Arithmetic of period maps, II (asymptotics)

(related to lectures M. Green & I will give)

1 Limits of period maps

$V :=$ fiber over a point

Let $\mathcal{V} = (\mathcal{V}, \mathbb{V}, Q, \mathcal{F}^\bullet)$ be a PVHS $/\Delta^*$ with MTG G, weight n & unipotent monodromy operator T,

$$\Phi : \Delta^* \to {}_{\Gamma}\backslash^{G(\mathbb{R})}/_{H} = {}_{\Gamma}\backslash D$$ the associated period map ($\ni T \in \Gamma \leq G(\mathbb{Q})$),

$$N := \log(T) = \sum_{k \geq 1} \frac{(-1)^{k-1}}{k}(T-I)^k \in \mathfrak{g}_{\mathbb{Q}} \subset \mathrm{End}(V)$$

$$W_\bullet := W(N)_\bullet \quad \left(= \text{unique incr. filt. on } V \text{ s.t. } \begin{cases} N(W_\bullet) \subset W_{\bullet-2} \\ N^\ell : Gr^W_{n+\ell} \xrightarrow{\cong} Gr^W_{n-\ell} \end{cases} \right)$$

$$\underbrace{\tilde{\mathbb{V}} := e^{-\frac{\log(s)}{2\pi i}N}\mathbb{V}}_{\text{extends to } \Delta} \rightsquigarrow \begin{cases} \mathcal{V}_e := \mathbb{V}_e \otimes \mathcal{O}_\Delta \supset \mathcal{F}_e^\bullet \\ V_{\lim} := \mathbb{V}_e|_0 \\ F^\bullet_{\lim} := \mathcal{F}^\bullet_e|_0 \end{cases}$$

(Schmid (nilpotent orb. thm.) $\mathcal{F}^\bullet$ extends to holo. subbundles)

Schmid (SL_2-orbit Thm.) $\Longrightarrow$ $(V_{\lim}, W_\bullet, F^\bullet_{\lim})$ is a MHS

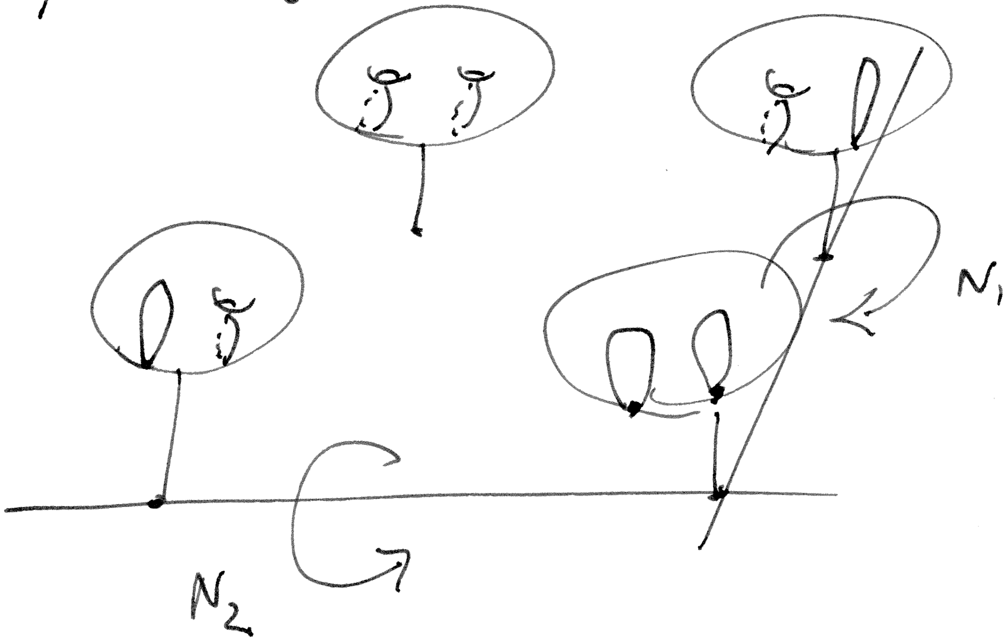
$=:$ LMHS

(A) Deligne: $\exists!$ bigrading $I^{p,q}$ of $V_{\lim,\mathbb{C}}$ s.t.

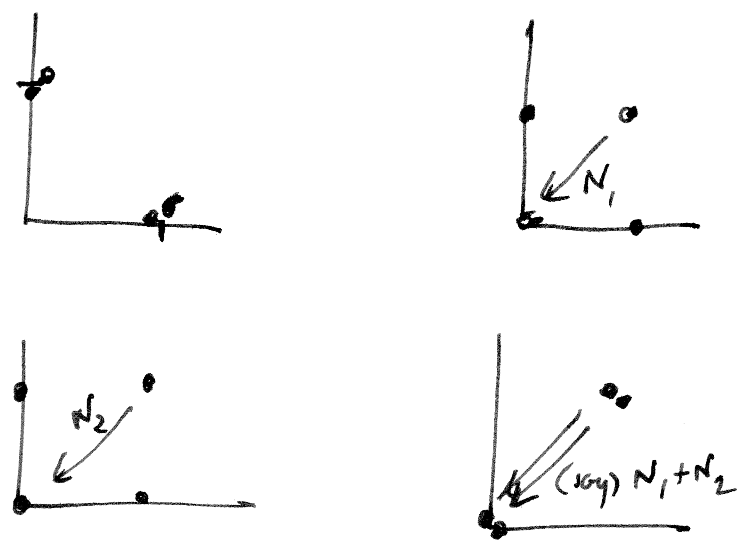
$$F^\bullet_{\lim} = \bigoplus_{\substack{p,q \\ p \geq \bullet}} I^{p,q}, \quad W_\bullet = \bigoplus_{\substack{p,q \\ p+q \leq \bullet}} I^{p,q}, \quad \overline{I^{p,q}} \equiv I^{q,p} \quad \left(\text{mod} \bigoplus_{\substack{a<p \\ b<q}} I^{a,b}\right)$$

Moreover $N : \oplus I^{p,q} \longrightarrow \oplus I^{p-1,q-1}$ can be completed to an sl_2-representation, which then decomposes $V_{lim,0}$ into isotypical components ("compatibly" with the bigrading). We can visualize all this by using dots to depict the dimensions of the $\{I^{p,q}\}$'s and arrows for the action of N.

