 we promised a filtration on $CH^n((\mathbb{P}^1,\{0,\infty\})^n(k))$ and maps from the graded pieces to $\oplus_{\sigma_i}H^i_{\mathcal{D}}(\eta_S,\mathbb{Q}(i))$. Write $X^n(k)=(\mathbb{P}^1_k,\{0,\infty\})^n$, and $\mathbf{f}_\sigma$ for an element of $\otimes^i\mathbb{Z}[\mathbb{P}^1_{\mathbb{Q}(S)}\setminus\{0,\infty\}]$ such that $B_i(\pi^i_\sigma(\mathcal{Z}))$ below spreads to $B_i(\gamma_{\mathbf{f}_\sigma})$ (the "i-box" of the graph).

Then by composing with projections to the standard i-plane we have now

$$\begin{array}{ccccccc} Gr_F^i CH^n(X^n(k)) & \underset{\cong}{\xrightarrow{\oplus\pi_\sigma^i}} & \oplus_\sigma Gr_F^i CH^i(X^i(k)) & \underset{\cong}{\longrightarrow} & \overset{\binom{n}{i}}{\oplus} \; K_i^M(k\cong\mathbb{Q}(S)) & \xrightarrow{\mathcal{R}} & \overset{\binom{n}{i}}{\oplus} \; H^i_{\mathcal{D}}(\eta_S,\mathbb{Q}(i)) \\ \mathcal{Z} & \longmapsto & \sum_{\sigma_i}\pi_\sigma^i(\mathcal{Z}) \underset{\text{rat}}{\equiv} \sum_\sigma B(\pi_\sigma^i(\mathcal{Z})) & \longmapsto & \sum_\sigma \{\mathbf{f}_\sigma\} & \longmapsto & \sum_\sigma (\Omega_{\mathbf{f}_\sigma}, R_{\mathbf{f}_\sigma}) \end{array}$$

or in the formal notation of §5.1.3

$$[\zeta]_i = \sum_{\sigma_i} \left([\wedge^i \mathrm{dlog} z_{\sigma_i}]^\vee \otimes\right)[\wedge^i \mathrm{dlog} f_{\sigma_i}] \in Gr^i_{\mathcal{L}} Hg^n(X^n \times \eta_S)$$

and if this vanishes

$$[AJ\zeta]_{i-1} = \sum_{\sigma_i} \left([\wedge^i \mathrm{dlog} z_{\sigma_i}]^\vee \otimes\right)[R_{\mathbf{f}_{\sigma_i}}] \in Gr^i_{\mathcal{L}} J^n(X^n \times \eta_S).$$

For simplicity we restrict ourselves to examining the map

$$F^n CH^n(X^n(k)) \longrightarrow H^n_{\mathcal{D}}(\eta_S,\mathbb{Q}(n)),$$

first reviewing the sense in which the invariants are computed as cycle-class/AJ of the spreads of n-boxes

$$\mathcal{Z} \underset{\text{rat}}{\equiv} B_n(\mathcal{Z}) = B\left(\sum m_j(\alpha_{1j},\ldots,\alpha_{nj})\right)$$

to $\zeta = B(\gamma_{\mathbf{f}})$, $\{\mathbf{f}\} = \prod\{f_{1j},\ldots,f_{nj}\}^{m_j} \in K_n^M(\mathbb{Q}(S))$ where $\mathbf{f}_{ij} \longleftrightarrow \alpha_{ij}$ under $\mathbb{Q}(S)\cong k$. In fact if[9]

$$[\wedge^n \mathrm{dlog}\mathbf{f}] \in H^n(\eta_S,\mathbb{Q}(n)) \cap F^n H^n(\eta_S,\mathbb{C})$$

vanishes then so does[10] $[T_{\mathbf{f}}] \in H_{2(\dim S)-n}(S_{\text{rel}},\mathbb{Z})$, so that one may exhibit $B(\gamma_{\mathbf{f}})$ as a topological boundary of $\Gamma_{\mathbf{f}} = \partial^{-1}(B(\gamma_{\mathbf{f}})) = \theta(B(\gamma_{\mathbf{f}})) + (S^1)^n \times \partial^{-1}_{(S,D)}T_{\mathbf{f}}$ in a sense described precisely in §1.2 – 1.4 (and §1.4.5). To find $AJ(B(\gamma_{\mathbf{f}}))$ one integrates over this

$$\int_{\Gamma_{\mathbf{f}}} \wedge_{i=1}^n \mathrm{dlog} z_i \wedge \pi_S^*\omega = \int_S R'_{\mathbf{f}} \wedge \omega$$

or replacing forms ω by $(n-1)$-cycles $\mathcal{C}$ (on η_S), $\int_{\mathcal{C}} R_{\mathbf{f}}$ (mod $\mathbb{Z}(n)$).

Rather than conceiving of $[B(\gamma_{\mathbf{f}})]$ and $[AJ(B(\gamma_{\mathbf{f}}))]$ in this sense as two separate invariants, we would like to combine them into one *geometrically defined* unit. In §2.2.2 we observed that since $\mathrm{d}[R_{\mathbf{f}}] = \Omega_{\mathbf{f}} - (2\pi i)^n T_{\mathbf{f}}$, $\int_{\mathcal{C}} R_{\mathbf{f}}$

[9] $\wedge^n \mathrm{dlog}\mathbf{f}$ can also be thought of in terms of Kähler differentials on the original cycle, without spreading to S.

[10] $S_{\text{rel}} := \varinjlim_{D/\mathbb{Q}\subsetneq S} (S,D)$

defined a holomorphic $\mathbb{C}/\mathbb{Z}(n)$-valued differential character (and thus an element of $H^n_{\mathcal{D}}(\eta_S, \mathbb{Q}(n)))$ even when $[\Omega_{\mathbf{f}}] \neq 0$. The geometric basis behind this involves membrane integrals. To produce the membrane we start with a lift

$$(\pi_S^{\zeta})^{-1}\mathcal{C} \;=\; B(\gamma_{\mathbf{f}}) \cap \{(\mathbb{P}^1)^n \times \mathcal{C}\}$$

of $\mathcal{C}$ to each component of $B(\gamma_{\mathbf{f}})$ (with multiplicity/sign). As classes $(\pi_S^{\zeta})^{-1}\mathcal{C}$ [dimension $n-1$] and $T^n_{\square} \times S$ [codimension n] do not intersect,[11] and so applying θ *directly*[12] gives an initial membrane on $(\mathbb{C}^*)^n \times S$, $\theta((\pi_S^{\zeta})^{-1}\mathcal{C})$ with boundary $(\pi_S^{\zeta})^{-1}\mathcal{C}$. Now we project this to $(\mathbb{C}^*)^n$ and define

$$\Gamma_{\mathcal{C}} := \pi_X \left\{ \theta\left((\pi_S^{\zeta})^{-1}\mathcal{C} \right) \right\}$$

so that, writing $f(\mathcal{C})$ for $\pi_X^{\zeta}((\pi_S^{\zeta})^{-1}\mathcal{C})$,

$$\partial\Gamma_{\mathcal{C}} \;=\; f(\mathcal{C}) \qquad \text{on } (\mathbb{C}^*)^n.$$

The key to understanding how the form $\wedge^n \mathrm{dlog} z_i$ on $(\mathbb{P}^1, \{0, \infty\})^n$ controls rational equivalence, and to the generalization to smooth X in §5.3, is then

$$\int_{\Gamma_{\mathcal{C}}} \wedge^n \mathrm{dlog} z_i \;=\; \int_{\mathcal{C}} R_{\mathbf{f}}$$

and that any other choice of membrane changes the value by a period of $\wedge^n \mathrm{dlog} z_i$ ($\in \mathbb{Z}(n)$). Such membrane-integral constructions in fact always yield differential characters.

In the case $\dim S = 1$, $n = 2$, the first figure in §1.4.3 shows essentially what $\Gamma_{\mathcal{C}}$ would look like for a couple of loops [=(I)+(III) for one loop and (II) for the other]. Although this picture is drawn for $\gamma_{\mathbf{f}}$ on $(\square^2, \partial\square^2) \times \eta_S$ instead of $B(\gamma_{\mathbf{f}})$ on $(\mathbb{P}^1 \times \mathbb{P}^1, \#) \times \eta_S$, the only difference (which is formal) is that $\Gamma_{\mathcal{C}}$ bounds also on the components $B_{\gamma_{\mathbf{f}}}$ *in* $(\{1\} \times \mathbb{P}^1 \cup \mathbb{P}^1 \times \{1\}) \times \eta_S = \mathbb{I}^2 \times \eta_S$ instead of using it as a "topological trashcan".

5.3. Zero-cycles on a Product of Curves

5.3.1. Reduction of the target spaces. In the first two subsections we establish the "reduced" invariants we will use, in the more general case of $X/\mathbb{Q}$ smooth projective. We take $\dim X = n$ and $\mathcal{Z} \in Z^n(X(k))$ a 0-cycle, the coefficients of whose defining equations generate k. Since the equations defining X have coefficients in $\mathbb{Q}$, X does not spread with $\mathcal{Z}$ and so one has the situation $/\mathbb{Q}$

[11] In the case where $(\mathbb{P}^1, \{0, \infty\})^n$ is replaced by X smooth projective (in §5.3) so that the invariants $[\zeta]_i$ are defined, this step is replaced by the following: because we consider a cycle $\mathcal{Z} \in F^n CH^n$, $[\zeta]_{n-1}$ must be zero, so that $[\pi_X^{\zeta}\{(\pi_S^{\zeta})^{-1}\mathcal{C}\}] = 0$ automatically. The point is that $[\zeta]_{n-1} \in \mathrm{hom}_{\mathbb{Q}}\{H_{n-1}(S, \mathbb{Q}),\, H_{n-1}(X, \mathbb{Q})\}$ describes that action of $[\pi_X^{\zeta} \circ (\pi_S^{\zeta})^{-1}]$ on $(n-1)$-cycles (which is exactly what $\mathcal{C}$ is).

[12] rather than to all of $B(\gamma_{\mathbf{f}})$, under the condition $[\Omega_{\mathbf{f}}] = 0$ (which we want to avoid here).

$$\begin{tikzcd}
X & X \times \eta_S \arrow[l, "\pi"] \arrow[d, "\rho"] & \zeta \arrow[l, hook'] \arrow[ll, bend right, "\pi_\zeta"'] \arrow[dl, "\rho_\zeta"] \\
 & \eta_S &
\end{tikzcd}$$

where as usual $S/\mathbb{Q}$ is smooth projective with $\mathbb{Q}(S) \cong k$. We will show how to simplify essentially a quotient of the graded pieces of $\underline{H}^{2n}_{\mathcal{H}}(X \times \eta_S, \mathbb{Q}(n))$ in this case, obtain maps into them and then specialize to products of curves and obtain very explicit formulas for (the "quotients" of) the maps Ψ_i.

Let $H_{\mathbb{Q}}$ be a MHS, and write $H_{\mathbb{C}} = H_{\mathbb{Q}} \otimes_{\mathbb{Z}} \mathbb{C}$, $W_0 H_{\mathbb{C}} = (W_0 H_{\mathbb{Q}}) \otimes \mathbb{C}$ and so on. Then (see **[Ca]**)

$$\begin{aligned}
\hom_{MH}(\mathbb{Q}(0),\ H_{\mathbb{Q}}) &= F^0 W_0 H_{\mathbb{C}} \cap W_0 H_{\mathbb{Q}}, \\
\mathrm{Ext}^1_{MH}(\mathbb{Q}(0),\ H_{\mathbb{Q}}) &= \frac{W_0 H_{\mathbb{C}}}{W_0 H_{\mathbb{Q}} + F^0 W_0 H_{\mathbb{C}}}
\end{aligned}$$

so that

$$\begin{aligned}
\hom_{MH}(\mathbb{Q}(0),\ H_{\mathbb{Q}} \otimes \mathbb{Q}(n)) &= \left\{ F^n W_{2n} H_{\mathbb{C}} \cap W_{2n} H_{\mathbb{Q}} \right\}(n) \\
\mathrm{Ext}^1_{MH}(\mathbb{Q}(0),\ H_{\mathbb{Q}} \otimes \mathbb{Q}(n)) &= \left\{ \frac{W_{2n} H_{\mathbb{C}}}{W_{2n} H_{\mathbb{Q}} + F^n W_{2n} H_{\mathbb{C}}} \right\}(n).
\end{aligned}$$

We will omit the (n) frequently when it is not important.

