Arithmetic of period maps, I

$\boxed{\text{OVERVIEW}}$

The last 40 years have seen the development of rich theories of $\underbrace{\text{Hodge theory at the boundary}}_{\text{(next talk)}}$ and $\underbrace{\text{symmetries of Hodge structures}}_{\text{(this talk)}}$ which are themes of this conference. Recent work of Griffiths et al $\Rightarrow$ the Hodge Conj. can now be stated in terms of asymptotics of such period integrals; while Mumford-Tate (i.e. symmetry) groups of HS's have led to proofs of Hodge & Beilinson-Hodge conjectures in special cases. The other closely related theme of the conference is the arithmetic of periods, of which Euler's work on rel's between multiple zeta values is an early example.

At the heart of current thought on the Hodge Conjecture, two intertwined programs have emerged:

1. The approach just referred to reduces the conj. to $\exists$ of singularities for certain several-variable admissible normal functions obtained from Hodge classes. While

this criterion pertains a priori to degenerations of normal functions, a result of Schnell reveals the importance of estimates on the dimension of ~~their~~ their zero-loci, which have recently been proven algebraic (generalizing Cattani-Deligne-Kaplan's result on the locus of Hodge classes).

(2) Another approach, championed by Voisin, is to break the Hodge conj. into 2 pieces: first, to show that the locus of Hodge classes in a VHS arising from alg. geom/$\mathbb{Q}$ is defined over a number field; then second, to prove the Hodge conjecture on arithmetic varieties (i.e. those /$\bar{\mathbb{Q}}$). Key to this approach is showing that a given family of Hodge classes is absolute, extending Deligne's theorem for abelian varieties discussed in Laza's 2nd lecture.

1) Spreads of period maps

Let D := period (or M-T) domain
$= G(\mathbb{R})/H$,

$\mathcal{I} \subset \Omega^\bullet(D)$ IPR (pulls back to 0 under local lifting of any VHS, by defn.)

Case 1: $\mathcal{I} = 0$ ($\Rightarrow$ D Herm. sym.)

$\Gamma \leq G(\mathbb{Q})$ arithmetic $\xRightarrow{\Downarrow \text{ thm. of Baily/Borel}}$ $\Gamma\backslash D$ has a proj. embedding by automorphic functions, & parametrizes a VHS (which is known to be motivic outside E_6/E_7 cases)

motivic case: automorphic fcns. provide the highly transcendental passage from

τ comes from A.F.

$$\text{periods}/\text{HS's} \xrightarrow[\text{(auto. fcns.)}]{(*)} \text{coeffs. of defining eqns. of algebraic varieties w/ these HS}$$

giving an inverse of the period map.

Ex/ $\tau \longrightarrow g_4(\tau), g_6(\tau)$

(other ex's: Clingher-Doran for lattice-polarized K3s (type IV)
Holzapfel/Shiga for Picard curves (ball))

Case 2 : $d \neq 0$

$$\begin{array}{ccccc} \mathcal{X} & & \mathcal{V} & & \\ \downarrow \pi \text{ (smooth proper)} & \rightsquigarrow & \downarrow \text{ VHS} & \rightsquigarrow & \\ \text{(cx. alg. manifold) } \mathcal{S} / \mathbb{C} & & \mathcal{S} & & \end{array}$$

consider some $R^n \pi_* \mathbb{Z}$

assoc. period map

$$\begin{tikzcd} & \mathcal{S}^{un} \arrow[r] \arrow[d, dashed] & D \arrow[d, dashed] & \ni \varphi_0 \arrow[d, mapsto] \\ \Phi : & \mathcal{S} \arrow[r] & \Gamma \backslash D & \\ & s_0 \arrow[rr, mapsto] & & = \end{tikzcd}$$

lift gives integral manifold through φ_0.