Ex/ (an only 2-parameter example, over a bidisk Δ^2)



LMHS pics :



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(B) <u>Nilpotent orbits</u> = PVHS $\mathcal{V}_{nilp} := \left(\mathcal{V}, \mathbb{V}, Q, e^{-\frac{\log(s)}{2\pi i}N} F^{\bullet}_{\lim}\right)$

(defined over Δ^* after possibly shrinking the radius) = "most trivial PVHS having the same LMHS as $\mathcal{V}$"

$\underbrace{\text{If } MTG(\mathcal{V}) = G, \text{ then } MTG(\mathcal{V}_{nilp}) \subseteq G}$, so period map will still go into $\Gamma\backslash D$

<u>Sketch</u>: Monodromy acts on Hodge tensors of a <u>polarized</u> VHS thru the <u>$\mathbb{Z}$-pts.</u> <u>of a special orthogonal group</u> = finite group!

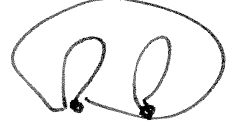
$\Rightarrow$ pulling back by a finite cover, we may assume they are <u>invariant</u>! So in each $\mathcal{V}^{\otimes k} \otimes \check{\mathcal{V}}^{\otimes l}$, the Hodge tensors give a trivial subVHS, which splits off by semisimplicity. Hence it is <u>unaffected</u> by taking LMHS or assoc. nilp. orbit. $\square$

The definition of LMHS depends upon a <u>choice</u> of s (or more generally, $(s_1, \ldots, s_n)$); that of $\underbrace{\text{nilpotent orbit}}$ does not.

or equiv., LMHS/reparametrization

(replace $F^{\bullet}_{\lim} \to e^{\sum \alpha_i N_i} F^{\bullet}_{\lim}$.)

Ex/ for the  LMHS, the extension class of $\mathbb{Q}(-1)^{\oplus 2}$ by $\mathbb{Q}(0)^{\oplus 2}$ has (in this setting, b/c of the polarization) $\underline{\underline{3}}$ degrees of freedom, 2 of which are zeroed out by $e^{\alpha_1 N_1 + \alpha_2 N_2}$. What remains is the CR of the 4 pts. in a resolution

$\mathbb{P}^1 \longrightarrow$  $= \dfrac{\mathbb{P}^1}{a \equiv b,\ c \equiv d}$ //

(c) Clemens – Schmid : when $\mathcal{V}$ arises from a semistable degen.

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \Delta \\ \cup & & \cup \\ \bigcup Y_i = X_0 & \longrightarrow & \{0\} \end{array}$$

we have a long exact sequence

$$\cdots \underset{\substack{\uparrow \\ \text{usually } 0}}{\longrightarrow} H^m(X_0) \to H^m_{lim}(X_s) \xrightarrow{N} H^m_{lim}(X_s)(-1) \to H_m(X_0)(-m-1) \to \cdots$$

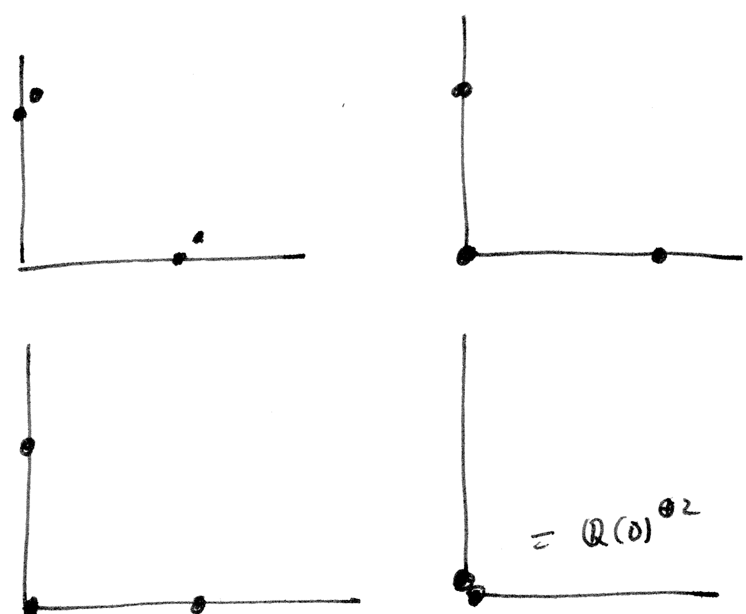
Computed by spectral sequence of double complex

$C^j(Y^{[i]})$, where C^j = "cochains with pullback".

and $Y^{[i]} := \coprod_{|K| = i+1} Y_{k_1} \cap \dots \cap Y_{k_i}$.