Reduction of $Gr^i_{\mathcal{L}} Hg^n(X \times \eta_S)$. Set

$$H_{\mathbb{Q}} := H^i(\eta_S,\ R^{2n-i}_{\rho_*}\mathbb{Q}) = H^i(\eta_S, \mathbb{Q}) \otimes H^{2n-i}(X, \mathbb{Q}),$$

so that the fact that $H^{2n-i}(X, \mathbb{Q})$ has pure weight $2n - i \implies$

$$W_{2n} H_{\mathbb{Q}} = W_i H^i(\eta_S, \mathbb{Q}) \otimes H^{2n-i}(X, \mathbb{Q}) = \underline{H}^i(\eta_S, \mathbb{Q}) \otimes H^{2n-i}(X, \mathbb{Q})$$

where

$$\underline{H}^i(\eta_S, \mathbb{Q}) := \mathrm{im}\left\{ H^i(S, \mathbb{Q}) \to H^i(\eta_S, \mathbb{Q}) \right\}.$$

Therefore

$$\begin{aligned}
Gr^i_{\mathcal{L}} Hg^n(X \times \eta_S) &= \hom_{MH}(\mathbb{Q}(0),\ H_{\mathbb{Q}} \otimes \mathbb{Q}(n)) \\
&= F^n \left\{ \underline{H}^i(\eta_S, \mathbb{C}) \otimes H^{2n-i}(X, \mathbb{C}) \right\} \cap \left\{ \underline{H}^i(\eta_S, \mathbb{Q}) \otimes H^{2n-i}(X, \mathbb{Q}) \right\}
\end{aligned}$$

$$= \left\{ \sum_{0 \le k \le i} F^k \underline{H}^i(\eta_S, \mathbb{C}) \otimes F^{n-k} H^{2n-i}(X, \mathbb{C}) \right\} \cap \left\{ \underline{H}^i(\eta_S, \mathbb{Q}) \otimes H^{2n-i}(X, \mathbb{Q}) \right\}.$$

The sum is a bit overwhelming, but if we quotient by

$$F^{n-i+1}_X Gr^i_{\mathcal{L}} Hg^n(X \times \eta_S) := Gr^i_{\mathcal{L}} Hg^n(X \times \eta_S) \cap \left\{ \underline{H}^i(\eta_S, \mathbb{Q}) \otimes F^{n-i+1} H^{2n-i}(X, \mathbb{C}) \right\}$$

then all but the $k=i$ term of the sum gets swallowed. So

$$Gr^i_{\mathcal{L}} Hg^n(X\times\eta_S) \twoheadrightarrow Gr^i_{\mathcal{L}} Hg^n(X\times\eta_S)\,/F_X^{n-i+1}\{\text{num}\}$$

$$= \left\{F^i\underline{H}^i(\eta_S,\mathbb{C})\otimes\frac{(F^{n-i})}{F^{n-i+1}}H^{2n-i}(X,\mathbb{C})\right\}\cap\left\{\underline{H}^i(\eta_S,\mathbb{Q})\otimes H^{2n-i}(X,\mathbb{Q})\right\}$$

$$= \hom_{\mathbb{C}}\left\{F^iH^i(X,\mathbb{C}),\ F^i\underline{H}^i(\eta_S,\mathbb{C})\right\}\cap\hom_{\mathbb{Q}}\left\{\underline{H}_i(\eta_S,\mathbb{Q}),\ H_i(X,\mathbb{Q})\right\}$$

where $\underline{H}_i(\eta_S,\mathbb{Q}) := \text{coim}\{H_i(\eta_S,\mathbb{Q})\to H_i(S,\mathbb{Q})\}$ consists of topological cycles avoiding certain[13] divisors $D/\mathbb{Q}\subset S$, modulo those which are ~ 0 under $\hookrightarrow S$. We strategically rewrite this $\cap$, setting $V := (F^iH^i(X,\mathbb{C}))^\vee$ and $\Lambda = \text{im}(H_i(X,\mathbb{Q}))\hookrightarrow V$ (these are the periods[14]), as

$$F^i\underline{H}^i(\eta_S,V)\cap\underline{H}^i(\eta_S,\Lambda)$$

and explain why later.

Explanation of the Ext/Hom quotient. Now let

$$H_{\mathbb{Q}} := H^{i-1}(\eta_S, R^{2n-i}_{\rho_*}\mathbb{Q}) = H^{i-1}(\eta_S,\mathbb{Q})\otimes H^{2n-i}(X,\mathbb{Q})$$

so $W_{2n-1}H_{\mathbb{Q}} = \underline{H}^{i-1}(\eta_S,\mathbb{Q})\otimes H^{2n-i}(X,\mathbb{Q})$, and $\tilde{H}_{\mathbb{Q}} := H^{2n-1}(X\times\eta_S,\mathbb{Q})\twoheadrightarrow H_{\mathbb{Q}}$ by Künneth projection. We want to do the same sort of thing for

$$Gr^{i-1}_{\mathcal{L}}J^n(X\times\eta_S) = \frac{\text{Ext}^1_{MH}(\mathbb{Q}(0),\ \mathcal{W}_{-1}(H_{\mathbb{Q}}\otimes\mathbb{Q}(n)))}{\hom_{MH}(\mathbb{Q}(0), Gr^W_0(\tilde{H}_{\mathbb{Q}}\otimes\mathbb{Q}(n)))}$$

$$= \frac{\text{Ext}^1_{MH}(\mathbb{Q}(0),\ (W_{2n-1}H_{\mathbb{Q}})\otimes\mathbb{Q}(n))}{\hom_{MH}(\mathbb{Q}(0),\ (Gr^W_{2n}\tilde{H}_{\mathbb{Q}})\otimes\mathbb{Q}(n))} \cong \frac{\text{Ext}^1_{MH}(\mathbb{Q}(n),\ W_{2n-1}H_{\mathbb{Q}})}{\hom_{MH}(\mathbb{Q}(n),\ Gr^W_{2n}\tilde{H}_{\mathbb{Q}})}$$

as we did above for $Gr^i_{\mathcal{L}}Hg^n(X\times\eta_S)$.

Although we will not have much use for it we first explain how the denominator maps naturally into the numerator. In fact this amounts to computing the connecting homomorphism

$$\hom_{MH}(\mathbb{Q}(n),\ Gr^W_{2n}\tilde{H}_{\mathbb{Q}}) \xrightarrow{\Delta} \text{Ext}^1_{MH}(\mathbb{Q}(n),\ W_{2n-1}\tilde{H}_{\mathbb{Q}})$$

in the long *Ext*-sequence associated to the short exact sequence

$$0\to W_{2n-1}\tilde{H}_{\mathbb{Q}}\to W_{2n}\tilde{H}_{\mathbb{Q}}\to Gr^W_{2n}\tilde{H}_{\mathbb{Q}}\to 0$$

of mixed Hodge structures (and then composing with the map induced by $\tilde{H}_{\mathbb{Q}}\twoheadrightarrow H_{\mathbb{Q}}$). Δ is obtained by composing $\xi\in\hom_{MH}(\mathbb{Q}(n),\ Gr^W_{2n}\tilde{H}_{\mathbb{Q}})$ with the extension class

$$e\in\text{Ext}^1_{MH}(Gr^W_{2n}\tilde{H}_{\mathbb{Q}},\ W_{2n-1}\tilde{H}_{\mathbb{Q}}) := \frac{\hom_{MH}(Gr^W_{2n}\tilde{H}_{\mathbb{Q}},\ W_{2n-1}\tilde{H}_{\mathbb{Q}})\otimes_{\mathbb{Z}}\mathbb{C}}{\hom_{MH}(Gr^W_{2n},\ W_{2n-1}) + F^0\{\text{num}\}}$$

---

[13](but not all; this would be impossible. One takes a representative cycle on some $S\setminus D$, that survives *in the direct limit*; more on this later).

[14]which for once are not in $(2\pi i)^n\mathbb{Z}$

to get $e \circ \xi \in \mathrm{Ext}^1_{MH}(\mathbb{Q}(n),\, W_{2n-1}\tilde{H}_{\mathbb{Q}})$. Therefore e is the important object; we may describe it (as a lift) in $\hom_{MH}(Gr^W_{2n}\tilde{H}_{\mathbb{Q}}, [(W_{2n-1}\tilde{H}_{\mathbb{C}})^\vee]^\vee)$ as follows. First, $Gr^W_{2n}\tilde{H}_{\mathbb{Q}}$ is represented by differential forms with dlog-poles along divisors $D/\mathbb{Q} \subset S$ but not along their intersections (no $\mathrm{dlog} z_i \wedge \mathrm{dlog} z_j$ locally), modulo forms with *no* residues. Writing $S_{rel} := \varinjlim_{D/\mathbb{Q} \subset S} (S,\, D)$, under the isomorphism

$$H^{2n-1}(X \times \eta_S, \mathbb{Q}) \cong H_{2t-2n+1}(X \times S_{rel}, \mathbb{Q})$$

such forms are Poincaré dual to certain (relative) topological cycles $\mathcal{C}$ bounding on $X \times D/\mathbb{Q} \subset X \times S$. Let $[\mathcal{C}] \in Gr^W_{2n}\tilde{H}_{\mathbb{Q}}$ be such a class and pull it back to $W_{2n}\tilde{H}_{\mathbb{Q}}$; this will pair with the pull-back of $[\omega] \in (W_{2n-1}\tilde{H}_{\mathbb{C}})^\vee$ to $(W_{2n}\tilde{H}_{\mathbb{C}})^\vee$ (in the dualized short exact sequence, $\otimes\mathbb{C}$). [Note that

$$(W_{2n-1}\tilde{H}_{\mathbb{C}})^\vee = \left(\mathrm{im}\left\{H^{2n-1}(X \times S, \mathbb{C}) \to H^{2n-1}(X \times \eta_S, \mathbb{C})\right\}\right)^\vee$$

$$= \mathrm{coim}\left\{H^{2t-2n+1}(X \times S_{rel}, \mathbb{C}) \to H^{2t-2n+1}(X \times S, \mathbb{C})\right\}$$

is spanned by certain $\omega \in \Gamma\left(\Omega^{2t-2n+1}_{(X \times S \setminus D)^\infty}\right)$, whose classes survive in the $\varprojlim$.]

Conclusion. One should think of $\mathrm{Ext}^1_{MH}(\mathbb{Q}(n),\, W_{2n-1}H_{\mathbb{Q}})$ as functionals on $[\omega] \in (W_{2n-1}H_{\mathbb{C}})^\vee$ (modulo periods/etc.), and $\hom_{MH}(\mathbb{Q}(n),\, Gr^W_{2n}\tilde{H}_{\mathbb{Q}})$ as a certain subset of the (Poincaré dual classes of) cycles $[\mathcal{C}] \in Gr^W_{2n}\tilde{H}_{\mathbb{Q}}$, whose image (in the double-dual) is computed by the functionals $\int_{\mathcal{C}} \omega$.

Reduction of $Gr^{i-1}_{\mathcal{L}} J^n(X \times \eta_S)$. Now that we know that the image of the $\hom_{MH}$-denominator makes sense, we proceed to eliminate it. Observe that this image factors through the Künneth projection of the denominator to

$$\hom_{MH}(\mathbb{Q}(0),\, (Gr^W_{2n}H_{\mathbb{Q}}) \otimes \mathbb{Q}(n)) \cong \frac{F^n W_{2n}H_{\mathbb{C}} \cap W_{2n}H_{\mathbb{Q}}}{\{\mathrm{num}\} \cap W_{2n-1}H_{\mathbb{Q}}}$$

$$\cong \frac{\left\{\sum_{1 \le j \le i-1} F^j W_i H^{i-1}(\eta_S, \mathbb{C}) \otimes F^{n-j}H^{2n-i}(X, \mathbb{C})\right\} \cap \left\{W_i H^{i-1}(\eta_S, \mathbb{Q}) \otimes H^{2n-i}(X, \mathbb{Q})\right\}}{\{\mathrm{num}\} \cap W_{2n-1}H_{\mathbb{Q}}}$$

since $H^{2n-i}(X, \mathbb{C})$ is pure, and[15] $j \ge i \implies F^j H^{i-1}(\eta_S, \mathbb{C}) = \{0\}$. On the other hand, by the calculation at the beginning the numerator

$$\mathrm{Ext}^1_{MH}(\mathbb{Q}(0),\, (W_{2n-1}H_{\mathbb{Q}}) \otimes \mathbb{Q}(n)) = \frac{(W_{2n})W_{2n-1}H_{\mathbb{C}}}{W_{2n-1}H_{\mathbb{Q}} + F^n W_{2n-1}H_{\mathbb{C}}}$$

$$= \frac{\underline{H}^{i-1}(\eta_S, \mathbb{C}) \otimes H^{2n-i}(X, \mathbb{C})}{\underline{H}^{i-1}(\eta_S, \mathbb{Q}) \otimes H^{2n-i}(X, \mathbb{Q}) \;+\; \sum_{0 \le k \le i-1} F^k \underline{H}^{i-1}(\eta_S, \mathbb{C}) \otimes F^{n-k}H^{2n-i}(X, \mathbb{C})}.$$

[15](it is also tempting to use $j < \frac{i}{2} \implies 2(n-j) > 2n - i \implies F^{n-j}H^{2n-i}(X, \mathbb{C}) \cap H^{2n-i}(X, \mathbb{Q}) = 0$, but this does not apply in the $\otimes$.)