Spread out : since $\mathcal{X} \to \mathcal{S}$ actually def'd / some K f.g. / $\bar{\mathbb{Q}}$,

$\exists\ \mathcal{S}' / \bar{\mathbb{Q}}$ affine & very general $p \in \mathcal{S}'(\mathbb{C})$ s.t. $ev_p : \bar{\mathbb{Q}}(\mathcal{S}') \xrightarrow{\cong} K$.

Pulling back the defining equations under ev_p & clearing denominators yields

$$\begin{tikzcd} \tilde{\mathcal{X}} \arrow[rr, "\tilde{\pi}"] \arrow[dr] & & \tilde{\mathcal{S}} \arrow[dl] \\ & \mathcal{S}' & \end{tikzcd} \quad / \bar{\mathbb{Q}} .$$

The resulting period map $\tilde{\Phi}$ (assoc. to $\tilde{\pi}$) still gives an integral manifold of the IPR through φ_0 (This is proper in D since $d \neq 0$). Since there are only countably many families of alg. vars. defined $/\bar{\mathbb{Q}}$, only countably many integral manifolds of the IPR come from AG $\implies$ nothing like $(*)$ (on the last pg.).

Problem : So, the "motivic" HS's in D, as a set, have measure 0.

Find one explicit HS not in that set! (this is open!)

2) Absoluteness of Hodge classes

X = smooth proj. var$/k \subset \mathbb{C}$,

$Hg^m(X) = F^m H^{2m}(X^{an}_{\mathbb{C}}, \mathbb{C}) \cap H^{2m}(X^{an}_{\mathbb{C}}, \mathbb{Q}(m))$.

Let $\sigma \in \mathrm{Aut}(\mathbb{C})$, identify $F^m H^{2m}_{(dR)}(X^{an}_{\mathbb{C}}, \mathbb{C}) \cong \underbrace{\mathbb{H}^{2m}_{Zar}(X_{\bar{k}}, \Omega^{\bullet \geq m}_{X_{\bar{k}}}) \otimes_{\bar{k}} \mathbb{C}}_{\text{(alg. diff'l forms)}}$

$\rightsquigarrow \sigma_* : F^m H^{2m}(X^{an}_{\mathbb{C}}, \mathbb{C}) \longrightarrow F^m H^{2m}(({}^{\sigma}X)^{an}_{\mathbb{C}}, \mathbb{C})$

(obtained by letting σ act on coeffs. of defining eqns. of X)

Define

$$AHg^m(X) := \{ \xi \in Hg^m(X) \mid \sigma_*(\xi) \in Hg^m(X) \ (\forall \sigma \in \mathrm{Aut}(\mathbb{C})) \}$$

$$\rightsquigarrow \underbrace{\mathrm{cl}(Z^m(X)) \subset AHg^m(X) \underbrace{\subset}_{\text{AHC}} Hg^m(X)}_{\text{HC}}$$

<u>Theorem (Deligne, 1982)</u>: AHC holds if X is an abelian variety.

(Deligne needed this in order to establish $\exists$ of canonical models for Shimura varieties of Hodge type.)

<u>Consequence of AHC</u> for $\mathcal{X} \xrightarrow{\pi} \mathcal{S}/_k$, $\Phi : \mathcal{S} \to {}_{\Gamma}\backslash D$ as above:

(1) Suppose Φ factors through ${}_{\Gamma_M}\backslash D_M$; then so does $\tilde{\Phi}$.

[<u>Pf</u>: This is a corollary of the following applied to $\tilde{\pi}$, $\tilde{\Phi}$ (suppose otherwise ...)]

(2) Let $\mathcal{D}$:= irreducible component of the preimage $\bar{\Phi}^{-1}(\Gamma_m\backslash D_m)$.

Then $\mathcal{D}$ is defined $/\bar{k}$. (Note: $\mathcal{D}$ is algebraic by Cattani–Deligne–Kaplan.)

