

Arithmetic of period maps, I

①

OVERVIEW

The last 40 years have seen the development of rich theories of Hodge theory of the boundary and symmetries of Hodge structures
(next talk) (this talk)

which are themes of this conference. Recent work of Griffiths et al \Rightarrow the Hodge Conj. can now be stated in terms of asymptotics of such period integrals; while Mumford-Tate (i.e. symmetry) groups of HS's have led to proofs of Hodge & Beilinson-Hodge conjectures in special cases. The other closely related theme of the conference is the arithmetic of periods, of which Euler's work on rel's between multiple zeta values is an early example.

At the heart of current thought on the Hodge Conjecture, two intertwined programs have emerged:

- ① The approach just referred to reduces the conj. to \exists of singularities for certain several-variable admissible normal functions obtained from Hodge classes. While

(2)
This criterion pertains a priori to degenerations of normal functions, a result of Schnell reveals the importance of estimates on the dimension of their zero-loci, which have recently been proven algebraic (generalizing Cattani-Dezhaire-Koplan's result on the locus of Hodge classes).

(2) Another approach, championed by Voisin, is to break the Hodge Conj. into 2 pieces: first, to show that the locus of Hodge classes in a VHS arising from alg. geom/ \mathbb{Q} is defined over a number field; then second, to prove the Hodge Conjecture on arithmetic varieties (i.e. those/ \mathbb{Q}). Key to this approach is showing that a given family of Hodge classes is absolute, extending Deligne's theorem for abelian varieties discussed in Laza's 2nd lecture.

1) Spreads of period maps

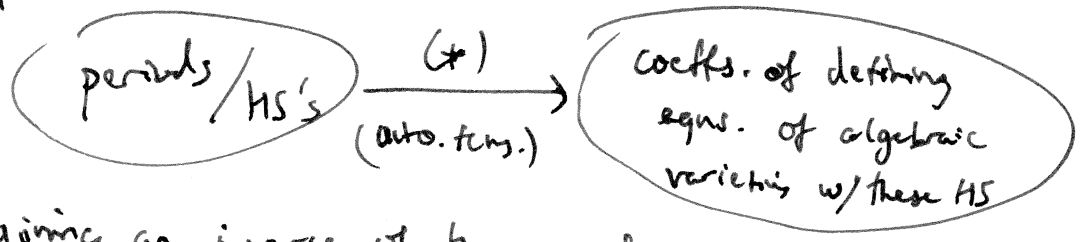
Let $D :=$ period (or $M-T$) domain
 $= \mathbb{C}(\mathbb{R}) / \mathcal{H}$

$\mathcal{J} \subset \Omega^*(D)$ IPR (pulls back to 0 under local lifting of any VHS, by defn.)

Case 1: $\mathcal{J} = 0$ ($\Rightarrow D$ Heron. sym.)

$\Gamma \leq G(\mathbb{Q})$ arithmetic $\implies \mathbb{P}^1/D$ has a proj. embedding by automorphic functions, & parametrizes a VHS (which is known to be not outside E_6/E_7 cases) \uparrow comes from A.T.

not the case: automorphic fens. provide the highly transcendental passage from

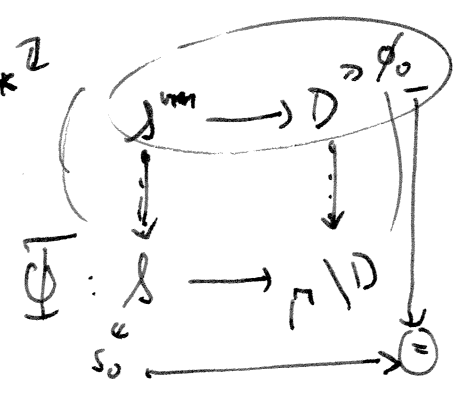
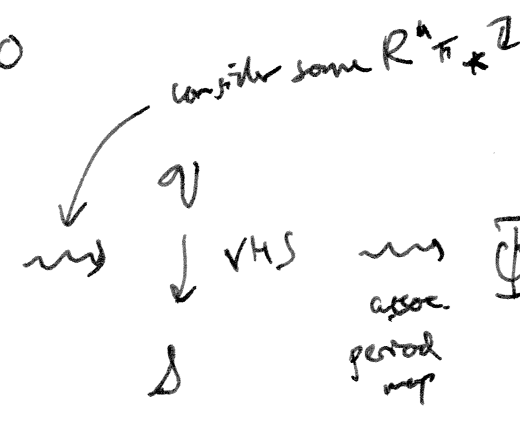
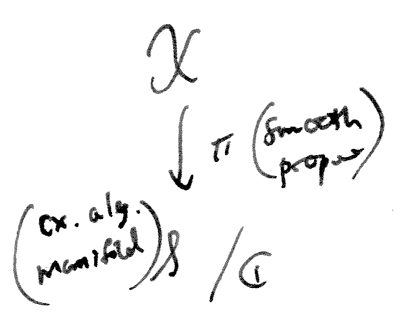