Now if we divide by

$$F_X^{n-i+1}Gr_{\mathcal{L}}^{i-1}J^n(X\times\eta_S) := \mathrm{im}\left\{\underline{H}^{i-1}(\eta_S,\mathbb{C})\otimes F^{n-i+1}H^{2n-i}(X,\mathbb{C})\to Gr_{\mathcal{L}}^{i-1}J^n(X\times\eta_S)\right\}$$

$$\cong \mathrm{im}\left\{\underline{H}^{i-1}(\eta_S,\mathbb{Q})\otimes F^{n-i+1}H^{2n-i}(X,\mathbb{C})\to Gr_{\mathcal{L}}^{i-1}J^n(X\times\eta_S)\right\}$$

then both sums (over j and k) get swallowed, since both j, $k\leq i-1$ (in all nonzero terms). Therefore

$$Gr_{\mathcal{L}}^{i-1}J^n(X\times\eta_S)\twoheadrightarrow Gr_{\mathcal{L}}^{i-1}J^n(X\times\eta_S)\,/F_X^{n-i+1}\{\mathrm{num}\}$$

$$=\frac{\underline{H}^{i-1}(\eta_S,\mathbb{Q})\otimes H^{2n-i}(X,\mathbb{C})}{\underline{H}^{i-1}(\eta_S,\mathbb{Q})\otimes F^{n-i+1}H^{2n-i}(X,\mathbb{C})\;+\;\underline{H}^{i-1}(\eta_S,\mathbb{Q})\otimes H^{2n-i}(X,\mathbb{Q})}$$

$$\cong \hom\left(\underline{H}_{i-1}(\eta_S,\mathbb{Q}),\;\frac{H^{2n-i}(X,\mathbb{C})}{F^{n-i+1}H^{2n-i}(X,\mathbb{C})+H^{2n-i}(X,\mathbb{Q})}\right)$$

$$=\hom\left(\underline{H}_{i-1}(\eta_S,\mathbb{Q}),\;\left(F^iH^i(X,\mathbb{C})\right)^\vee/\mathrm{im}H_i(X,\mathbb{Q})\right)$$

$$=:\underline{H}^{i-1}(\eta_S,\,V\,/\Lambda).$$

5.3.2. "Reduced" higher Abel-Jacobi maps for 0-cycles on a smooth projective variety/$\mathbb{Q}$. Let

$$\Lambda(i)_{\mathcal{D}}:=\Lambda\hookrightarrow\mathcal{O}_S\otimes V\to\Omega_S^1\otimes V\to\ldots\to\Omega_S^{i-1}\otimes V\to 0$$

and define

$$H_{\mathcal{D}}^*(S,\Lambda(i)):=\mathbb{H}^*(S,\Lambda(i)_{\mathcal{D}}).$$

If we put

$$\underline{H}_{\mathcal{D}}^i(\eta_S,\Lambda(i)):=\mathrm{im}\left\{H_{\mathcal{D}}^i(S,\Lambda(i))\to H_{\mathcal{D}}^i(\eta_S,\Lambda(i))\right\},$$

some algebra shows the top row of

$$\begin{tikzcd}
0\to\underline{H}_{\mathcal{D}}^{i-1}(\eta_S,V/\Lambda) \arrow[r,"\nu"] & \underline{H}_{\mathcal{D}}^i(\eta_S,\Lambda(i)) \arrow[r,"\eta"] & F^i\underline{H}^i(\eta_S,V)\cap\underline{H}^i(\eta_S,\Lambda)\to 0 \\
0\to Gr_{\mathcal{L}}^{i-1}J^n(X\times\eta_S) \arrow[u,"q"] \arrow[r,"\nu"] & Gr_{\mathcal{L}}^i\underline{H}_{\mathcal{H}}^{2n}(X\times\eta_S,\mathbb{Q}(n)) \arrow[r,"\eta"] & Gr_{\mathcal{L}}^iHg^n(X\times\eta_S)\to 0 \arrow[u,"p"]
\end{tikzcd}$$

is exact. The two extension classes could still be different, though it seems unlikely, but we can get around this anyway. If $\mathcal{L}^iZ^n(X(k))$ denotes cycles with class in $\mathcal{L}^iCH^n(X(k))$, then below we will define (once again spreading $\mathcal{Z}\mapsto\zeta$)

$$\chi_i:\;\mathcal{L}^iZ^n(X(k))\longrightarrow\underline{H}_{\mathcal{D}}^i(\eta_S,\Lambda(i))$$

as a holomorphic differential character. This gives rise to invariants $\overline{[\zeta]}_i := \eta(\chi_i(\mathcal{Z}))$, and *if this is zero* $\chi_i(\mathcal{Z}) =: \nu(\overline{[AJ\zeta]_{i-1}})$. Recall that we have mapps

$$\Psi_i : \mathcal{L}^i Z^n(X(k)) \longrightarrow Gr^i_{\mathcal{L}} \underline{H}^{2n}_{\mathcal{H}}(X \times \eta_S, \mathbb{Q}(n))$$

which kill $\mathcal{L}^{i+1}$ by definition (and in particular this includes all $\mathcal{Z} \underset{\text{rat}}{\equiv} 0$). The one can show that $p[\zeta]_i = \overline{[\zeta]_i}$, and *if* $[\zeta]_i = 0$ (a stronger condition than $\overline{[\zeta]_i} = 0$), $q[AJ\zeta]_{i-1} = \overline{[AJ\zeta]_{i-1}}$. This is enough to show that χ_i kills $\mathcal{L}^{i+1}$ (and $\therefore$ in particular is defined on $\mathcal{L}^i CH^n(X(k))$), since $[\zeta]_i = [AJ\zeta]_{i-1} = 0 \implies \overline{[\zeta]_i} = \overline{[AJ\zeta]_{i-1}} = 0$.

We now define for each $\mathcal{Z} \in \mathcal{L}^i Z^n(X(k))$ a holomorphic V/Λ-valued differential character $\chi_i(\mathcal{Z})$. This must be defined on all topological $(i-1)$-cycles $\mathcal{C}$ on $S \setminus D$ for some divisor $D/\mathbb{Q}$ in the limit (we will explain why later). Since $\mathcal{Z} \in \mathcal{L}^i$, its spread has $[\zeta]_{i-1} = 0$, in particular as an element of $\hom_{\mathbb{Q}}\left(\underline{H}_{i-1}(\eta_S, \mathbb{Q}),\ H_{i-1}(X, \mathbb{Q})\right)$. Therefore $\pi_\zeta(\rho_\zeta^{-1}(\mathcal{C}))$ has trivial class in $H_{i-1}(X, \mathbb{Q})$, and so may be written $\partial\Gamma$ (for Γ defined up to an i-cycle on X). If $\{\omega_\ell\}$ is a basis for $F^i H^i(X, \mathbb{C})$ then the integrals

$$\chi_i(\mathcal{Z})\mathcal{C} := \left\{ \int_\Gamma \omega_\ell \right\}$$

give a "value" in V/Λ. If $\mathcal{C} = \partial\mathcal{K}$ on S then (again modulo i-cycles on X) $\Gamma = \pi_\zeta(\rho_\zeta^{-1}(\mathcal{K}))$ and so

$$\chi_i(\mathcal{Z})\partial\mathcal{K} \underset{\text{mod } \Lambda}{\equiv} \left\{ \int_{\mathcal{K}} \rho_{\zeta *} \pi_\zeta{}^* \omega_\ell \right\}$$

amounts to integrals of holomorphic i-forms (or *a* V-valued holomorphic i-form) over $\mathcal{K}$. Thus $\chi_i(\mathcal{Z})$ is a differential character and (slightly extending a standard fact, see **[Ga]**) gives an element of $\underline{H}^i_{\mathcal{D}}(\eta_S, \Lambda(i))$, as promised.

Formulas for $\overline{[\zeta]_i}$ and $\overline{[AJ\zeta]_{i-1}}$ come out of this construction; the minor miracle here is that $\overline{[AJ\zeta]_{i-1}}$ is defined "before" $[AJ\zeta]_{i-1}$ is. Although $\overline{[\zeta]_i} = 0 \;\not\Longrightarrow\; [\zeta]_i = 0$ (and so $[AJ\zeta]_{i-1}$ is not yet defined), the fact that $\chi_i(\mathcal{Z})$ is a well-defined element of $\underline{H}^i_{\mathcal{D}}(\eta_S, \Lambda(i))$ *does* mean that $\overline{[\zeta]_i} = 0$ is enough to pull $\chi_i(\mathcal{Z})$ back to define $\overline{[AJ\zeta]_{i-1}}$. Now $\overline{[\zeta]_i}$ is computed by the "V-valued holomorphic i-form" $\{\rho_{\zeta *}\pi_\zeta{}^*\omega_\ell\}$ or in more down-to-earth language $\overline{[\zeta]_i} =$

$$\sum_\ell [\omega_\ell]^\vee \otimes [\rho_{\zeta *}\pi_\zeta{}^*\omega_\ell] \in \hom_{\mathbb{C}}\left(F^i H^i(X, \mathbb{C}),\ F^i \underline{H}^i(\eta_S, \mathbb{C})\right) \cap \operatorname{im}\{\hom_{\mathbb{Q}}\}.$$

Vanishing of this invariant does *not* mean $[\zeta]_i$ is zero as an element of $\hom_{\mathbb{Q}}(\underline{H}_i(\eta_S, \mathbb{Q}),\ H_i(X, \mathbb{Q}))$; it merely means that the classes $[\rho_{\zeta *}\pi_\zeta{}^*\omega_\ell] = 0$, which by easy Hodge theory $\implies$ the forms $\rho_{\zeta *}\pi_\zeta{}^*\omega_\ell = 0$ exactly. This has the effect of making the differential character pull up to a cohomology class, since if $\mathcal{C} = \partial\mathcal{K}$ on S ($\iff [\mathcal{C}] = 0$ in $\underline{H}_{i-1}(\eta_S, \mathbb{Q})$) the integrals over

$\mathcal{K}$ are zero. Therefore in $\overline{[\zeta]}_i = 0$ the integrals $\left\{\int_\Gamma \omega_i\right\}$ yield a cohomology class

$$\overline{[AJ\zeta]_{i-1}} \in \hom\left(\underline{H}_{i-1}(\eta_S,\mathbb{Q}),\ \frac{V}{\Lambda} = \frac{(F^i H^i(X,\mathbb{C}))^\vee}{\mathrm{im} H_i(X,\mathbb{Q})}\right).$$

One slight problem is that we may have $\dim_{\mathbb{Q}} \Lambda > \dim_{\mathbb{R}} V$, which makes it difficult to tell when $\overline{[AJ\zeta]_{i-1}} \neq 0$ in practice – there are "too many periods". But in return for this inconvenience, it is easy to set up a situation in which $\overline{[\zeta]}_i = 0$: namely, $\dim S\,(= \mathrm{trdeg}(k/\mathbb{Q})) < i$. We also remark that everything we have done up to this point should work (with different numbers) for codimension $p\,(< n)$ cycles $\mathcal{Z}$ on $X(k)$ as well.

In summary, we have two invariants for zero-cycles $\mathcal{Z} \in \mathcal{L}^i CH^n(X(k))$, which we write

$$\Psi_i : Gr^i_{\mathcal{L}} CH^n(X(k)) \overset{(\otimes\mathbb{Q})}{\hookrightarrow} Gr^i_{\mathcal{L}} \underline{H}^{2n}_{\mathcal{H}}(X \times \eta_S, \mathbb{Q}(n))$$

and

$$\chi_i : Gr^i_{\mathcal{L}} CH^n(X(k)) \longrightarrow \underline{H}^i_{\mathcal{D}}(\eta_S, \Lambda(i)),$$

where both maps are well-defined (kill $\mathcal{L}^{i+1}$) but the bottom one is *not* injective: $\ker(\chi_i) \supsetneq \mathcal{L}^{i+1} CH^n$ in general. As for invariants into which these "split", $\overline{[\zeta]}_i = p[\zeta]_i$, and $\overline{[AJ\zeta]_{i-1}} = q[AJ\zeta]_{i-1}$ when both are defined ($[\zeta]_i = 0$). However we will be interested below in a situation where $\overline{[\zeta]}_i = 0$ (but possibly $[\zeta]_i \neq 0$), and where we can show $\overline{[AJ\zeta]_{i-1}} \neq 0$, so that $\mathcal{Z} \underset{\text{rat}}{\not\equiv} 0$.

[This is meant to be analogous to Milnor regulator currents $R_{\mathbf{f}}$, $\{\mathbf{f}\} \in K_n^M$, on $(n-1)$-dimensional varieties (so that $\Omega_{\mathbf{f}} = 0$).] We emphasize that arguing $\mathcal{Z} \underset{\text{rat}}{\not\equiv} 0$ in this way, does *not* rely on BBC^q.

5.3.3. The situation for products of curves. In fact, we now specialize to $X = C_1 \times \ldots \times C_n/\mathbb{Q}$ for two reasons.