Proof: Consider the $\bar{k}$-spread $\mathcal{R}$ of an arbitrary $p \in \mathcal{D}(\mathbb{C})$. (This is the Zariski closure of the set of pts. $q \in \mathcal{D}(\mathbb{C})$ s.t. $X_q = {}^{\sigma}X_p$ for some $\sigma \in Aut(\mathbb{C}/\bar{k})$.) These $\{\sigma\}$ produce a continuous family of $\cong$'s $H^n_{dR}(X_p) \xrightarrow{\cong} H^n_{dR}(X_q)$ inducing (by AHC) $\cong$'s defined $/\mathbb{Q}$ of spaces of Hodge tensors

$\Rightarrow$ Hodge tensor spaces are constant (w.r.t. $\mathbb{Q}$-Betti structure)

$\underset{\mathcal{R} \text{ irred.}}{\Longrightarrow} \mathcal{R} \subset \mathcal{D}$

$\Rightarrow$ $\bar{k}$-spread of $\mathcal{D} = \mathcal{D}$. $\square$

Evidence for (2) (Voisin): Suppose $\tilde{T} \subset \mathcal{S}$ is an irred. subvariety$/\mathbb{C}$ such that (i) ${}^{\sigma}\tilde{T}$ = irred. component of Hodge locus of some $\alpha \in (F^m \cap \mathbb{V}_{\mathbb{Q}})_{t_0}$ (VHS attached to $H^{2m}(\mathcal{X}/\mathcal{S})$) & (ii) $\pi_1({}^{\sigma}\tilde{T}, t_0)$ fixes only the line generated by α

Then $\tilde{T}$ is defined $/\bar{k}$.

Sketch: (Except in trivial case, the hypotheses force $\dim \tilde{T} > 0$.) Extend α to ∇-flat family $/\tilde{T}$, so (by algebraicity of ∇ & invariance) ${}^{\sigma}\alpha$ is ∇-flat family $/{}^{\sigma}\tilde{T}$ $\underset{\text{(ii)}}{\Longrightarrow}$ ${}^{\sigma}\alpha = \lambda\beta$, β $\mathbb{Q}$-Betti (i.e. Hodge).

Consider $\bar{k}$-spread: by continuity, $\sigma \in Aut(\mathbb{C}/\bar{k}) \Rightarrow {}^{\sigma}[\alpha] = [\alpha]$. $\square$

(In fact, we have more:

$Q(\alpha,\alpha) = Q({}^{\sigma}\alpha, {}^{\sigma}\alpha) = \lambda^2 Q(\beta,\beta) \Rightarrow \lambda^2 \in \mathbb{Q} \underset{\text{continuity}}{\Rightarrow} {}^{\sigma}\alpha = \alpha$!)

3) Zero locus of normal function

MHS analogue of Hodge locus: vanishing of Ext^1 class
$\equiv$ presence of Hodge class

$$0 \to \underset{(-1)}{H} \to V \to \mathbb{Q}(0) \to 0$$

?

Let $\nu : \mathcal{S} \to \mathcal{J}(\mathcal{H})$ NF arising from family of primitive cycles on $\{X_s\}/\underline{\underline{k}}$.

$Z(\nu) :=$ 0-locus (algebraic by Brosnan-Pearlstein / Schnell / Kato-Nakayama-Usui)

<u>Proposition (Charles)</u>: Assume $\mathbb{H}_{\mathbb{C}}$ has no nonzero global sections $/Z(\nu)$.
Then $Z(\nu)$ is defined $/\bar{k}$.

<u>Sketch</u>: $Z_0 \subset Z(\nu)$ irred. component, $\mathfrak{z} =$ cycle$|_{Z_0}$
$\sigma \in Aut(\mathbb{C}/k)$, ${}^{\sigma}\mathfrak{z} :=$ cycle$|_{{}^{\sigma}Z_0}$.

Infinitesimal invariant of NF algebraic: so
its vanishing for $\nu_{\mathfrak{z}} \Rightarrow$ vanishing for $\nu_{({}^{\sigma}\mathfrak{z})}$.
$\Rightarrow \nu_{({}^{\sigma}\mathfrak{z})}$ lives in fixed part of $\mathcal{J}(\mathcal{H})|_{{}^{\sigma}Z_0}$.