giving an inverse of the period map.

Ex / $\tau \longmapsto g_4(\tau), g_6(\tau)$

(Other ex's: Chingher-Doran for lattice-polarized $K3$ s (type IV)
Molzenfel/Shtiza for period curves (b41))

Case 2: $\Delta \neq 0$



lift gives integral manifold through ϕ_0 .

Spread out: since $\mathcal{X} \rightarrow \mathcal{S}$ actually def'd / some K f.g. / $\overline{\mathbb{Q}}$,

$\exists \mathcal{S}' / \overline{\mathbb{Q}}$ affine & very general $p \in \mathcal{S}'(\mathbb{C})$ s.t. $ev_p: \overline{\mathbb{Q}}(\mathcal{S}') \xrightarrow{\cong} K$.

Pulling back the defining equations under ev_p & clearing denominators yields $\tilde{\mathcal{X}} \xrightarrow{\tilde{\pi}} \tilde{\mathcal{S}} / \overline{\mathbb{Q}}$

The resulting period map $\tilde{\Phi}$ (assoc. to $\tilde{\pi}$) still gives an integral manifold of the IPR through ϕ_0 (this is paper in \mathcal{D} since $\Delta \neq 0$). Since there are only countably many families of alg. vars. defined / $\overline{\mathbb{Q}}$, only countably many integral manifolds of the IPR come from AG \implies nothing like (*) (on the last pg.)

Problem: So, the "motivic" HS's in \mathcal{D} , as a set, have measure 0.

Find one explicit HS not in that set! (this is open!)

(5)

2) Absoluteness of Hodge classes

$X = \text{smooth proj. var}/k \subset \mathbb{C}$,

$$Hq^m(X) = F^m H^{2m}(X_{\mathbb{C}}^{\text{an}}, \mathbb{C}) \cap H^{2m}(X_{\mathbb{C}}^{\text{an}}, \mathbb{Q}(m)).$$

Let $\sigma \in \text{Aut}(\mathbb{C})$, identify $F^m H_{(de)}^{2m}(X_{\mathbb{C}}^{\text{an}}, \mathbb{C}) \cong \underbrace{H_{2\text{an}}^{2m}(X_{\mathbb{C}}^{\text{an}}, \mathcal{O}_{X_{\mathbb{C}}^{\text{an}}}^{\otimes 2m})}_{\substack{\text{alg. diff'l forms} \\ \otimes \mathbb{C}}}$

$\rightarrow \sigma_* : F^m H^{2m}(X_{\mathbb{C}}^{\text{an}}, \mathbb{C}) \rightarrow F^m H^{2m}(\underbrace{\sigma(X)}_{\mathbb{C}}^{\text{an}}, \mathbb{C})$
 (obtained by letting σ act on coeffs. of defining eqns. of X)

Define

$$AHq^m(X) := \left\{ \xi \in Hq^m(X) \mid \sigma_*(\xi) \in Hq^m(X) \ (\forall \sigma \in \text{Aut}(\mathbb{C})) \right\}$$

$$\rightarrow \text{cl}(Z^m(X)) \subset AHq^m(X) \xrightarrow{\text{AHC}} Hq^m(X)$$

(AHC)

Theorem (Deligne, 1982): AHC holds if X is an abelian variety.

(Deligne needed this in order to establish \exists of canonical models for Shimura varieties of Hodge type.)