The first is that in this case, $\mathcal{L}^i CH^n(X(k))$ is easily described via the product structure. (This is necessary because we cannot, e.g. for $\mathcal{Z} \in \mathcal{L}^{i-1}$, use $\chi_i(\mathcal{Z}) = 0$ to push $\mathcal{Z}$ into $\mathcal{L}^i$; we need some other means for obtaining elements in $\mathcal{L}^i$.) As in §5.1 (for the relative varieties) one uses the projections

$$\pi^i_\sigma : X \longrightarrow C_{\sigma_i(1)} \times \ldots \times C_{\sigma_i(i)} =: X_{\sigma_i}$$

to induce isomorphisms

$$H^i(X,\mathbb{Q}) \underset{\cong}{\overset{\oplus \pi^{i\,*}_\sigma}{\longleftarrow}} \oplus_{\sigma_i} H^i(X_{\sigma_i},\mathbb{Q})$$

$$H^{2n-i}(X,\mathbb{Q}) \underset{\cong}{\longrightarrow} \oplus_{\sigma_i} H^i(X_{\sigma_i},\mathbb{Q})$$

and (together with the formulas for the $Gr^*_{\mathcal{L}} \underline{H}^*_{\mathcal{H}}$) a diagram

$$\begin{tikzcd}
\mathcal{L}^i CH^n(X(k)) \arrow[r, "\oplus[\pi^i_\sigma]_*"] \arrow[d, "\Psi_i(n)"] & \oplus_{\sigma_i}\mathcal{L}^i CH^i(X_{\sigma_i}(k)) \arrow[d, hook, "\oplus\Psi_i(i)", "\mathrm{BBC}^q"'] \\
Gr^i_{\mathcal{L}}\underline{H}^{2n}_{\mathcal{H}}(X\times\eta_S,\mathbb{Q}(n)) \arrow[r, "\cong"] & \oplus_{\sigma_i}Gr^i_{\mathcal{L}}\underline{H}^{2i}_{\mathcal{H}}(X_{\sigma_i}\times\eta_S,\mathbb{Q}(i)).
\end{tikzcd}$$

Setting

$$F^i_\times CH^n(X(k)) := \ker(\oplus[\pi^i_\sigma]_*) \subseteq CH^n(X(k))\,,$$

in particular

$$F^n_\times CH^n(X(k)) := \left\{ \begin{array}{c} \text{0-cycles whose projections to} \\ \text{products of } n-1 \text{ curves, are } \underset{\text{rat}}{\equiv} 0 \end{array} \right\} \Big/ \underset{\text{rat}}{\equiv}\ ,$$

one has from this diagram

$$F^i_\times CH^n(X(k)) \subseteq \mathcal{L}^i CH^n(X(k))$$

and equality if BBC^q holds (or without BBC^q for $i \leq 2$). (Here the $\mathcal{L}^i$ are defined, as usual, via kernels of successive Ψ_i; note that $\mathrm{BBC}^q \implies \mathcal{L}^{n+1}CH^n = 0$, while $F^{n+1}_\times CH^n = 0$ by definition.) It is easily shown by an argument like that in §5.2, that $F^n_\times CH^n(X(k))$ is spanned by cycles of the form

$$\mathcal{Z}_{\mathbf{a}} = \mathcal{Z}_{(a_1,\ldots,a_n)} := (a_1,\ldots,a_n) - \sum_j (a_1,\ldots,p_j,\ldots,a_n) +$$

$$\sum_{j_1<j_2}(a_1,\ldots,p_{j_1},\ldots,p_{j_2},\ldots,a_n) \;-\; \ldots \;+\; (-1)^n\cdot(p_1,\ldots,p_n)$$

for all $\{a_j\} \in C_j(k)$, where $\{p_j\} \in C_j(\bar{\mathbb{Q}})$ are fixed base points. So such cycles are *in* $\mathcal{L}^n CH^n(X(k))$ (and if BBC^q holds, they span it); therefore we may in principle compute $\overline{[\zeta]_n}$, $\overline{[AJ\zeta]_{n-1}}$ for these "n-boxes".

The second reason for specializing to products of curves, is that we may compute these invariants, in particular $\overline{[AJ\zeta]_{n-1}}$, as ($V/\Lambda$-valued collections of) $(n-1)$-currents on the base S. The idea is to extend the "standard homotopy θ" of §1.3 from $(\mathbb{C}^*)^n$ to $C_1\times\ldots\times C_n$ using the fact (as with $\mathbb{C}^*$) that eaxg C_j can be cut into a fundamental domain. One then applies this $'\theta$ to the spread of $\mathcal{Z}_{\mathbf{a}}$ (in exactly the same way as θ was applied to n-box-graphs in §1.4.5) to standardize the chains Γ used in the differential character construction above. (We will not produce $'\theta$ explicitly but it will hopefully be clear from our construction of the $(n-1)$-currents below, how to do so.) So we can push χ_n down to a collection of $(n-1)$-currents on S, one for each

$$\omega_\ell \in F^n H^n(C_1\times\ldots\times C_n) \cong H^0(\Omega^1_{C_1})\otimes\ldots\otimes H^0(\Omega^1_{C_n})\,,$$

which we write by abuse of notation $\overline{[AJ\zeta]_{n-1}}$. The fact that χ_n is a holomorphic differential character is expressed by

$$\mathrm{d}\overline{[AJ\zeta]_{n-1}} = \overline{[\zeta]_n} + T_\zeta$$

where T_ζ is a {topological $(2t-n+1)$-chain}$\otimes\Lambda$ giving a Λ-valued current. If $\overline{[\zeta]_n} = 0$ (e.g. if $t = \dim S < n$), then $\overline{[AJ\zeta]_{n-1}}$ no longer abuses notation and gives a cohomology class

$$\overline{[AJ\zeta]_{n-1}} \in \underline{H}^{n-1}\left(\eta_S, \frac{\{F^n H^n(C_1 \times \ldots \times C_n)\}^\vee}{\text{periods}}\right).$$

As a point of reference to Chapters 1 and 2, $\{F^n H^n(\mathbb{C}^* \times \ldots \times \mathbb{C}^*)\}^\vee/\text{periods} = \mathbb{C}/\mathbb{Z}(n)$.

Now to finish a point left open above, one obtains this cohomology class by integrating over topological $(n-1)$-cycles $\mathcal{C}$, which can be moved (in their class on S) to avoid an unspecified set of divisors $D/\mathbb{Q}$. (This is the leaning of the limit; the set of such cycles is just the group $\underline{H}_{n-1}(\eta_S, \mathbb{Q})$ from above.) From the perspective of **[GG5]**, this is so that the class $[AJ\zeta]_{n-1}$ is unaffected by any modifications of the full spread $\tilde{\zeta}$ (on $X \times S$) over any $D/\mathbb{Q}$. Algebraically,

$$\underline{H}^{n-1}(\eta_S, V/\Lambda) \cong_{\text{nc}} \operatorname{coim}\left\{H^{n-1}(S, V/\Lambda) \to H^{n-1}(\eta_S, V/\Lambda)\right\}$$

$$\cong \varinjlim_{D \subset S} \left\{\frac{\hom(H_{n-1}(S,\mathbb{Q}),\ V/\Lambda)}{\operatorname{im}\{\hom(H_{n-3}(D,\mathbb{Q}),\ V/\Lambda)\}}\right\}$$

where the image is computed by $\cap D : H_{n-1}(S) \to H_{n-3}(D)$. If S is a curve the $\underline{H}_1(\eta_S, \mathbb{Q}) \cong H_1(S, \mathbb{Q})$ while if[16] $S = C_1 \times C_2$ the $\underline{H}_2(\eta_S, \mathbb{Q}) \cong H_1(C_1) \otimes H_1(C_2)$. So there are plenty such cycles.

An analogy (because of the lack of residues along D) between $\overline{[AJ\zeta]_{n-1}}$ and "holomorphic" Milnor regulator currents (coming from $\{\mathbf{f}\} \in K_n^M(S)$), is certainly apt. Fortunately, however, nothing like the vanishing theorem could hold because all S's in the present context are defined over $\mathbb{Q}$ (and $\therefore$ not general).

5.3.4. Formulas for the invariants and a concrete example. We now show how to write down general formulas for the currents representing $\overline{[AJ\zeta]_{n-1}}$, first for $n = 3$. Let $C_{i=1,2,3}$ be curves defined over $\mathbb{Q}$, and p, q, r be points in the respective $C_i(k)$, $k \subset \mathbb{C}$. Writing 0 for all three base points/$\mathbb{Q}$, consider the "3-box" cycle

$$\mathcal{Z} = \mathcal{Z}_{(p,q,r)} = (p,q,r) - (0,q,r) - (p,0,r) - (p,q,0) + (0,0,r) + (0,q,0) + (p,0,0) - (0,0,0)$$

in $Z_0((C_1 \times C_2 \times C_3)(k))$ with class $[\mathcal{Z}] \in \mathcal{L}^3 CH^3((C_1 \times C_2 \times C_3)(k))$, by the above discussion. As usual we will consider its spread ζ to η_S, for S some projective variety $/\mathbb{Q}$ with $\mathbb{Q}(S) \cong k$.

[16]as might occur if the cycle being spread involved general points on each of two curves in a product $X = C_1 \times C_2 \times C_3$.

Let $\alpha_i^{j=1,\cdots,2g_i}$ (with g_i the genus) be topological cycles on C_i spanning $H_1(C_i,\mathbb{Q})$ and all based at 0. We take $\hat{\alpha}_i^j$ to be a dual basis of "markings" avoiding 0, and based at another point which the α_i^j avoid. Things are set up as usual so that $[\hat{\alpha}_i^j]=[\alpha_i^{j+g_i}]$ (but $\hat{\alpha}_i^j \neq \alpha_i^{j+g_i}$), and $\alpha_i^{j_1}\cap\alpha_i^{j_2}$ is empty unless $j_1=j_2$ (and then it is a point). The interior of our fundamental domain (with "center" at 0) is then given by $C_i \setminus \cup_j \hat{\alpha}_i^j$, and we denote its closure by $\mathbb{D}_i$.

The forms $\{\omega_i^{k=1,\cdots,g_i}\}\in\Omega^1(C_i)$ integrate to give functions z_i^k on $\mathbb{D}_i$ which are zero at 0. They may be viewed as discontinuous, complex-valued functions on C_i with cuts at the markings, i.e. as 0-currents like $\log z$ on $\mathbb{C}^*$, so that

$$\mathrm{d}[z_i^k]=\omega_i^k-\sum_{j=1}^{2g_i}\omega_i^k(\alpha_i^j)\cdot\delta_{\hat{\alpha}_i^j}$$

where $\omega(\alpha):=\int_\alpha \omega$. Now the spreads of the points p, q, r give maps $s_i : S\to C_i$, and by composition with the z_i^k, zero-currents $f_i^k := z_i^k\circ s_i \in {}'\mathcal{D}^0(S)$, or functions on S with branch cuts $T_i^j=s_i^{-1}(\hat{\alpha}_i^j)$. We have

$$\mathrm{d}[f_i^k]=df_i^k-\sum_{j=1}^{2g_i}\omega_i^k(\alpha_i^j)\cdot\delta_{T_i^j}$$

where $df_i^k := s_i^*\omega_i^k$. (This is of course the analogue of the formula $\mathrm{d}[\log f]=\mathrm{dlog}f-2\pi i\cdot\delta_{T_f}$.)

Provided $\overline{[\zeta]_3}=0$ (see below) one then has for $\overline{[AJ\zeta]_2}$ the vector-valued current

$$f_1^{k_1}df_2^{k_2}\wedge df_3^{k_3}\;+\;\sum_{j_1=1}^{2g_1}(\omega_1^{k_1}(\alpha_1^{j_1}))f_2^{k_2}df_3^{k_3}\cdot\delta_{T_1^{j_1}}\;+\;\sum_{j_1,j_2}(\omega_1^{k_1}(\alpha_1^{j_1}))(\omega_2^{k_2}(\alpha_2^{j_2}))f_3^{k_3}\cdot\delta_{T_1^{j_1}\cap T_2^{j_2}}$$

which gives a well-defined class in $\underline{H}^2(\eta_S,\{F^3H^3(C_1\times C_2\times C_3,\mathbb{C})\}^\vee/\Lambda)\cong \hom(\underline{H}_2(\eta_S,\mathbb{Q}),\,V/\Lambda)$. (The "vectors" here consist of $g_1\times g_2\times g_3$ entries, each of which corresponds to a choice of k_1, k_2, k_3.) Notice that d of this current is a lattice-valued current (trivial) plus $df_1^{k_1}\wedge df_2^{k_2}\wedge df_3^{k_3}$, its "infinitesimal invariant" or $\overline{[\zeta]_3}$; if this is not zero the "AJ" current gives a differential character instead of a cohomology class, but we still write $\overline{[AJ\zeta]_2}$ anyway.

Similarly, for $\mathcal{Z}_{(p,q)}\in Z_0((C_1\times C_2)(k))$ one has for $\overline{[AJ\zeta]_1}$

$$[f_1^{k_1}df_2^{k_2}\;+\;\sum_{j_1=1}^{2g_1}(\omega_1^{k_1}(\alpha_1^{j_1}))f_2^{k_2}\cdot\delta_{T_1^{j_1}}]\in\underline{H}^1(\eta_S,\{F^2H^2(C_1\times C_2,\mathbb{C})\}^\vee/\Lambda)$$

provided the class of $df_1^{k_1}\wedge df_2^{k_2}$ vanishes, as for instance in the case where $\dim S=\mathrm{trdeg}(k/\mathbb{Q})=1$. We now give an example of such a case, where $\overline{[AJ\zeta]_1}\neq 0$.