Ex/

$H^1(A_0)$
$\shortparallel$
$\ker(N)$



definitely misses the ext. class assoc. to the CR

D <u>Adjoint reduction</u>: w/o changing D, we can replace

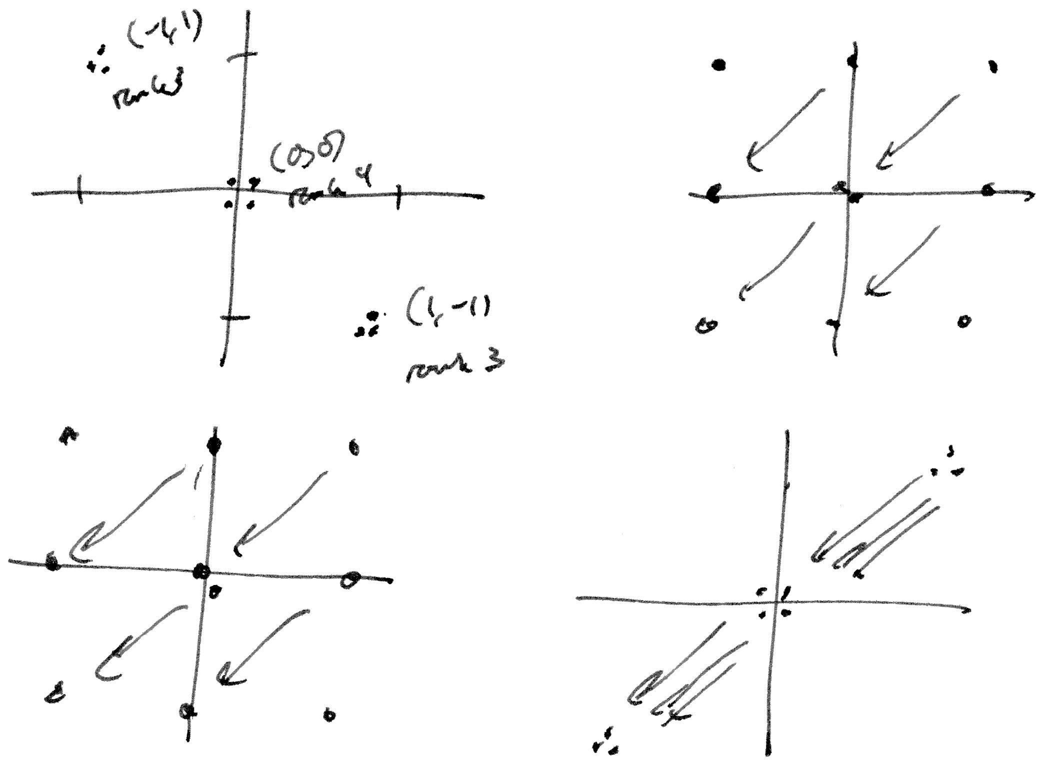
G by $M := G^{ad}$

V by $\mathfrak{m} := \mathrm{Lie}(M)$ $\left(\subset \mathrm{End}(V) \cong V^\vee \otimes V \right)$

HS $\varphi : S^1 \to G$ "on V" by $\mathrm{Ad} \circ \varphi : S^1 \to M$ "on $\mathfrak{m}$" (weight $= 0$)

LMHS picture is compatible with tensors ($\otimes$) (& subalgebras cut out by Hodge tensors) $\rightsquigarrow$ carries over to $\mathfrak{m}$.

Ex/



(E) Boundary components

Given $N \in \mathfrak{m}_{\mathbb{Q}}$ nilpotent,

(LMHS) $\hookrightarrow$ $\tilde{B}(N) := \{ F^\bullet \in \check{D} \mid \mathrm{Ad}(e^{\tau N}) F^\bullet \in D$ for $\mathrm{Im}(\tau) \gg 0$ and $N(F^\bullet) \subseteq F^{\bullet-1} \}$

(nilp. orbits) $\hookrightarrow$ $B(N) := e^{\mathbb{C}\langle N \rangle} \backslash \tilde{B}(N)$ (left quotient)

$\bar{B}(N) := \Gamma_N \backslash B(N)$, $\Gamma_N :=$ largest sgp. of Γ stabilizing the line $\langle N \rangle$.

Set $Z_N := \{\text{centralizer of } N\} \leq M$: "locally" $Z_N(\mathbb{C})$ acts transitively on $B(N)$

Lie algebra: $\mathfrak{z}_N = \ker(\mathrm{ad}\, N) \leq \mathfrak{m}$ — visualize as bottom of $\mathfrak{sl}_2$-chains

$G_N := Gr_0^W Z_N$, $D(N) := G_N(\mathbb{R}).(\mathrm{Ad} \circ \varphi)_{\text{split}}$.

Then $\underset{\text{"boundary component"}}{B(N)} \twoheadrightarrow \cdots \twoheadrightarrow B(N)_{(k)} \twoheadrightarrow B(N)_{(k-1)} \twoheadrightarrow \cdots \twoheadrightarrow \underset{\text{"Levi MT domain"}}{D(N)}$

with $\{\text{tangent space to } k^{\text{th}} \text{ fiber}\} = Gr_{-k}^W \mathfrak{z}_{N,\mathbb{C}} / F^0 + \langle N \rangle$,