Consequences of AHC for $X \xrightarrow{\pi} S/k$, $\Phi : S \rightarrow \prod \mathbb{D}$ as above:

(1) Suppose Φ factors through $\prod_m \mathbb{D}_m$; then so does $\tilde{\Phi}$.

[Pf: this is a corollary of the following applied to $\tilde{\pi}, \tilde{\mathbb{Q}}$ (suppose otherwise...)]

(2) Let $D :=$ irreducible component of the preimage $\bar{\Phi}^{-1}(T_m | D_m)$.
 Then D is defined / \bar{k} . (Note: D is observed by Göttsche-Deligne-Kaplan.)

Proof: Consider the \bar{k} -spread \mathcal{Q} of an arbitrary $p \in D(\mathbb{C})$.

(This is the Zariski closure of the set of pts. $q \in D(\mathbb{C})$ s.t.

$X_q = \sigma X_p$ for some $\sigma \in \text{Aut}(\mathbb{C}/\bar{k})$.) These pts produce a

continuous family of \cong 's $H_{\mathbb{Q}\mathbb{R}}^n(X_p) \cong H_{\mathbb{Q}\mathbb{R}}^n(X_q)$ inducing (by AHC)
 \cong 's defined / \mathbb{Q} of spaces of Hodge tensors

\Rightarrow Hodge tensor spaces are constant (w.r.t. \mathbb{Q} -Betti structure)

$\Rightarrow \mathcal{Q} \subset D$
 \mathcal{Q} irred.

$\Rightarrow \bar{k}$ -spread of $D = \mathcal{Q}$.

□

Evidence for (2) (Voisin): Suppose $T \subset S$ is an irred. subvariety / \mathbb{C}
 such that (i) $\sigma_T =$ irred. component of Hodge locus of some

$\alpha \in (F^m \cap V_{\mathbb{Q}})_{t_0}$ (VHS attached to $H^{2m}(X/S)$)

(ii) $\pi_1(\sigma_T, t_0)$ fixes only the line generated by α

Then T is defined / \bar{k} .

Sketch: (Except in trivial case, the hypotheses force $\dim T > 0$.)

Extend α to \mathbb{D} -flat family / σ_T , α (by algebraicity of \mathbb{D} & invariance)

σ_T is \mathbb{D} -flat family / $\sigma_T \Rightarrow \sigma \alpha = \lambda \beta$, β \mathbb{Q} -Betti (i.e. Hodge).

Consider \bar{k} -spread: by continuity, $\sigma \in \text{Aut}(\mathbb{C}/\bar{k}) \Rightarrow \sigma[\alpha] = [\alpha]$. □

(In fact, we have more:

$$Q(\alpha, \alpha) = Q(\sigma \alpha, \sigma \alpha) = \lambda^2 Q(\beta, \beta) \Rightarrow \lambda^2 \in \mathbb{Q} \Rightarrow \sigma \alpha = \alpha \quad ! \quad)$$

\uparrow continuity

3) Zero locus of hermit function

(7)

MHS analogue of Hodge locus: vanishing of Ext^1 class
 \equiv presence of Hodge class

$$0 \rightarrow H \rightarrow V \rightarrow \mathbb{Q}(a) \rightarrow 0$$

$\textcircled{-1}$ $\uparrow \dots$

let $v: S \rightarrow Q(H)$ NF arising from family of primitive cycles on $\{X_s\}/k$.

$Z(v) := 0$ -locus (algebraic by Brosnan-Pearlstein/Schnell/Kato-Nakayama-Urui)

Proposition (Charles): Assume H_0 has no nonzero global sections/ $Z(v)$.
 Then $Z(v)$ is defined/ \bar{k} .

Sketch: $Z_0 \subset Z(v)$ irred. component, $z := \text{cycl}|_{Z_0}$
 $\sigma \in \text{Aut}(F/k)$, $\sigma z := \text{cycl}|_{\sigma Z_0}$.

Infinitesimal invariant of NF algebraic: so
 its vanishing for $v_z \Rightarrow$ vanishing for $v_{(\sigma z)}$.
 $\Rightarrow v_{(\sigma z)}$ lines in fixed part of $Q(H)|_{\sigma Z_0}$.