We mentioned in §5.3.2 that $\dim_{\mathbb{Q}} \Lambda$ might be larger than $\dim_{\mathbb{R}}\{F^2H^2(C_1 \times C_2, \mathbb{C})\}^\vee$; one situation where this does *not* happen is for $C_1 = C_2 = E$ an elliptic curve with complex multiplication. In particular, take $E = \{y^2 = x^3 - 5x\}$ with $\omega = dx/y \in \Omega^1(E)$ and topological 1-cycles α, β; for the 0-currents resulting from the integrals $\int_0 \omega$ on $\mathbb{D}$ we will write simply z_1, z_2. (These are just the standard "plane coordinates" on the factors $E \times E$.) For the periods on E we write $\Omega_1 = \Omega = \int_\alpha \omega$ and $\Omega_2 = i\Omega = \int_\beta \omega$ (where $\Omega \in \mathbb{C}$ is some transcendental number), so that $\int_{\alpha\times\beta} dz_1 \wedge dz_2 = (i\Omega)\Omega = i\Omega^2$, $\int_{\beta\times\beta} dz_1 \wedge dz_2 = (i\Omega)(i\Omega) = -\Omega^2$, $\int_{\alpha\times\alpha} dz_1 \wedge dz_2 = \Omega^2$ give the period lattice $\langle \Omega^2, i\Omega^2 \rangle = \Lambda \subset V = F^2H^2(E \times E)^\vee = \langle dz_1 \wedge dz_2 \rangle^\vee = \mathbb{C}$.

Now take a base point 0 defined over $\mathbb{Q}$, and a general point p defined over $k \cong \mathbb{Q}(E)$ (so that S and E will be birationally equivalent). There also exists a nontorsion rational point ξ, which means that $\int_0^\xi \omega$ is not torsion[17] in $\mathbb{C}/\langle \Omega, i\Omega \rangle_{\mathbb{Z}}$. We study the cycle

$$\mathcal{Z} = \mathcal{Z}_{(p,p-\xi)} = (p,\, p-\xi) - (0,\, p-\xi) - (p,\, 0) + (0,\, 0),$$

with class $[\mathcal{Z}] \in \mathcal{L}^2CH^2((E\times E)(k))$ and spread $\tilde{\zeta} \subseteq E \times E \times E$, and $\overline{[\zeta]_2} = 0$ (since $\dim S = 1$).

The spreads of p and $p - \xi$ give rise to maps $(s_1, s_2) = (id,\, id - \xi) : S \simeq E \to E \times E$. We choose our cuts $\hat{\alpha}$, $\hat{\beta}$ so that $z = \int_0 \omega$ takes values in the square $-\Omega/2 \le \Im(z) \le \Omega/2$, $-\Omega/2 \le \Re(z) \le \Omega/2$ centered at 0, and pick α, β to have support along the real and imaginary axes, respectively. Composing $(id, id - \xi)$ with z gives zero-currents f and g on $S \simeq E$, with cuts T_f^α, T_f^β (and T_g^α, T_g^β). Using df, dg (both $= dz$) for the pullbacks of ω (by s_1, s_2), we can now write down the basic 1-current

$$\overline{[AJ\zeta]_1} = [f\,dg + \Omega\, g \cdot \delta_{T_f^\alpha} + i\Omega\, g \cdot \delta_{T_f^\beta}] \in \hom\left(\underline{H}_1(\eta_E),\, \mathbb{C}/\langle \Omega^2, i\Omega^2 \rangle\right).$$

Now in fact, since $f = z \circ id$, $T_f^\alpha = \hat{\alpha}$ and $T_f^\beta = \hat{\beta}$. So integrating the current term by term over $\alpha \in H_1(E) \cong \underline{H}_1(\eta_E)$, we have

$$\int_\alpha f\,dg \;=\; \int_\alpha z\,dz \;=\; \int_{-\Omega/2}^{\Omega/2} z\,dz \;= 0,$$

$$\sum_{p \in T_f^\alpha \cap \alpha} \Omega \cdot g(p) \;=\; \sum_{p \in \hat{\alpha} \cap \alpha} \Omega \cdot g(p) \;=\; \Omega \cdot g(\pm\Omega/2) \;=\; \pm\frac{\Omega^2}{2} - \xi\Omega,$$

$$\sum_{p \in T_f^\beta \cap \alpha} i\Omega \cdot g(p) \;=\; 0 \;\text{ since }\; \hat{\beta} \cap \alpha = \emptyset \;\text{ (they are parallel).}$$

The only nonzero entry is nontorsion in $\mathbb{C}/\langle \Omega^2, i\Omega^2 \rangle_{\mathbb{Z}}$, which is to say nonzero in V/Λ, and so $0 \ne \overline{[AJ\zeta]_1}$, which $\implies$ $0 \ne [\mathcal{Z}] \in \mathcal{L}^2CH^2((E \times E)(k))$. So we have just used basic calculus to show a cycle $\mathcal{Z}_{(p,p-\xi)} \in$

[17] or nonzero in $\mathbb{C}/\langle \Omega, i\Omega \rangle$ (unless otherwise indicated all lattices are $\mathbb{Q}$-lattices).

$\ker(Alb)$ is not rationally equivalent to zero.[18] For purposes of comparison (and this is easy to show explicitly by solving a diagram as in §5.2.2), the cycle $2\mathcal{Z}_{(p,p)}$ *is* rationally equivalent to zero.

Though it follows from the theory we have developed above, we wish to emphasize that there is nothing conjectural about this argument. If $\mathcal{Z}$ were rationally equivalent to 0 then its spread ζ over η_S would also be $\underset{\text{rat}}{\equiv} 0$, and so one could not have a concrete piece of the AJ-map coming up nonzero. In fact, it's possible to see by a direct geometric argument (using a homotopy $'\theta$) that integrating the above current over α is the same as taking $\zeta = \partial\Gamma$ (on $X \times \eta_S$) and integrating a cohomology class in $H^3(X \times \eta_S)$ over Γ. But Γ becomes much harder to find (at least directly) for even the next example, that of $\mathcal{Z} = \mathcal{Z}_{(p,q-p,q-\xi)}$ on $X = E \times E \times E$, say, for q and p algebraically independent general points on $E(\mathbb{C})$. Yet the current constructed for the $n = 3$ case remains easy to compute with, and one can easily show $\overline{[AJ\zeta]_2} \neq 0$.

The next step would be to apply this to a more interesting X, perhaps a $K3$ by viewing it as an elliptic fibration. One wants a more general construction for pushing AJ down to currents on η_S, a formula valid for more than just X =products, and in principle this should be possible. Also the question arises, of how to compute the remaining portions of Ψ_i; it would be interesting to see what form a cycle with all $\chi_i(\mathcal{Z}) = 0$, but say $\Psi_n(\mathcal{Z}) \neq 0$, would take.

[18]The $n = 2$ example just given (but not the $n = 3$ one below) appears to be covered by the very general result regarding 0-cycles on products of two curves in **[RS]**. (That result generalizes Nori's examples, which exclude CM curves and thus do not include our cycle.) However, we feel that this very concrete approach to computing (quotients of) the higher Ψ_i is in itself an attractive alternative to the more abstract theory of M. Saito.

Appendix: An Elementary Proof of Suslin Reciprocity

Let X be a compact Riemann Surface. We define abelian groups

$$K_2(\mathbb{C}(X)) := \frac{\mathbb{C}(X)^* \bigwedge_{\mathbb{Z}} \mathbb{C}(X)^*}{\langle f \wedge (1-f) \rangle},$$

with elements written as products of "symbols" $\prod_\alpha \{f_\alpha, g_\alpha\}$. The "$\bigwedge_{\mathbb{Z}}$" means that (i) $\{f,g\} = \{g,f\}^{-1}$ and (ii) $\{f^n, g\} = \{f, g^n\} = \{f,g\}^n$ ("multiplicative bilinearity" – this is the "$\mathbb{Z}$"). We also have (iii) $\{f, 1-f\} = 1$; these (sometimes together with (i) and (ii)) are called the "Steinberg relations", and the notation above means that we quotient out by the ideal they generate. Similarly set

$$K_2(\mathbb{C}) := \frac{\mathbb{C}^* \bigwedge_{\mathbb{Z}} \mathbb{C}^*}{\langle a \wedge (1-a) \rangle}.$$

Now let f, g, $h \in \mathbb{C}(X)^*$ with $h = 1$ on $|(f)| \cup |(g)|$.

Proposition. *(Suslin Reciprocity)*

$$\prod_{p_i \in |(h)|} \{f(p_i), g(p_i)\}^{\nu_{p_i}(h)} = 1 \in K_2(\mathbb{C}),$$

that is, the expression can be rewritten as a product of Steinberg relations (i), (ii), and (iii).

In the following two sections we develop the ideas of regulator and norm on (Milnor) K-theory which are employed in the proof.

I. Regulator

Define a map[19]

$$R_X : K_2(\mathbb{C}(X)) \to \varinjlim_{Z \subset X} H^1(X - Z, \mathbb{C}^*) =: H^1(\eta_X, \mathbb{C}^*)$$

by sending

$$\{f,g\} \longmapsto \left\{ \gamma \in H_1(X - |(f)| \cup |(g)|, \mathbb{Z}) \mapsto e^{\frac{1}{2\pi i} \int_\gamma \log f \,\mathrm{dlog} g - \log g(p_0) \mathrm{dlog} f} \right\},$$

[19]Here η_X is the "generic point" of X: $Z \subseteq Z' \implies X - Z \supseteq X - Z' \implies H^1(X-Z, \mathbb{C}^*) \subseteq H^1(X - Z', \mathbb{C}^*)$, so the direct limit is of course highly nontrivial. Its prettiest strategic side-effect: in checking $R_X\{f,g\} = R_X\{f',g'\}$, we may have the paths *avoid* a finite point set, say $|(f)| \cup |(g)| \cup |(f')| \cup |(g')|$.

where p_0 is the base point from which we continue $\log f$, which is to say it will function as the branch cut for $\log f$ along γ (since this is not a regulator on 1-forms but merely on 1-cycles, the choice of branch of $\log g$ doesn't matter). This map is extended "$\times$-linearly" to products of terms $\prod_\alpha \{f_\alpha, g_\alpha\}$ by using the multiplication induced on $\varinjlim H^1(\eta_X, \mathbb{C}^*)$ (by multiplication in $\mathbb{C}^*$) as its abelian group structure. We now show that it is well-defined. Some facts in this direction:

$\int_\gamma (\dots)$ **is independent of the choice of** $p_0 \in |\gamma|$**, branch of** $\log f$ **and "branch" of** $\log g(p_0)$**.** Indeed, if p_0 and p_1 are two points on γ,

$$\int_{\gamma_{p_0}} \{(\log f)_0 \mathrm{dlog} g - \log g(p_0) \mathrm{dlog} f\} - \int_{\gamma_{p_1}} \{(\log f)_1 \mathrm{dlog} g - \log g(p_1) \mathrm{dlog} f\}$$

$$= -\int_\gamma \mathrm{dlog} f \int_{p_0}^{p_1} \mathrm{dlog} g + [\log g(p_1) - \log g(p_0)] \int_\gamma \mathrm{dlog} f \quad = \quad 0.$$

The first step uses the fact that $(\log f)_0$ and $(\log f)_1$ differ only from p_0 to p_1, where the difference is $-\int_\gamma \mathrm{dlog} f$, and the second follows from the bracketed quantity being equal to $\int_{p_0}^{p_1} \mathrm{dlog} g$.

R(1-f,f)=1. It's sufficient to check the case where $f \sim z^\nu$ locally, and γ encircles 0. In fact we may assume $f = z^\nu$ locally since $f = z^\nu g$, $g(0) = 1 \Longrightarrow 1 - f = (1 - z^\nu)G$, $G(0) = 1$, so that g and G drop out of the integral: in

$$\log[(1 - z^v)G] \mathrm{dlog}(z^\nu g) - \log[z(p_0)^\nu g(p_0)] \mathrm{dlog}[(1 - z^\nu)G],$$

the G, g and $1 - z^\nu, g$ terms integrate to 0 trivially, while

$$\int_\gamma \log G \mathrm{dlog} z^\nu - \log z(p_0)^\nu \mathrm{dlog} G$$

vanishes since $G(0) = 1$. So we are left with

$$\int_\gamma \log(1 - z^\nu) \mathrm{dlog} z^\nu - \log p_0^\nu \mathrm{dlog}(1 - z^\nu),$$

which for $v \geq 0$ again vanishes by the residue theorem. If $v < 0$, replace ν by $-\nu$, write $1 - z^{-\nu} = \frac{z^\nu - 1}{z^\nu}$, and give $\log(1 - z^\nu)$ the corresponding product branch; the integral becomes

$$\int_\gamma \nu \log(z^\nu - 1) \mathrm{dlog} z - \nu^2 \log z \mathrm{dlog} z - \nu \log p_0 \mathrm{dlog}(z^\nu - 1) + \nu^2 \log p_0 \mathrm{dlog} z.$$

To compute this we shall take $radius(\gamma) = \frac{1}{2}$ (not 1, since it must avoid the zero of $1 - f$), and $p_0 = \frac{1}{2}$; then

$$\int_\gamma \text{1st term} = 2\pi i \nu \log(-1) \xrightarrow{e^{(\frac{1}{2\pi i}(\cdot))}} (-1)^\nu,$$

$$\int_\gamma \text{2nd term} = -\nu^2 \frac{1}{2}\int_\gamma d\left[(\log z)^2\right] = -\frac{\nu^2}{2}\left[(2\pi i + \log\frac{1}{2})^2 - (\log\frac{1}{2})^2\right]$$
$$= -\frac{\nu^2}{2} 2\pi i \left[2\log\frac{1}{2} + 2\pi i\right] = 2\pi i\left[\nu^2\log 2 - \nu^2 \pi i\right],$$
$$\int_\gamma \text{3rd term} = 0, \ \int_\gamma \text{4th term} = \nu^2 2\pi i \log\frac{1}{2} = -2\pi i \nu^2 \log 2.$$

And so
$$\int_\gamma \text{last 3 terms} = 2\pi^2\nu^2 \xrightarrow{e^{(\frac{1}{2\pi i}(\cdot))}} (-1)^{\nu^2}.$$

But $(-1)^\nu(-1)^{\nu^2} = (-1)^{\nu(\nu+1)} = 1$, which is what we wanted.