$\{\text{tangent space to } D(N)\} = Gr_0^W \mathfrak{z}_{N,\mathbb{C}} / F^0$

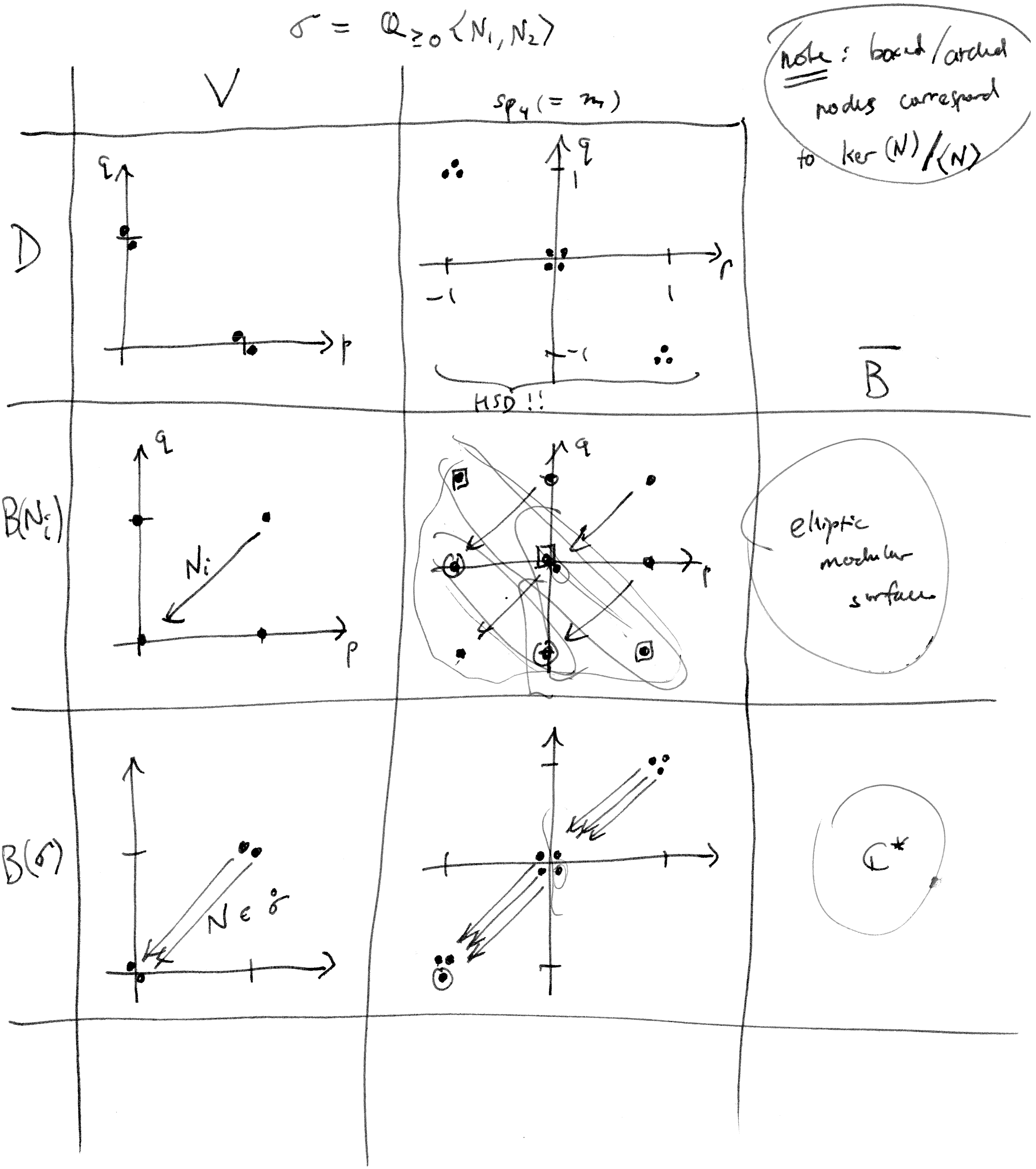
This tower passes to the quotient by Γ_N ; the fibers are generalized intermediate Jacobians.

Some examples follow:

Example 0 : $\dim V = 4$, weight $= 1$, $\underline{h} = (2,2)$

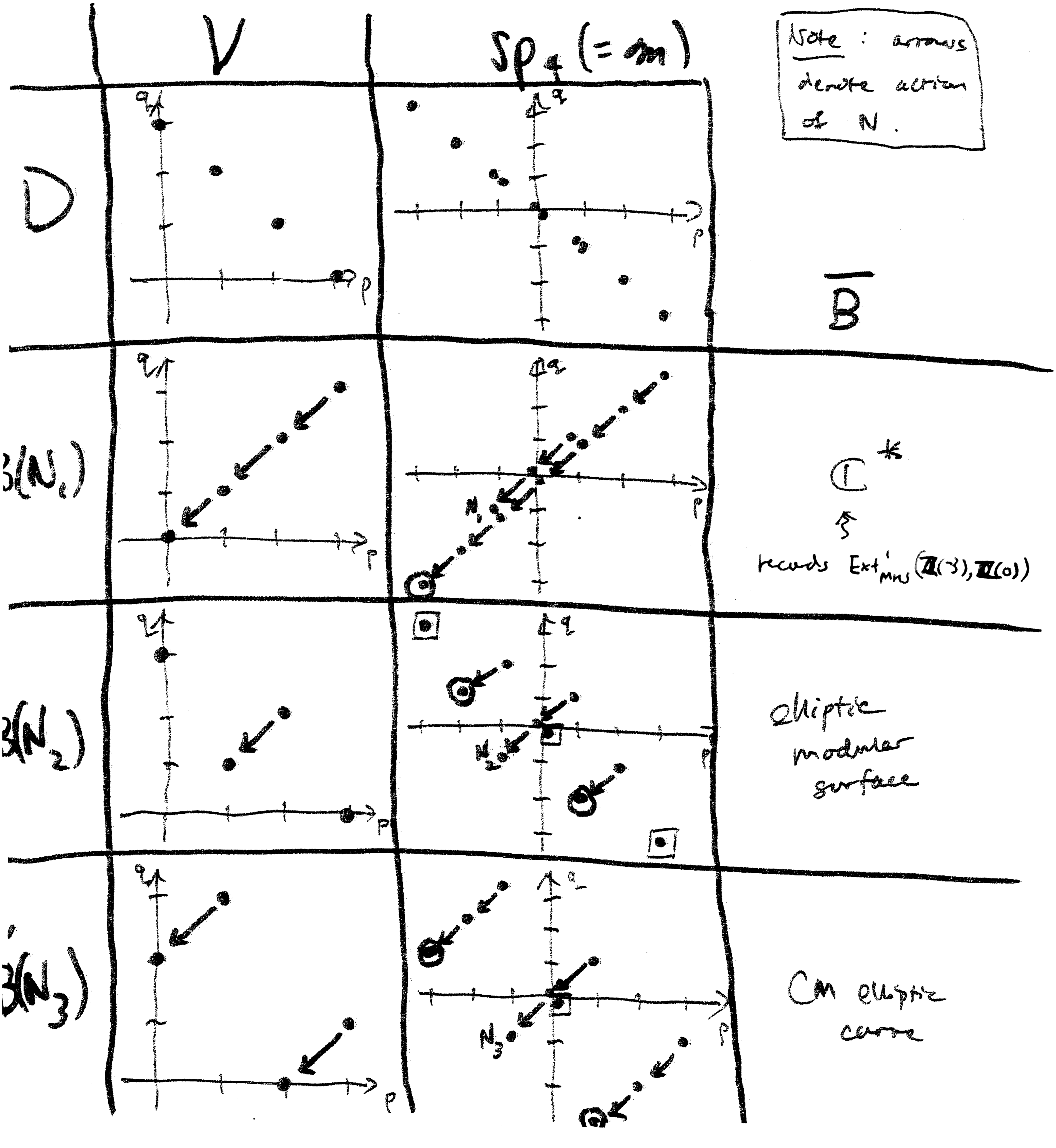
$D \cong Sp_4(\mathbb{R}) / U(2)$ [period domain]