∇ algebraic + $H_0|_{Z_0}$ no global sections $\Rightarrow H_0|_{Z_0}$ none
 \Rightarrow fixed part = $\{0\}$
 $\Rightarrow \sigma Z_0 \subset Z(v)$

Now $Z(v)$ algebraic \Rightarrow <do components $\Rightarrow Z_0$ /dense ext. of k . □

Connection of this to Bloch-Beilinson conjectures

K f.g. $\bar{\mathbb{Q}}$; $\bar{\mathbb{Q}}$ -spread gives $Z \subset X$
 \downarrow
 \downarrow
 \downarrow and

$$\Psi: CH^m(X/K) \xrightarrow{\cong} \text{im} \left\{ CH^m(\bar{X}/\bar{\mathbb{Q}}) \rightarrow \varinjlim_{\substack{U \subset S/\bar{\mathbb{Q}} \\ \text{zar. open}}} CH^m(X_U) \right\}$$

$$\rightarrow \text{im} \left\{ H_{\mathbb{Q}}^{2m}(\bar{X}_0^{an}, \mathbb{Q}(m)) \rightarrow \varinjlim_U H_{\mathbb{Q}}^{2m}((X_U)_0^{an}, \mathbb{Q}(m)) \right\}$$

Now there exists a Leray filtration on this;

define $L^i CH^m(X/K)$ by $\Psi^{-1}(L^i)$.

$Gr_{\mathbb{Z}}^i$ invariants:

$[z]$	v_z	...
$Gr_{\mathbb{Z}}^0$	$Gr_{\mathbb{Z}}^1$	$Gr_{\mathbb{Z}}^2$ (more mysterious) ...

\Rightarrow to be in L^2 is equiv. to $v_z = 0$.

Clearly this $\Rightarrow AJ(z) = 0$.

Proposition: The converse holds in general $\Leftrightarrow Z(v_z/\bar{\mathbb{Q}}) \text{ def'd } / \bar{\mathbb{Q}}$.
 "F_{BB}² CH = ker AJ"

Sketch: same idea as last few pp.: spreading out a point in the zero locus should remain in the zero locus. □

4) CM points

(9)

A HS is $(M \stackrel{\text{def.}}{=} M_\rho)$ is abelian (i.e. an algebraic torus)
 (V, g) \Leftrightarrow $\{g\}$ itself is a subdomain (0-dim!) ("CM point")
 "lots of endomorphisms"

These are always dense in D

Construction: L CM field of degree $2g$ (tot. imag. ext. of tot. real field)

$$\text{Hom}(L, \mathbb{C}) = \{\theta_1, \dots, \theta_g; \bar{\theta}_1, \dots, \bar{\theta}_g\} = \coprod \Pi^{p, q}$$

$2g$ -dim \mathbb{Q} -vect. sp. $V := L \xrightarrow{\text{mult. by } l \in L}$

choice $\Pi^{p, q}$ s.t. $\bar{\Pi}^{p, q} = \Pi^{q, p}$

$$V_{\mathbb{C}} \cong L \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{\theta \in \text{Hom}(L, \mathbb{C})} E_{\theta}(V_{\mathbb{C}})$$

(eigensp. on which l acts as mult. by $\theta(l)$)

\rightarrow Set $V^{p, q} := \bigoplus_{\theta \in \Pi^{p, q}} E_{\theta}(V_{\mathbb{C}})$;

the resulting HS we'll call $V_{(L, \Pi)}$

Theorem: (i) Any HS of this form is polarizable, and any polarizable CMHS decomposes as a \bigoplus of these.

(ii) [Abdunko, 2006] Any polarized CMHS is of algebro-geometric origin.

[Sketch: let (L, Π) be as above, $\Theta(\Pi) :=$ set of CM types refining Π ,

$$A := \prod_{\Theta \in \Theta(\Pi)} A_{(L, \Theta)}^{\times m_{\Theta}}. \text{ Then } V_{(L, \Pi)} \subset \text{subHS } H^n(A).]$$

Note that the HC is not known for "degenerate" CM ab. vars. (Weil)

(absolute Hodge thm. for ab. vars.)