∇ algebraic + $\mathbb{H}_{\mathbb{C}}|_{Z_0}$ no global sections $\Rightarrow \mathbb{H}_{\mathbb{C}}|_{{}^{\sigma}Z_0}$ none
$\Rightarrow$ fixed part $= \{0\}$
$\Rightarrow {}^{\sigma}Z_0 \subset Z(\nu)$

Now $Z(\nu)$ algebraic $\Rightarrow$ $<\infty$ components $\Rightarrow$ Z_0/finite ext. of k. $\square$

Connection of this to Bloch–Beilinson conjectures

K f.g. $/\bar{\mathbb{Q}}$; $\bar{\mathbb{Q}}$-spread gives $\mathfrak{Z} \subset \begin{matrix}\mathcal{X}\\ \downarrow \\ \mathcal{S}\end{matrix}$ and

$$\underline{\Psi} : CH^m(X/K) \xrightarrow{\cong} \operatorname{im}\left\{ CH^m(\bar{\mathcal{X}}/\bar{\mathbb{Q}}) \to \varinjlim_{\substack{U \subset_f \mathcal{S}/\bar{\mathbb{Q}} \\ \text{Zar. open}}} CH^m(\mathcal{X}_U) \right\}$$

$$\to \underbrace{\operatorname{im}\left\{ H^{2m}_{\mathcal{D}}(\bar{\mathcal{X}}^{an}_{\mathbb{C}}, \mathbb{Q}(m)) \to \varinjlim_{U} H^{2m}_{\mathcal{H}}((\mathcal{X}_U)^{an}_{\mathbb{C}}, \mathbb{Q}(m)) \right\}}.$$

Now there exists a Leray filtration on this ;

define $\mathcal{L}^i CH^m(X/K)$ by $\underline{\Psi}^{-1}(\mathcal{L}^i)$.

$Gr^i_{\mathcal{L}}$ invariants:	$[Z]$	$\nu_{\mathfrak{Z}}$	......
	$Gr^0_{\mathcal{L}}$	$Gr^1_{\mathcal{L}}$	$Gr^2_{\mathcal{L}}$ (more invariants) ....

$\Rightarrow$ to be in $\mathcal{L}^2$ is equiv. to $\nu_{\mathfrak{Z}} = 0$.

Clearly this $\Rightarrow AJ(Z) = 0$.

Proposition : The converse holds in general $\Leftrightarrow Z(\nu_{\mathfrak{Z}/\bar{\mathbb{Q}}})$ def'd $/\bar{\mathbb{Q}}$.
"$F^2_{BB} CH = \ker AJ$"

Sketch : same idea as last few pp. : spreading out a point in the zero locus should remain in the zero locus! $\square$

4) CM points

"lots of endomorphisms"

A HS (V,φ) is <u>CM</u> $\overset{\text{def.}}{\iff}$ M_φ (MTG) is abelian (i.e. an algebraic torus)

$\iff$ $\{\varphi\}$ itself is a ($\mathbb{Q}$-def'd) subdomain ("CM point")

These are always dense in D

<u>Construction</u>: L CM field of degree $2g$ (tot. imag. ext. of tot. real field)

$$\mathrm{Hom}(L,\mathbb{C}) = \{\theta_1,\ldots,\theta_g;\bar\theta_1,\ldots,\bar\theta_g\} = \underset{\text{choice}}{\coprod_{p+q=n}} \Pi^{p,q} \quad \text{s.t. } \overline{\Pi^{p,q}} = \Pi^{q,p}$$

$$V := L \circlearrowleft \text{ mult. by } \ell\in L \qquad (2g\text{-dim'l } \mathbb{Q}\text{-vect. sp.})$$

$$V_{\mathbb{C}} \cong L\otimes_{\mathbb{Q}}\mathbb{C} = \bigoplus_{\theta\in\mathrm{Hom}(L,\mathbb{C})} E_\theta(V_{\mathbb{C}})$$

(eigensp. on which ℓ acts as mult. by $\theta(\ell)$)

$\rightsquigarrow$ set $V^{p,q} := \bigoplus_{\theta\in\Pi^{p,q}} E_\theta(V_{\mathbb{C}})$;

the resulting HS we'll call $\underline{V_{(L,\Pi)}}$.