$\sigma = \mathbb{Q}_{\geq 0}\langle N_1, N_2 \rangle$



Example 1: $\dim V = 4$, weight $= 3$, $\underline{h} = (1,1,1,1)$

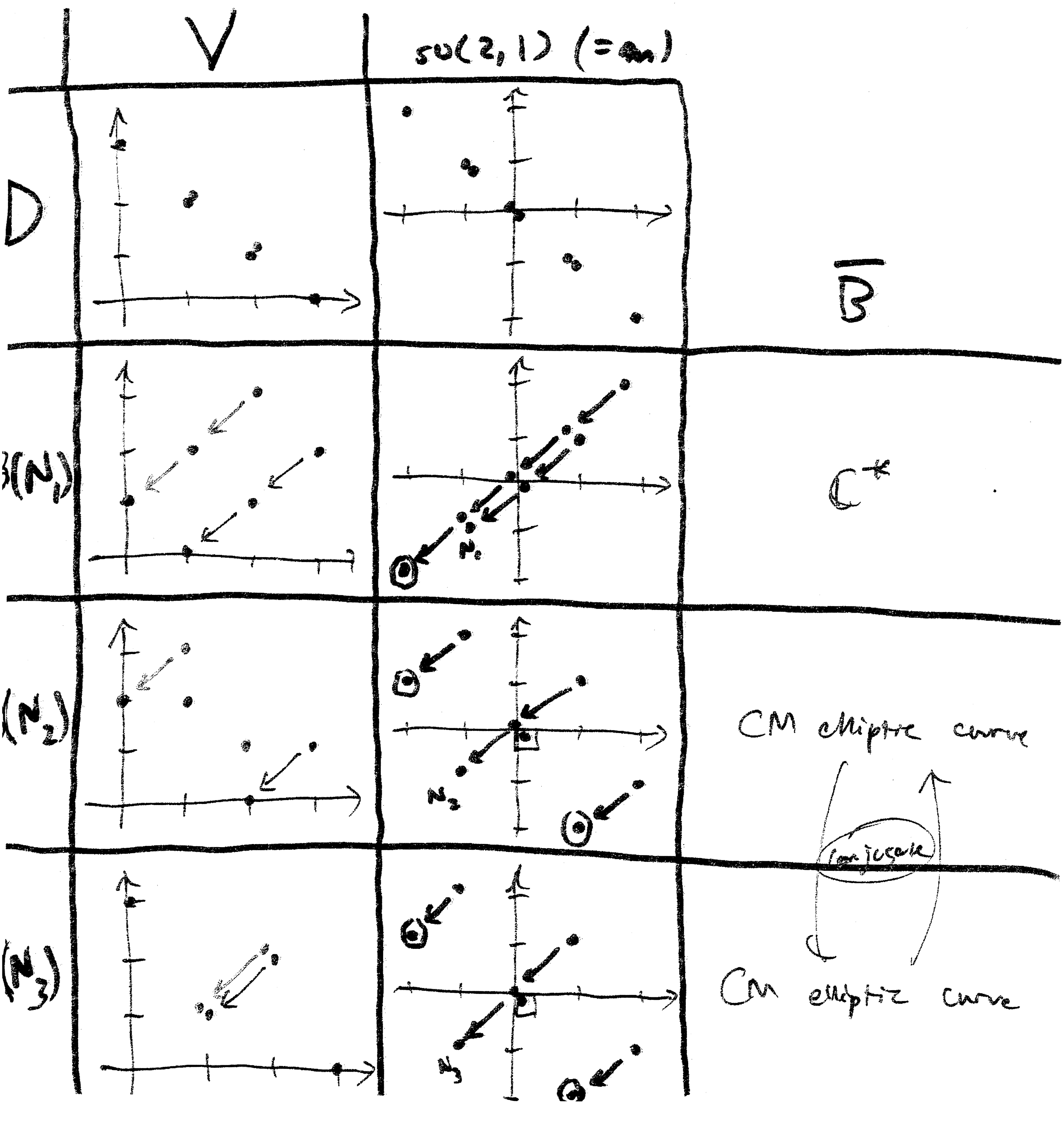
$$D \cong Sp_4(\mathbb{R}) / U(1)^{\times 2} \quad \text{[period domain]}$$