R(f,g)=R(g,f)$^{-1}$. As γ starts and ends at p_0, $\log\frac{g}{g(p_0)}$ and $\log\frac{f}{f(p_0)}$ – which are zero at p_0 – each change by a multiple of $2\pi i$. Hence
$$\int_\gamma d\left[\log\frac{g}{g(p_0)}\log\frac{f}{f(p_0)}\right] \equiv 0\,(mod\,(2\pi i)^2\mathbb{Z}),$$
and so
$$\int_\gamma \log g\,\mathrm{dlog}f - \log f(p_0)\mathrm{dlog}g \equiv -\int_\gamma \log f\,\mathrm{dlog}g - \log g(p_0)\mathrm{dlog}f.$$

Taking $e^{\frac{1}{2\pi i}(\text{each side})}$ gives the result.

R(f′f,g)=R(f′,g)×R(f,g). This is obvious.

So R is well-defined and it makes sense to write $R_X\{f,g\}$, or more generally $R_X\prod_\alpha\{f_\alpha,g_\alpha\}$. Now if this yields 1 (i.e., is trivial) on 1-cycles
$$\gamma \in ker\{\varprojlim{}_{Z\subset X} H_1(X-Z,\mathbb{Z}) \to H_1(X,\mathbb{Z})\}$$
(loops around points), then we say $\prod_\alpha\{f_\alpha,g_\alpha\} \in K_2(X)$. Such elements constitute a *subgroup* of $K_2(\mathbb{C}(X))$, and we have the series of inclusions $K_2(\mathbb{C}(X)) \supseteq K_2(X) \supseteq ker(R_X) \supseteq K_2(\mathbb{C})$. What if R_X is trivial on *all* 1-cycles?

CONJECTURE. *$ker(R_X) = K_2(\mathbb{C})$.*

We prove this for $X = \mathbb{P}^1$. The interplay of (local) analysis and global algebra (on the function field) will show why this is so hard in general (for X of higher genus). We manage to get around this later (for the purposes of the "norm" algorithm) by working with "K_2 of meromorphic functions on branches of X" (since there we are only concerned with the information that the algorithm "commutes" with the local evaluation and regulator maps on K_2). But here we need a real *global* computation (1.6).

First of all, since $H^1(\mathbb{P}^1) = 0$, $K_2(\mathbb{P}^1) = ker(R_{\mathbb{P}^1})$. So we'll prove $K_2(\mathbb{P}^1) = K_2(\mathbb{C})$.

Local Analysis (for all Riemann surfaces X). Let $\beta \in X$ be some point and write $f, g \in \mathbb{C}(X)^*$ locally as $f = (z-\beta)^{\nu_\beta(f)}\tilde{f}$, $g = (z-\beta)^{\nu_\beta(g)}\tilde{g}$. We compute $R\{f,g\}(\gamma_\beta)$, where γ_β is a very small path about β, and pick $p_0 \in X$ so that, in this local parametrization, $p_0 - \beta = 1$. (Note in particular that this $\implies g(p_0) = \tilde{g}(p_0)$.) The integral is

$$\frac{1}{2\pi i}\int_{\gamma_\beta} \{\, \log((z-\beta)^{\nu_\beta(f)}\tilde{f})\mathrm{dlog}((z-\beta)^{\nu_\beta(g)}\tilde{g}) - \log g(p_0)\mathrm{dlog}((z-\beta)^{\nu_\beta(f)}\tilde{f}) \,\}$$

$$= \frac{1}{2\pi i}\int_{\gamma_\beta} \{\, \nu_\beta(g)\log\tilde{f}\mathrm{dlog}(z-\beta) - \nu_\beta(f)\log g(p_0)\mathrm{dlog}(z-\beta)$$

$$+\nu_\beta(f)\log(z-\beta)\mathrm{dlog}\tilde{g} + \nu_\beta(f)\nu_\beta(g)\log(z-\beta)\mathrm{dlog}(z-\beta) + \text{inessential terms} \,\}$$

[now use the residue theorem; also, on third term above use $\int$ by parts to get the last term below; and in the last term above use $p_0 - \beta = 1$ plus integrating $d\{\log^2(z-\beta)\}$ to get the third term below]

$$= \nu_\beta(g)\log\tilde{f}(\beta) - \nu_\beta(f)\log g(p_0) + \nu_\beta(f)\nu_\beta(g)\,\pi i$$

$$+\frac{\nu_\beta(f)}{2\pi i}\int_{\gamma_\beta} \{\, d[\log(z-\beta)\log\tilde{g}] - \log\tilde{g}\mathrm{dlog}(z-\beta) \,\}$$

$$= \nu_\beta(g)\log\tilde{f}(\beta) - \nu_\beta(f)\log\tilde{g}(p_0) + \nu_\beta(f)\nu_\beta(g)\,\pi i + \nu_\beta(f)\log\tilde{g}(p_0) - \nu_\beta(f)\log\tilde{g}(\beta)$$

$$= \nu_\beta(g)\log\tilde{f}(\beta) - \nu_\beta(f)\log\tilde{g}(\beta) + \nu_\beta(f)\nu_\beta(g)\,\pi i.$$

So, taking $e^{(\frac{1}{2\pi i}(\cdot))}$,

$$R\{f,g\}(\gamma_\beta) = \lim_{z\to\beta}(-1)^{\nu_\beta(f)\nu_\beta(g)}\frac{f(z)^{\nu_\beta(g)}}{g(z)^{\nu_\beta(f)}} =: T_\beta\{f,g\},$$

and we call $T_\beta\{f,g\}$ the "tame symbol of f and g (evaluated at β)". Now *Weil reciprocity* says that

$$\prod_{\beta\in X} T_\beta\{f,g\} = 1,$$

i.e. some kind of "global reciprocity" law *always* holds. Our computation implies, on the other hand, that if a pointwise "local reciprocity" $T_\beta\{f,g\} = 1$ holds at β for two functions, then the corresponding K-theory element must have trivial regulator around β. We restate this more generally in the following

PROPOSITION. $\prod_\alpha\{f_\alpha, g_\alpha\} \in K_2(X) \iff$

$$\lim_{z\to\beta}(-1)^{\sum_\alpha \nu_\beta(f_\alpha)\nu_\beta(g_\alpha)}\left(\prod_\alpha \frac{f_\alpha(z)^{\nu_\beta(g_\alpha)}}{g_\alpha(z)^{\nu_\beta(f_\alpha)}}\right) =: T_\beta\prod_\alpha\{f_\alpha, g_\alpha\} = 1 \ (\forall\beta\in X).$$

This holds for all X. What follows does not.

Global Arithmetic in $K_2(\mathbb{C}(\mathbb{P}^1))$. We establish yet another

PROPOSITION. $T_\beta \prod_\alpha \{f_\alpha, g_\alpha\} = 1$ $(\forall \beta \in \mathbb{P}^1) \Leftrightarrow$

$$\prod_\alpha \{f_\alpha, g_\alpha\} \in K_2(\mathbb{C}) \left(\subseteq K_2(\mathbb{C}(\mathbb{P}^1))\right).$$

Combined with the previous result, this will prove $K_2(\mathbb{P}^1) = K_2(\mathbb{C})$.

The implication "$\Leftarrow$" is of course trivial since constants have no poles or zeroes (and so the $\nu_\beta(.)$ are all 0). We shall begin the other direction with a single term

$$\{f, g\} = \left\{ \prod_i (z - a_i)^{m_i}, \prod_i (z - b_j)^{n_j} \right\},$$

where a_i and b_j are all distinct, and the following

LEMMA. $\{z - a, z - b\} = \{z - a, a - b\}\{b - a, z - b\}$.

PROOF. Put $A = z - a$, $B = z - b$, $C = a - b$. We have $B = A + C$, i.e. $1 = \frac{A}{B} + \frac{C}{B}$, which implies by the Steinberg relations that

$$1 = \{\frac{A}{B}, \frac{C}{B}\} = \{A, C\}\{A, B\}^{-1}\{B, C\}^{-1}\{B, B\},$$

and so

$$\{A, B\} = \{A, C\}\{C, B\}\{B, B\}.$$

Now $\{B, B\} = \{B, B\}^{-1} = \frac{\{1-B, B\}}{\{B, B\}} = \{\frac{1-B}{B}, B\} = \{\frac{1}{B} - 1, B\} = \{\frac{1}{B} - 1, \frac{1}{B}\}^{-1} = \{-1, \frac{1}{B}\}^{-1}\{1 - \frac{1}{B}, \frac{1}{B}\}^{-1} = \{-1, B\}$. So

$$\{A, B\} = \{A, C\}\{C, B\}\{-1, B\} = \{A, C\}\{-C, B\},$$

which is the desired equality. $\square$

CASE 4. *Assume one term, f and g monic with $|(f)| \cap |(g)| = \emptyset$ or $\{\infty\}$.* (We are assuming $T_\beta\{f, g\} = 1$ for all $\beta \in \mathbb{P}^1$.)

$$\{f, g\} = \prod_{i,j} \{z - a_i, z - b_j\}^{m_i n_j} = \prod_{i,j} \left(\{z - a_i, a_i - b_j\}, \{b_j - a_i, z - b_j\}\right)^{m_i n_j}$$

$$= \prod_i \{z - a_i, \prod_j (a_i - b_j)^{n_j}\}^{m_i} \Big/ \prod_j \{z - b_j, \prod_i (b_j - a_i)^{m_i}\}^{n_j}$$

$$= \prod_i \{z - a_i, g(a_i)^{m_i}\} \Big/ \prod_j \{z - b_j, f(b_j)^{n_j}\}$$

$$= \prod_i \{z - a_i, 1\} \Big/ \prod_j \{z - b_j, 1\} = 1,$$

where the second-to-last step comes from local reciprocity, since a_i and b_j are distinct. Two quick proofs that $\{A,1\}=1$: either use $\{A,1\}=\{A,1^0\}=\{A,1\}^0=1$ or $\{A,1\}=\frac{\{A,A\}}{\{A,A\}}=1$. Trivially $1\in K_2(\mathbb{C})$ so we're done.

Case 5. *Remove the assumption on divisors.*

Assume, with all a_i, b_j, c_k distinct, that

$$f=\prod_k(z-c_k)^{q_k}\prod_i(z-a_i)^{m_i}\ \textit{and}\ g=\prod_\ell(z-c_\ell)^{r_\ell}\prod(z-b_j)^{n_j}$$

satisfy local reciprocity at each β. Then $\{f,g\}$

$$=\prod_k\left(\{z-c_k,z-c_k\}^{q_kr_k}\prod_{\ell\neq k}\{z-c_k,z-c_\ell\}^{q_kr_\ell}\prod_j\{z-c_k,z-b_j\}^{q_kn_j}\prod_i\{z-a_i,z-c_k\}^{m_ir_k}\right)$$

$$\times\prod_{i,j}\{z-a_i,z-b_j\}^{m_in_j}$$

$$=\prod_k\{z-c_k,-1\}^{q_kr_k}\times\prod_{k,\ell\neq k}\{z-c_k,c_k-c_\ell\}^{q_kr_\ell}\times\prod_{\ell,\,k\neq\ell}\{c_\ell-c_k,z-c_\ell\}^{q_kr_\ell}$$

$$\times\prod_{k,j}(\{z-c_k,c_k-b_j\}\{b_j-c_k,z-b_j\})^{q_kn_j}\times\prod_{k,i}(\{z-a_i,a_i-c_k\}\{c_k-a_i,z-c_k\})^{m_ir_k}$$

$$\times\prod_{i,j}(\{z-a_i,a_i-b_j\}\{b_j-a_i,z-b_j\})^{m_in_j}$$

[now switch k and ℓ in the third factor above]

$$=\prod_k\left\{z-c_k,(-1)^{q_kr_k}\frac{\left(\prod_{\ell\neq k}(c_k-c_\ell)^{r_\ell}\prod_j(c_k-b_j)^{n_j}\right)^{q_k}}{\left(\prod_i(c_k-a_i)^{m_i}\prod_{\ell\neq k}(c_k-c_l)^{q_\ell}\right)^{r_k}}\right\}$$

$$\times\prod_i\left\{z-a_i,\left(\prod_j(a_i-b_j)^{n_j}\prod_k(a_i-c_k)^{r_k}\right)^{m_i}\right\}\times\left(\prod_j\left\{z-b_j,\left(\prod_i(b_j-a_i)^{m_i}\prod_k(b_j-c_k)^{q_k}\right)^{n_j}\right\}\right)^{-1}$$

$$=\prod_k\{z-c_k,1\}\prod_i\{z-a_i,1\}\Big/\prod_j\{z-b_j,1\}\quad=\quad 1\quad\in K_2(\mathbb{C}).$$

Case 6. *Separate $|(f)|$ and $|(g)|$ again but remove the requirement that f and g be monic.*

That is, let $f=\xi\tilde{f}$ and $g=\eta\tilde{g}$, where $\tilde{f}=\prod_i(z-a_i)^{m_i}$ and $\tilde{g}=\prod_j(z-b_j)^{n_j}$. Then

$$\{f,g\}=\{\xi,\eta\}\prod_j\{\xi,z-b_j\}^{n_j}\times\prod_i\{z-a_i,\eta\}^{m_i}\times\prod_{i,j}\{z-a_i,z-b_j\}^{m_in_j}$$

$$= \{\xi,\eta\}\prod_j\{\xi^{n_j}, z-b_j\}\times\prod_i\{z-a_i,\eta^{m_i}\}\times\left(\prod_i\{z-a_i,\tilde{g}(a_i)^{m_i}\}\Big/\prod_j\{z-b_j,\tilde{f}(b_j)^{n_j}\}\right)$$

$$= \{\xi,\eta\}\frac{\prod_i\{z-a_i,(\eta\tilde{g}(a_i))^{m_i}[=1]\}}{\prod_j\{z-b_j,(\xi\tilde{f}(b_j))^{n_j}[=1]\}} \;=\; \{\xi,\eta\}\;\in\; K_2(\mathbb{C}).$$

Combining the Cases. (Remove all assumptions on f and g.)