<u>Theorem</u>: (i) Any HS of this form is polarizable, and any polarizable CMHS decomposes as a $\oplus$ of these.

(ii) [Abdulali, 2006] Any polarized CMHS is of algebro-geometric origin.

[<u>Sketch</u>: let (L,Π) be as above, $\Theta(\Pi) :=$ set of CM types refining Π,

$$A := \bigtimes_{\Theta\in\Theta(\Pi)} A_{(L,\Theta)}^{\times m_\Theta}. \text{ Then } V_{(L,\Pi)} \underset{\text{sub-HS}}{\subset} H^n(A).]$$

Note that the HC is <u>not</u> known for "degenerate" CM ab. vars. (Weil)

(absolute Hodge thm. for ab. vars.)

Idea of pf of Deligne AH : [review from Laza's lectures]

① Start with $\mathcal{A} \to \mathcal{S}$ family of ab. vars. over a connected Shimura variety of Hodge type

② CM pts. dense in $\mathcal{S}$. By algebraicity of ∇, AHC for generic A_s reduces to AHC for CM ab. vars.

③ $\underbrace{\text{Fixed tensors of "absolute MTG"}}_{\underset{\text{MTG}}{\cup}\ (\dagger)}$ are AH $\Rightarrow$ suff. to find AH classes cutting out the MTG ($\Rightarrow$ ($\dagger$) an equality)

④ Weil Hodge classes are absolute. (these are the ones for which HC not known.)

Observation: When the tautological VHS / Shimura variety comes from (say) $\mathcal{H}^n_{X/\mathcal{S}}$, the $s \in \mathcal{S}$ with $H^n(X_s)$ CM are dense. [e.g., H^3 of CY vars. / ball quotient]. If Abdulali's inclusion $H^n(X_s) \subset H^n(\text{CM Ab. var. } A)$ is induced by an AH class in $H^{2n}(X_s \times A)$ [for each such s], then AHC holds for H^n of all X_s. //

André-Oort type Conj. (re. ∃ of density of CM pts. in a VHS)

(✱) $X \xrightarrow{\pi} \mathcal{S} / \bar{\mathbb{Q}}$, $\Phi : \mathcal{S} \to \Gamma \backslash D$ (image quasi-proj. algebraic by Sommese)

Conjecture: The Zariski closure of the set of CMHS in $\Phi(\mathcal{S})$ is a union of Shimura varieties. $\underbrace{\left(\begin{matrix} \uparrow \rho \\ D \end{matrix} \right)}$

Ex / $\Phi : \mathbb{P}^1 \backslash \{0,1,\infty\} \to \Gamma \backslash D_{(1,1,1,1)}$
MQ VHS

$\underset{\text{conj.}}{\Rightarrow}$ #(CM pts) $< \infty$.

5) Transcendence of periods

Ex / E = ell. curve$/\overline{\mathbb{Q}}$, with period ratio τ.

Then $[\mathbb{Q}(\tau):\mathbb{Q}] = \begin{cases} 2 & , \text{ if } H^1(E) \text{ CM} \\ \text{or} & \\ \infty & , \text{ if } MTG(\text{''}) = SL_2 \end{cases}$

(Put differently: if $H^1(E)$ is not contained in a proper subdomain of $\mathfrak{h} = D$, then it gives a period pt. whose spread is all of D.)