Example 2: $\dim V = 6$, weight $= 3$, $\underline{h} = (1,2,2,1)$

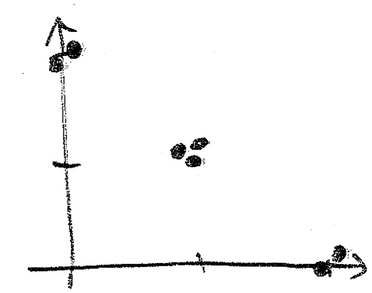
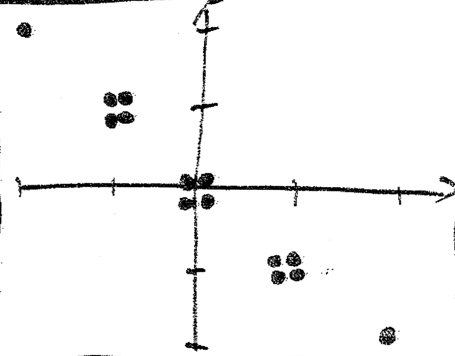
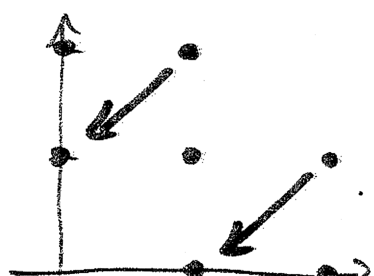
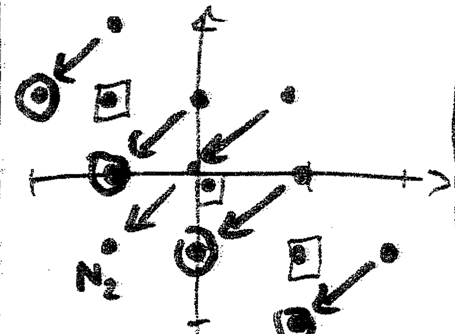
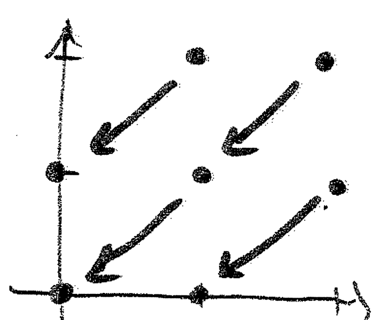
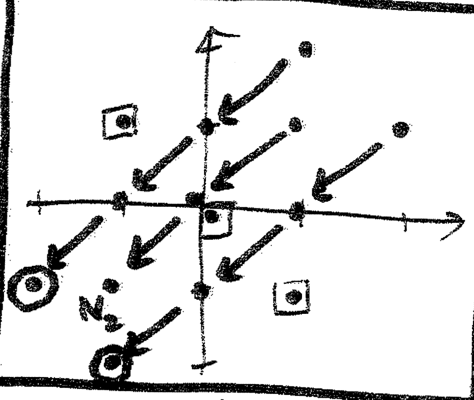
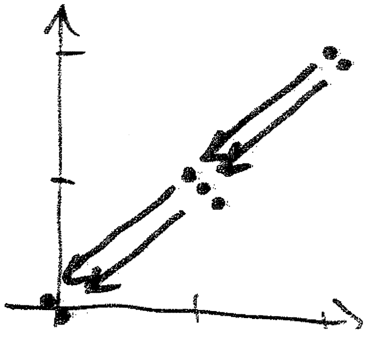
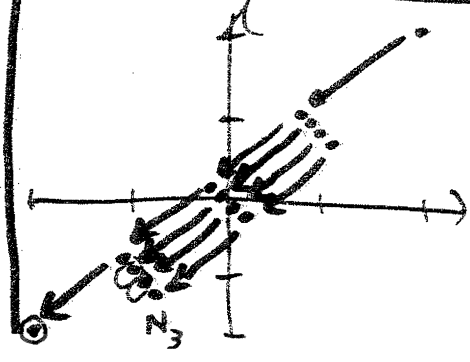
$V_{\mathbb{Q}(\sqrt{-d})} \overset{\mu}{=} V_+ \oplus V_-$, $\underline{h}_+ = (1,1,1,0)$

$D \cong U(2,1)/U(1)^{\times 3}$



Example 3: $\dim V = 7$, weight $= 2$, $\underline{h} = (2,3,2)$
+ distinguished Hodge 3-tensor

$$D \cong G_2(\mathbb{R})/U(2)$$

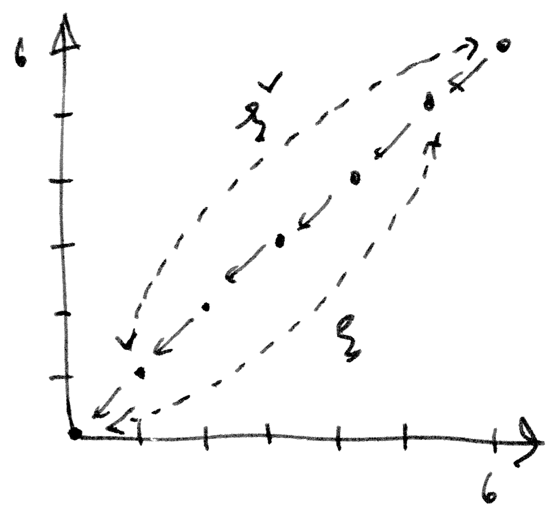
	V	$\mathfrak{g}_2$ ($\cong \mathfrak{m}$)	$\overline{B}$
D			
(N_1)			family of compact complex 2-tori over a modular curve
(N_2)			$(\mathbb{C}^*)^2$-fibration over a modular curve
(N_3)			$\mathbb{C}^* \times (\mathbb{C}^*)^2$ (assuming Γ neat)