So we have essentially $f=\xi\prod_i(z-a_i)\prod_k(z-c_k)^{q_k}$ and $g=\eta\prod_j(z-b_j)\prod_\ell(z-c_\ell)^{r_\ell}$. Defining for every $\beta\in X$

$$\tilde{g}_\beta := \frac{g}{(z-\beta)^{\nu_\beta(g)}} \text{ and } \tilde{f}_\beta := \frac{f}{(z-\beta)^{\nu_\beta(f)}},$$

from the previous computations it is clear that

$$\{f,g\} = \left(\prod_{\beta\in|(f)|\cup|(g)|\setminus\infty}\{z-\beta,(-1)^{\nu_\beta(f)\nu_\beta(g)}\frac{\tilde{g}_\beta(\beta)^{\nu_\beta(f)}}{\tilde{f}_\beta(\beta)^{\nu_\beta(g)}}\}\right)\times\{\xi,\eta\}.$$

For a product $\prod_\alpha\{f_\alpha,g_\alpha\}$ we have therefore in $K_2(\mathbb{C}(\mathbb{P}^1))$

$$\prod_{\beta\in|(f_\alpha)|\cup|(g_\alpha)|\setminus\infty}\left\{z-\beta,\prod_\alpha(-1)^{\nu_\beta(f_\alpha)\nu_\beta(g_\alpha)}\frac{\tilde{g}_{\alpha_\beta}(\beta)^{\nu_\beta(f_\alpha)}}{\tilde{f}_{\alpha_\beta}(\beta)^{\nu_\beta(g_\alpha)}}\right\}\times\{\xi_\alpha,\eta_\alpha\}$$

$$= \prod_\alpha\{\xi_\alpha,\eta_\alpha\}\;\in\;K_2(\mathbb{C}),$$

since the big product over α is just $T_\beta\prod_\alpha\{f_\alpha,g_\alpha\}$ ($=1$ by assumption). This completes the proof that $K_2(\mathbb{P}^1)=K_2(\mathbb{C})$.

II. Norm

From the Riemann-Roch theorem follows the existence of a "primitive pair" of meromorphic functions h, $x: X\to\mathbb{P}^1$. What we mean by "primitive" is the following:

(i) Geometrically, they give an embedding $X\hookrightarrow\mathbb{P}^1\times\mathbb{P}^1$ (*not* P^2 – there you get at least normal crossings in general – $\mathbb{P}^1\times\mathbb{P}^1$ has a bit "more" structure, being the compactification of $\mathbb{C}^*\times\mathbb{C}^*$ by four $\mathbb{P}^1$s rather than three). We write (z,w) for coordinates on $\mathbb{P}^1\times\mathbb{P}^1$, and think of $X\xrightarrow{h}\mathbb{P}^1$ as giving a branched covering of the z-sphere. When convenient we write z in lieu of h to denote the function on X (an exception would be "$h^{-1}(z)$").

(ii) Algebraically, they generate the function field: sending $w \mapsto x$ gives an isomorphism $\mathbb{C}(z)[w]/(\Phi(z,w)) \xrightarrow{\cong} \mathbb{C}(X)$, where $\Phi(z,w) = w^n + w^{n-1}\mathcal{R}_1(z) + \ldots + \mathcal{R}_n(z)$ is the minimal polynomial of x, and $\mathcal{R}_i(z) \in \mathbb{C}(z)$ are rational functions. The "graph" of (i) is the solution set $X_\Phi = \{(z,w) \mid \Phi(z,w) = 0\}$ (every Riemann surface is algebraic!). Since $\mathbb{C}(z) \cong \mathbb{C}(\mathbb{P}^1)$, this expresses $\mathbb{C}(X)$ as an extension of $\mathbb{C}(\mathbb{P}^1_z)$.

Galois K_2-norm for splitting field extensions. Preliminary remarks on strategy in the function-field case. We will describe an algorithm similar to the Galois norm which maps $K_2(\mathbb{C}(X)) \to K_2(\mathbb{C}(\mathbb{P}^1))$. Simplifying for a moment to subfields of $\mathbb{C}$, suppose we have a splitting field extension $\mathcal{L}/\mathcal{K}$, $\mathcal{L} = \mathcal{K}(x)$ ($x \in \mathbb{C}$), with $\Phi(x)[= 0]$ the mimimal polynomial of x over $\mathcal{K}$, with roots $\{x = x_1\}, x_2, \ldots, x_n$. Sending $w \mapsto x$ gives an isomorphism $\mathcal{K}[w]/(\Phi(w)) \xrightarrow{\cong} \mathcal{L}$, and so we may write F, $G \in \mathcal{L}$ as $f(x)$, $g(x)$, where $\deg(f(w))$, $\deg(g(w)) < n$. ($F = f(x)$ and $G = g(x)$ are numbers, $f(w)$ and $g(w)$ are polynomials.) Define

$$N_{\mathcal{L}/\mathcal{K}}\{F,G\} := \prod_{i=1}^{n} \{f(x_i), g(x_i)\}.$$

Notice that, while the extension $\mathcal{L}/\mathcal{K}$ has degree n, $(\mathcal{K}[w]/(f(w)))/\mathcal{K} = \mathcal{K}(\sigma)/\mathcal{K}$ and $(\mathcal{K}[w]/(g(w)))/\mathcal{K} = \mathcal{K}(\tau)/\mathcal{K}$ are *lower*-degree extensions *not* contained in $\mathcal{L}$ and with degrees *not* necessarily dividing n. (Here σ and τ are complex numbers satisfying $f(\sigma) = 0$ and $g(\tau) = 0$ – that is, $g(w)$ and $f(w)$ are their minimal polynomials – with conjugates σ_j and τ_k.) So, if we could somehow exchange the role of Φ with that of f and/or g, we could pass from terms $\in K_2(\mathcal{L})$ to terms $\in K_2$(lower degree extensions) (or so I claim). We'll work this out completely in the function field case below.

Passing back to function fields, the *roots* x_i get replaced with the *branches* x_i of x over the z-sphere, which are no longer $\in \mathcal{L} = \mathbb{C}(X)$ (that's the only real difference). [Likewise for σ and τ.] So the computations which follow are not really in $K_2(\mathbb{C}(X))$; they merely constitute an algorithm. However, they are "correct" locally and pointwise (almost everywhere), enough to preserve (commute with) the regulator and $K_2(\mathbb{C})$-evaluation at z, in a sense to be described later.

The norm algorithm. This is based on an idea in [**GG3**]. Let

$$f(z,w) = \prod_{j=1}^{\ell\,(<n)} (w - \sigma_j(z)), \quad g(z,w) = \prod_{k=1}^{m\,(<n)} (w - \tau_k(z))$$

be general (monic, for simplicity) elements $\in \mathbb{C}(z)[x]/(\Phi(z,x))$, and of course

$$\Phi(z,w) = \prod_{i=1}^{n} (w - x_i(z)).$$

(The "functions" σ_j, τ_k, x_i all have branch cuts and so are not meromorphic over the z-sphere.) It's important in what follows that ℓ, $m < n$. Omitting the z-variable[20] we write "$\widetilde{N}_\Phi\{f,g\}$"[21]$:= \prod_i \{f(x_i), g(x_i)\}$

$$= \prod_i \left\{ \prod_j (x_i - \sigma_j), \prod_k (x_i - \tau_k) \right\}$$

[now use the Lemma from **I**]

$$= \prod_{i,j,k} \{x_i - \sigma_j, \sigma_j - \tau_k\}\{\tau_k - \sigma_j, x_i - \tau_k\}$$

$$= \prod_j \left\{ \prod_i (x_i - \sigma_j), \prod_k (\sigma_j - \tau_k) \right\} \times \prod_k \left\{ \prod_j (\tau_k - \sigma_j), \prod_i (x_i - \tau_k) \right\}$$

$$= \prod_j \{(-1)^n \Phi(\sigma_j), g(\sigma_j)\} \times \prod_k \{f(\tau_k), (-1)^n \Phi(\tau_k)\}$$

[now we reduce, e.g. in the first factor, $(-1)^n\Phi(w)$ and $g(w)$ modulo $f(w)$ to get, respectively, $\bar{\Phi}(w)$ and $\bar{g}(w)$, both of degrees $< \ell$. Since $f(\sigma_j) = 0$, $\bar{\Phi}(\sigma_j) = \bar{\Phi}(\sigma_j) + \phi(\sigma_j)f(\sigma_j) = (-1)^n\Phi(\sigma_j)$. Similarly $\bar{g}(\sigma_j) = g(\sigma_j)$.]

$$= \prod_j \{\bar{\Phi}(\sigma_j), \bar{g}(\sigma_j)\} \times \prod_k \{\bar{f}(\tau_k), \bar{\bar{\Phi}}(\tau_k)\}$$

$$= \widetilde{N}_f\{\bar{\Phi}, \bar{g}\} \times \widetilde{N}_g\{\bar{f}, \bar{\bar{\Phi}}\}.$$

These should be thought of as norms on $K_2(\mathbb{C}(X_f))$ and $K_2(\mathbb{C}(X_g))$ relative to the extensions $\mathbb{C}(X_f)/\mathbb{C}(\mathbb{P}^1_z)$ and $\mathbb{C}(X_g)/\mathbb{C}(\mathbb{P}^1_z)$ (rather than $\mathbb{C}(X = X_\Phi)/\mathbb{C}(\mathbb{P}^1_z)$), where e.g. $X_f = \{(z,w) | \, f(z,w) = 0\} \xrightarrow{(x,\sigma)} \mathbb{P}^1 \times \mathbb{P}^1$.[22]

Continuing this process, we reach degree 0 (in w, corresponding to a degree 1 [trivial] field extension of $\mathbb{C}(\mathbb{P}_z)$) so that everything is rational functions of z. Thus we land in $K_2(\mathbb{C}(\mathbb{P}))$, and define by abuse of notation

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[20]writing, for instance, $f(x_i)$ for $f(z, x_i(z))$

[21](really a placeholder, since what follows is not, strictly speaking, a quantity)

[22]X_f and X_g are *not* intermediate in the covering $X \to \mathbb{P}^1_z$. Rather all three are intermediate in some covering $Y \to \mathbb{P}^1_z$. The "dictionary" of meromorphic functions on these Riemann surfaces is:

$X = X_\Phi \leftrightarrow (z = h,\ f,\ g,\ w = x)$

$X_f \leftrightarrow (z = h_f,\ g = \bar{g},\ \Phi = \bar{\Phi},\ w = \sigma)$

$X_g \leftrightarrow (z = h_g,\ f = \bar{f},\ \Phi = \bar{\bar{\Phi}},\ w = \tau)$

$Y \leftrightarrow (z = h_Y,\ w,\ \sigma,\ \tau,\ x)$;$\sigma$, τ, x, together with z, give maps from Y to (the embedded images in $\mathbb{P}^1 \times \mathbb{P}^1$ of) X_f, X_g, X, respectively. The σ_j, τ_k, x_i are just the branches of w on X_f,X_g, X over $\mathbb{P}^1_z$, respectively. On Y one may write the branches of σ as $\sigma_{ijk}(z)$ (resp. τ as $\tau_{ijk}(z)$, x as $x_{ijk}(z)$), where changing i or k (resp. i or j, j or k) has no effect.