Conjecture (Grothendieck, André): In the setting $(*)$ above, let $p \in \mathcal{S}(\overline{\mathbb{Q}})$ & suppose that $\varphi \in D$ satisfies $\rho(\varphi) = \Phi(p)$. Then φ is very general in $D_{M_\varphi} = M_\varphi(\mathbb{R}).\varphi$, i.e. it is a point of max'l transc. deg. in the projective variety $\check{D}_{M_\varphi}$ (def'd $/\overline{\mathbb{Q}}$).

Remarks: (i) transcendental periods occurring should have arithmetic meaning. The conifold mirror quintic has type $(1,9,9,1)$, period ratio = quotient of two $\zeta_{10} + \{\Gamma(k/5)\}$ - linear combinations of ${}_4F_3$ special values.

where the singularity has been resolved

(ii) mixed HS analogue: Beilinson conj's on special values of L-functions.

Evidence for Groth. conj:

Schneider's Thm.: $E/\overline{\mathbb{Q}}$ and $\tau\in\overline{\mathbb{Q}} \Rightarrow E$ has CM (equiv, $[\mathbb{Q}(\tau):\mathbb{Q}]=2$)

[so: no cubic $\tau\in\overline{\mathbb{Q}}$ unless E transcendental)

generalization

(Tretkoff) ||

Cohen-Shiga-Wolfart: $\mathcal{A}\to\mathcal{S}$ family $/\overline{\mathbb{Q}}$ of ab. vars. over Shimura variety of PEL type (= MTD for HS of level one cut out by 2-tensors)

(π $\uparrow\rho$ $D\subset\check{D}$)

Thm $\boxed{A_s/\overline{\mathbb{Q}} \text{ \& } \varphi\in\check{D}(\overline{\mathbb{Q}}) \Rightarrow A_s \text{ has CM}}$

i.e. $s\in\overline{\mathbb{Q}}$, $s=\rho(\varphi)$

Tretkoff: gen. to fam. of CY / Shimura variety

What is behind all this?

Wüstholz analytic subgroup thm.: (the version is state is a corollary — his thm. is much more general)

Let G = connected $\overline{\mathbb{Q}}$-algebraic gp., $\mathfrak{h}\subset\mathfrak{g}_{\mathbb{C}}$ a proper subspace

with $\begin{cases}\mathfrak{h} \text{ defined } /\overline{\mathbb{Q}}\\ 0\neq v\in\mathfrak{h}\cap\ker(\exp).\end{cases}$ Then $\exists$ closed connected alg.

subgroup $G_0/\overline{\mathbb{Q}}\le G$ s.t. $v\in\mathfrak{g}_{0,\mathbb{C}}\subset\mathfrak{h}$.

Wüstholz $\Rightarrow$ Schneider: $E/\overline{\mathbb{Q}}$, $\omega\in H^0(E,\Omega^1_{E/\overline{\mathbb{Q}}})$, $\Lambda:=\mathbb{Z}\langle\pi_0,\pi_1\rangle$, $\tau:=\frac{\pi_1}{\pi_0}\in\overline{\mathbb{Q}}$.

Have short exact seq. $0\to\Lambda^2\to\mathbb{C}^2\to E^2=:G\to 0$, so $v:=(\pi_0,\pi_1)\in\ker(\exp)$.

Put $\mathfrak{h}:=\mathbb{C}\langle v\rangle\subseteq\mathfrak{g}_{\mathbb{C}} \underset{\text{Wüstholz}}{\Longrightarrow} \exists\, G_0/\overline{\mathbb{Q}}\subset E\times E$ s.t. $v\in\mathfrak{g}_{0,\mathbb{C}}\subset\mathbb{C}\langle v\rangle \Rightarrow \mathfrak{g}_{0,\mathbb{C}}=\mathbb{C}\langle v\rangle$

$\Rightarrow G_0=\exp(\mathfrak{h})$ is closed $\Rightarrow$ mult. by τ gives a correspondence $\Rightarrow E$ has CM. $\square$