<u>Example 4</u>: $\dim V = 7$, weight $= 6$, $\underline{h} = (1,1,1,1,1,1,1)$

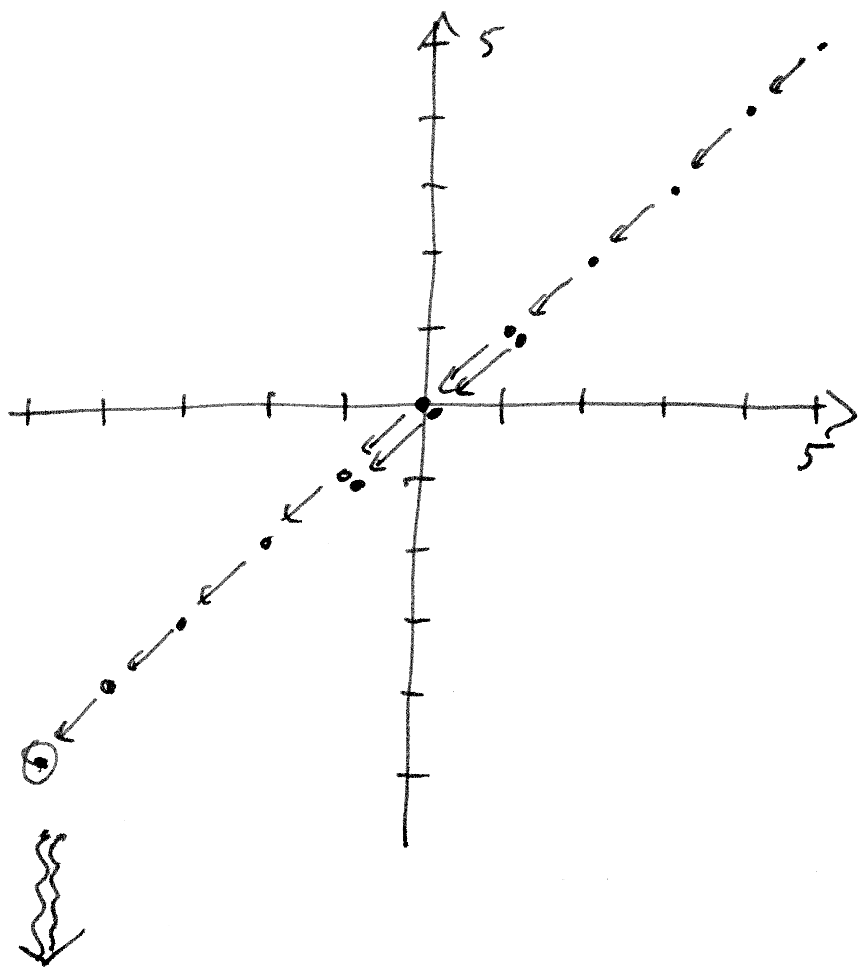
+ distinguished Hodge 3-tensor

$$D \cong G_2(\mathbb{R}) / U(1)^{\times 2}.$$

Here we just note that $N \in \mathfrak{g}_{\mathbb{Q}}$ exists which gives the following pictures:



V



$$\overline{B} \cong \mathbb{C}^*,$$

classifying extensions $\xi \in \mathrm{Ext}^1_{MHS}(\mathbb{Q}(-5), \mathbb{Q}(0))$ as shown.

(F) Arithmetic

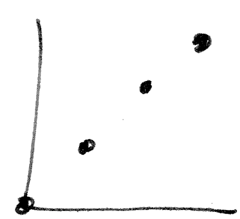
Let k = # field, & suppose $\mathcal{V}$ arises from a semistable degeneration $/k$: this means that $\mathcal{X} \to \Delta$ belongs to a larger family over $\mathbb{P}^1$, defined $/k$, and with the $\{Y_I\}$ def'd $/k$.

Conjecture: The LMHS is the Hodge realization of a (mixed) motive defined $/k$. In particular extension classes in $\mathrm{Ext}^1_{MHS}(\mathbb{Q}(0), \mathbb{Q}(n)) \cong \mathbb{C}/\mathbb{Q}$ are essentially Borel regulators of elements of $K_{2n-1}(k)$. If $k = \mathbb{Q}$, these classes are $\therefore$ rational mults. of $\frac{\zeta(n)}{(2\pi i)^n}$.

Ex/ The mirror quintic VHS $= H^3(X_s)$, $s = \xi^5$

$$X_s = \underset{\text{of}}{\text{sm. compact.}} \left\{ 1 - \xi \left(\sum_{i=1}^4 x_i + \frac{1}{\prod_{i=1}^4 x_i} \right) = 0 \right\} \subset (\mathbb{C}^*)^{\times 4}$$

It has LMHS at $s = 0$ of form



Writing $I^{3,3} = \mathbb{C}\langle e_3 \rangle$, and $\gamma_3, \gamma_2, \gamma_1, \gamma_0$ for a symplectic $\mathbb{Q}$-basis, we have $e_3 = \gamma_3 - \frac{200\,\zeta(3)}{(2\pi i)^3}\gamma_0$ (Candelas - de la Ossa - Green - Parkes)

Ex/ The 2^{nd} G_2 example above (LoHS) comes from AG/$\mathbb{Q}$ (Dettweiler–Reiter).