"$N_h\{f,g\}$":$= N_\Phi\{f,g\}$:=the element so obtained. So in retrospect, this can formally be seen as a recursive definition of an element in $K_2(\mathbb{C}(\mathbb{P}))$.

Behavior with respect to evaluation and regulator maps. For any Riemann surface Y one may verify[23] that pointwise evaluation $\Theta_Y\{f,g\}(p) :=$

[23] To show this is well defined one needs to prove the following fact:

$$\prod\{A_i,B_i\}^{m_i} = \prod\{'A_j,'B_j\}^{'m_j} \text{ in } K_2(\mathbb{C}(Y)) \text{ and } A_i,\, B_i,\, 'A_j,\, 'B_j \text{ all } \neq 0,\infty \text{ at } p$$

$$\implies \prod\{A_i(p),B_i(p)\}^{m_i} = \prod\{'A_j,'B_j\}^{'m_j} \text{ in } K_2(\mathbb{C}).$$

The nontrivial thing to show here is that it doesn't matter if the Steinberg relation *by which the* $K_2(\mathbb{C}(Y))$ *equivalence is accomplished*, contains terms with zeroes or poles at p.

Rewrite the hypothesis in $\mathbb{Z}[\mathbb{P}^1_{\mathbb{C}(Y)} \setminus \{0,\infty\}]$ as a term-for-term equality

$$(\#) \qquad \sum m_i A_i \otimes B_i - \sum {}'m_j{}'A_j \otimes {}'B_j \;=\; \sum_* (\omega_* \otimes \xi_*\eta_* - \omega_* \otimes \xi_* - \omega_* \otimes \eta_*)$$
$$+ \sum_* ({}'\omega_* \otimes {}'\xi_* + {}'\xi_* \otimes {}'\omega_*) \;+\; \sum_* {}'\eta_* \otimes (1 - {}'\eta_*).$$

Fix a function ϵ with a first order zero at p; if $Y = \mathbb{P}^1$ then it could be $(z-p)$. For $\alpha \in \mathbb{C}(Y)$ we will write $\alpha = \epsilon^a\tilde{\alpha}$ where $\tilde{\alpha}(p) \neq 0,\infty$. Some terminology: if $a = 0$ then α is "reduced"; if both α and β are reduced then $\alpha\otimes\beta$ is; and if all (1, 2, or 3) terms in a Steinberg are reduced then that Steinberg is. Furthermore, for any $\alpha\otimes\beta = \epsilon^a\tilde{\alpha}\otimes\epsilon^b\tilde{\beta}$ there is a fixed algorithm to produce a (very lengthy) sum of Steinbergs $\mathcal{S}(\alpha\otimes\beta)$ such that (term-for-term)

$$(\#\#) \quad \alpha\otimes\beta \;=\; \mathcal{S}(\alpha\otimes\beta) + \tilde{\alpha}\otimes\tilde{\beta} + \epsilon\otimes(-1)^{ab}\frac{\tilde{\alpha}^b}{\tilde{\beta}^a}.$$

If $\alpha\otimes\beta$ is already reduced then $\mathcal{S}(\alpha\otimes\beta) = 0$.

Now develop the r.h.s. of (#) as follows:

(i) set aside the Steinbergs that are reduced to begin with, and apply the fixed algorithm (##) to *every term* of each remaining Steinberg (the reduced terms among these will be unaffected).

(ii) The resulting (nonzero) $\mathcal{S}$'s are in 1-1 correspondence with all unreduced *terms* from the original r.h.s. of (#). Since these terms had to cancel to give the (entirely reduced) terms of the l.h.s., by the same cancellation scheme the $\mathcal{S}$-terms all cancel (oddly enough some of these will be reduced).

(iii) Since the only remaining terms containing ϵ are now of type $\epsilon\otimes(\cdots)$, and (obviously) none of these are reduced, they also neatly cancel out.

The upshot is that we have rewritten the r.h.s. of (#) (after some pair creation/annihilation)

$$\sum_* \left(\tilde{\omega}_* \otimes \tilde{\xi}_*\tilde{\eta}_* - \tilde{\omega}_* \otimes \tilde{\xi}_* - \tilde{\omega}_* \otimes \tilde{\eta}_*\right) + \sum_* \left({}'\tilde{\omega}_* \otimes {}'\tilde{\xi}_* + {}'\tilde{\xi}_* \otimes {}'\tilde{\omega}_*\right) + \sum_* {}'\tilde{\eta}_* \otimes (\widetilde{1 - {}'\eta_*}).$$

The first two sums are of reduced Steinbergs and therefore evaluate to Steinbergs at p. On the other hand, ${}'\tilde{\eta}_* \otimes (\widetilde{1 - {}'\eta_*})$ may *not* be a Steinberg. For example, if $a > 0$ and $\alpha = \epsilon^a\tilde{\alpha}$ then

$$\tilde{\alpha}\otimes\widetilde{1-\alpha} = \tilde{\alpha}\otimes(1-\epsilon^a\tilde{\alpha})$$

is not a Steinberg, while if instead $\alpha = \epsilon^{-a}\tilde{\alpha}$ then

$$\tilde{\alpha}\otimes\widetilde{1-\alpha} = \tilde{\alpha}\otimes(\epsilon^a - \tilde{\alpha})$$

$\{f(p), g(p)\}$ induces a well-defined map

$$\Theta_Y : K_2(\mathbb{C}(Y)) \longrightarrow \{\eta_Y \to K_2(\mathbb{C})\}.$$

Somewhat more exotically, we would like to be able to hit $\tilde{N}_\Phi$ and N_Φ (the beginning and end of the norm algorithm) both with Θ to obtain

$$(*) \qquad \prod_{p_i \in h^{-1}(z)} \{f(p_i), g(p_i)\} = [\Theta_{\mathbb{P}^1}(N_h\{f,g\})](z)$$

where the p_i are counted with multiplicity if z is a branch point. Unfortunately this is true only *almost everywhere*: while the norm algorithm commutes with evaluation (in the sense that the same manipulations would be correct in $K_2(\mathbb{C})$ over a fixed z_0), the introduction of $\sigma_j - \tau_k$ in the norm algorithm (via the Lemma from **I**) produces zeroes (and poles) where there weren't any.

On the other hand, if we knew *a posteriori* that $N_h\{f,g\}$ were of the form $K_2(\mathbb{C}) \subseteq K_2(\mathbb{P}^1)$, then we would know that these zeroes (and poles) had been removed either (i) in the remainder of the norm algorithm, or (ii) in the use of the Steinberg relations in $\otimes^2\mathbb{Z}[\mathbb{P}^1_{\mathbb{C}(\mathbb{P}^1)} \setminus \{0,\infty\}]$ to reach $\otimes^2\mathbb{Z}[\mathbb{P}^1_{\mathbb{C}} \setminus \{0,\infty\}]$. In either case $(*)$ holds for all $z \in \mathbb{P}^1$ for which the right hand term makes sense – that is, for which $h^{-1}(z) \cap (|(f)| \cup |(g)|) = \emptyset$. To see this one can simply repeat the algorithm of the well-definedness argument (extended to include fractional powers of ϵ) locally on $\mathbb{P}^1$.

So towards this objective we show that the norm algorithm commutes with something which (a) yields local information and (b) doesn't flinch at the sight of zeroes: the regulator (whose paths may avoid going *through* any specified number of points, because of the direct limit). We claim that

$$(**) \qquad [R_{\mathbb{P}^1}(N_h\{f,g\})](\gamma) = (R_X\{f,g\})(h^{-1}\gamma),$$

where $h^{-1}\gamma$ is a path in X with (possibly non-closed) branches γ_i over the z-sphere. There is absolutely no problem with the meaning of the left-hand side, because $N_h\{f,g\}$ is an element of $K_2(\mathbb{C}(\mathbb{P}^1))$.

In what follows we rewrite the regulator in an equivalent form, which results from changing $-\int_\gamma \log g(p_0)\mathrm{dlog} f$ to $2\pi i \sum_{q \in \gamma \cap T_f} \pm \log g(q)$. Here $T_f = f^{-1}(\mathbb{R}^-)$ is shorthand for the branch cuts of $\log f$ on X (unlike x_i or f_i, this has nothing to do with the branches of X over $\mathbb{P}^1$), and the sign is positive for a jump (along γ) from 0 to $2\pi i$ and negative for the opposite. Now, writing $f_i = f(z, x_i(z))$ and $g_i = g(z, x_i(z))$ for branches of f and g

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isn't either! However, evaluating them at p (since $\epsilon(p) = 0$) yields respectively

$$\tilde{\alpha}(p) \otimes 1 \qquad \text{and} \qquad \tilde{\alpha}(p) \otimes -\tilde{\alpha}(p)$$

which *are* Steinberg relations in $\mathbb{Z}[\mathbb{P}^1_{\mathbb{C}} \setminus \{0,\infty\}]$. Therefore we have expressed

$$\sum m_i A_i(p) \otimes B_i(p) - \sum{}' m_j{}' A_j(p) \otimes{}' B_j(p)$$

as a sum of Steinbergs, the corresponding element of $K_2(\mathbb{C})$ is zero, and we are done.

and working on the right-hand side of $(**)$, we have: $(R_X\{f,g\})(h^{-1}\gamma)$

$$= exp[(\frac{1}{2\pi\sqrt{-1}}\int_{h^{-1}\gamma}\log f \mathrm{dlog} g) + \sum_{q\in\gamma\cap T_f}\log g(q)]$$

$$= \prod_i exp[(\frac{1}{2\pi\sqrt{-1}}\int_{\gamma_i}\log f_i \mathrm{dlog} g_i) + \sum_{q\in\gamma\cap T_{f_i}}\log g_i(q)]$$

$$=: \widetilde{R}_{\mathbb{P}^1}\left(\prod_i\{f_i,g_i\}\right)(\gamma) \;=\; \widetilde{R}_{\mathbb{P}^1}\left(\widetilde{N}_h\{f,g\}\right)(\gamma).$$

Again, $\tilde{R}$ is just a placeholder rather than an actual regulator, although we do have a well-defined quantity here. Our claim is that the norm algorithm, applied to the expression $\prod_i\{f_i,g_i\}$ in parentheses (to obtain $N_h\{f,g\}$), preserves the value of this quantity while gradually turning it into an expression which *is* a regulator of something (on $\mathbb{P}^1$). We outline how to see this. If one backtracks through the proof of our algebraic Lemma, one finds that the Steinberg relations forgotten in the stage of the norm algorithm which we have written out are

$$\prod_{i,j,k}\left(\left\{\frac{x_i-\sigma_j}{x_i-\tau_k},\, 1-\frac{x_i-\sigma_j}{x_i-\tau_k}\right\}\times\{-(x_i-\tau_k),\, x_i-\tau_k\}\right).$$

We want to show that $[\tilde{R}_{\mathbb{P}}(.)](\gamma)$ applied to this gives 1. Referring to the footnote on p. 8 for the discussion of Y, this is (a power of)

$$\left[R_Y\left(\left\{\frac{x-\sigma}{x-\tau}, 1-\frac{x-\sigma}{x-\tau}\right\}\times\{-(x-\tau), x-\tau\}\right)\right](h_Y^{-1}\gamma),$$

which clearly *is* 1. So the right-hand side of $(**)$ becomes

$$\tilde{R}_{\mathbb{P}}\left(\tilde{N}_f\{\bar{\Phi},\bar{g}\}\times\tilde{N}_g\{\bar{f},\bar{\tilde{\Phi}}\}\right) \;=\; [R_{X_f}\{\bar{\Phi},\bar{g}\}](h_f^{-1}\gamma)\times[R_{X_g}\{\bar{f},\bar{\tilde{\Phi}}](h_g^{-1}\gamma)$$

by essentially the same computation as above in reverse. In this way we gradually "descend to $\mathbb{P}$" and the left-hand side of $(**)$.

The Proof of Suslin. This is now slick: suppose $h=1$ on $|(f)|\cup|(g)|$. Then $(R_X\{f,g\})(h^{-1}\gamma)=1$ for all γ on $\mathbb{P}^1$ avoiding 1 (simply slide γ to $\{0\}$ on $\mathbb{P}^1\setminus\{1\}$). By $(**)$, $N_h\{f,g\}\in\ker R_{\mathbb{P}^1}$, which by our work in part **I** is $K_2(\mathbb{C})$. So $N_h\{f,g\}$ consists of constants, and $\frac{\Theta_{\mathbb{P}^1}(N_h\{f,g\})(0)}{\Theta_{\mathbb{P}^1}(N_h\{f,g\})(\infty)}=1$. Moreover, since $N_h\{f,g\}\in K_2(\mathbb{C})$ and only $h^{-1}(1[=z])$ intersects $|(f)|$ and $|(g)|$, it follows from the discussion following $(*)$ that we may use $(*)$ at $z=0,\infty$. That is,

$$1=\frac{\Theta_{\mathbb{P}^1}(N_h\{f,g\})(0)}{\Theta_{\mathbb{P}^1}(N_h\{f,g\})(\infty)}=\frac{\prod_{p\in h^{-1}(0)}\{f(p),g(p)\}}{\prod_{q\in h^{-1}(\infty)}\{f(q),g(q)\}}=\prod_{p\in|(h)|}\{f(p),g(p)\}^{v_p(h)},$$

Q.E.D.

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