$\therefore$ we expect the limiting extension to be a rational multiple of $\zeta(5)/(2\pi i)^5$!!

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2 Limits of normal functions

(related to J. Lewis's lecture)

Consider a SSD

$$\begin{array}{ccccc} & & \text{dim } 2m & & \text{reduced NCD} \\ & & \text{smooth} & & \text{smooth, irred} \\ \mathcal{X}^* & \hookrightarrow & \mathcal{X} & \hookleftarrow & X_0 = \bigcup_i Y_i \\ \downarrow{\scriptstyle \pi \text{ smooth}} & & \downarrow{\scriptstyle \bar{\pi} \text{ proper, holo.}} & & \downarrow \\ \Delta^* & \overset{j}{\hookrightarrow} & \Delta & \hookleftarrow & \{0\} \end{array}$$

and an alg. cycle $\mathfrak{Z} \in Z^m(\mathcal{X})$ properly intersecting fibers.

$\rightsquigarrow$ $Z_s := \mathfrak{Z} \cdot X_s \in Z^m(X_s)$, $s \in \Delta$.

Assume $0 = [\mathfrak{Z}] \in H^{2m}(\mathcal{X})$ $\left(\Rightarrow 0 = [Z_s] \in H^{2m}(X_s) \right)$

Is there a sense in which

(*) $$\lim_{s \to 0} AJ_{X_s}(Z_s) = AJ_{X_0}(Z_0) \ ?$$

First of all, what do the 2 sides mean?

<u>LHS</u>: $AJ_{X_s}(Z_s)$ yields a section $\nu_{\mathfrak{Z}} \in \Gamma(\Delta^*, J)$

$\uparrow$ Jacobian bundle of VHS $\mathcal{H}$ assoc. to $\mathcal{H}^{2m-1}_{\mathcal{X}^*/\Delta^*}$

J has an extension defined by

not Hausdorff!!

$$0 \to j_* \mathbb{H} \to \frac{\mathcal{H}_e}{\mathcal{F}^m_e} \to J_e \to 0$$

and $\exists$ extension $\bar{\nu}_{\mathfrak{Z}} \in \Gamma(\Delta, J_e)$ (Zucker - El Zein).

their thm. applies to more general setting $[\mathfrak{Z}^*] = 0$ in $H^{2m}(\mathcal{X}^*)$.

Set $\lim_{s \to 0} AJ_{X_s}(Z_s) := \bar{\nu}_{\mathfrak{Z}}(0)$.

RHS: The singular variety $X_0 = \cup Y_i \overset{\iota_0}{\hookrightarrow} \mathcal{X}$ has substrata (of codim. $l = 0, \ldots, 2m-1$)

$$Y^{[l]} := \coprod_{|I| = l+1} Y_I \quad \text{where} \quad Y_I := \bigcap_{i \in I} Y_i .$$

Using these one may explicitly write down AJ maps on the motivic cohomology of X_0:

(vanishing $\ker(N) \subset H_{\lim}$)

$$(**) \quad \underbrace{H^{2m}_{\mathcal{M}}(X_0, \mathbb{Z}(m))_{\hom}} \xrightarrow{AJ_{X_0}} J^m(X_0) \text{ cx. Lie group} \quad \left(:= \frac{H^{2m-1}(X_0, \mathbb{C})}{F^m + H^{2m-1}(X_0, \mathbb{Z})} \right)$$

(built out of K-gps. of substrata $K_l(W^{[l]})$)

built out of "regulators" = Chern-class maps on $K_l(W^{[l]})$ } or "AJ maps for higher Chow gps."

Write also
- $CH^m(\mathcal{X})_{\hom} \xrightarrow{\iota_0^*} H^{2m}_{\mathcal{M}}(X_0, \mathbb{Z}(m))_{\hom}$
  $\mathfrak{Z} \longmapsto \mathfrak{Z}_0$
- $J^m(X_0) \xrightarrow{\Psi} (\mathcal{J}_e)_0$ [induced by map $H^{2m-1}(X_0) \to H^{2m-1}_{\lim}(X_s)$ in C-S sequence

Then $(*)$ holds w/ RHS replaced by $\Psi(AJ_{X_0}(\mathfrak{Z}_0))$.

Cor. 1: The extension $\bar{\nu}_{\mathfrak{Z}}$ is actually a section of $\hat{\mathcal{J}}_e$

↑ replace $(\mathcal{J}_e)_0$ by $J^m(X_0)$ — this actually makes it Hausdorff

Cor. 2: The regulators in $(**)$ imply arithmetic behavior for the limit of the AJ map.

Here is an example of what sort of thing happens in the case of the quintic 3-fold: here $X_0 = \cup\, 5\ \mathbb{P}^3$'s blown up along Fermat curves.

Ex/

$$\begin{tikzcd}
H^4_{\mathcal{M}}(X_0, \mathbb{Q}(2))_{hom} \arrow[r, "AJ_{X_0}"] & J^2(X_0) \arrow[r, hook, "\Psi"] & (\mathcal{J}_e)_0 \\
K_3^{ind}(\mathbb{C}) \arrow[u, phantom, "\cup"] \arrow[r, "\text{regulator}"] & \mathbb{C}/\mathbb{Z}(2) \arrow[u, phantom, "\cup"] \arrow[d, "\mathrm{Im}"] & \\
& \mathbb{R} &
\end{tikzcd}$$

(countable image, and related to L-fn. special value if limiting cycle def'd / # field.)

$\exists$ family of 1-cycles Z_t with Z_0 "lying in the K_3^{ind} subgroup", and $\mathrm{Im}\left(AJ_{X_0}(Z_0)\right) = D_2(\sqrt{-3}) \neq 0$.

Walcher will explain examples coming from differences of "van Geemen lines".