Idea of pf of Deligne AH: (review from Lazard's lecture)

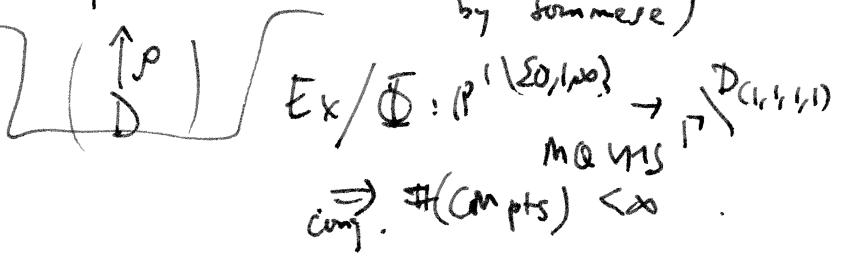
- ① Start with $A \rightarrow S$ family of ab. vars. over a connected Shimura variety of Hodge type
- ② CM pts. dense in S . By algebraicity of \mathcal{V} , AHC for generic A_s reduces to AHC for CM ab. vars.
- ③ Fixed tensors of "absolute MTG" are AH \Rightarrow suff. to find AH classes cutting out the MTG (\Rightarrow (†) an equality)
- ④ Weil Hodge classes are absolute. (these are the ones for which HC not known.)

Observation: When the tautological VMS / Shimura variety comes from (say) $H^n_{X/S}$, the set S with $H^n(X_s)$ CM are dense. (eg., H^3 of CY vars. / ball quotient). If Abels's inclusion $H^n(X_s) \subset H^n(\text{CM Ab. var. } A)$ is induced by an AH class in $H^{2n}(X_s \times A)$ [for each such s], then AHC holds for H^n of all X_s . //

Andre-Oort type Conj. (re. \exists density of CM pts. in a VMS)

(*) $X \xrightarrow{\pi} S/\mathbb{Q}$, $\mathcal{D} : S \rightarrow \mathbb{P}^D$ (image quasi-proj. algebraic by Serre)

Conjecture: The Zariski closure of the set of CM pts in $\mathcal{D}(S)$ is a union of Shimura varieties.



5) Transcendence of periods

(1)

Ex / $E = \text{ell. curve} / \bar{\mathbb{Q}}$, with period ratio τ .

$$\text{Then } [\mathbb{Q}(\tau) : \mathbb{Q}] = \begin{cases} 2, & \text{if } H^1(E) \text{ CM} \\ \text{or} \\ \infty, & \text{if } \text{MTC}(\tau) = \text{SL}_2 \end{cases}$$

(Put differently: if $H^1(E)$ is not contained in a proper subdomain of $h = D$, then it gives a period pt. whose spread is all of D.)

Conjecture (Grothendieck, André): In the setting (*) above, let

$p \in \mathcal{S}(\bar{\mathbb{Q}})$ & suppose that $\rho \in D$ satisfies $\rho(\rho) = \bar{\Phi}(\rho)$.

Then ρ is very general in $D_{M_p} = M_p(\mathbb{R}) \cdot \rho$, i.e. it is a point of max'l transc. deg. in the projective variety

\check{D}_{M_p} (det'd / $\bar{\mathbb{Q}}$).

Remarks: (i) transcendental periods occurring should have arithmetic meaning. The conifold mirror quintic

where the singularity has been resolved

has type $(1, 9, 9, 1)$, period ratio = quotient of two $S_{10} + \{F(\frac{1}{4})\}$ - linear combinations of $4 F_3$ special values.

(ii) mixed HS dialogue: Beilinson conj: on special values of L-functions.

Evidence for Groth. conj:

Schneider's Thm.: $E/\bar{\mathbb{Q}}$ and $\tau \in \bar{\mathbb{Q}} \Rightarrow E$ has CM (equiv, $[\mathbb{Q}(\tau):\mathbb{Q}]=2$)

Trotteroff } generalization [So: no cube $\tau \in \bar{\mathbb{Q}}$ unless E transcendental]

Cohen-Shige-Wolfart: $\mathcal{V} \rightarrow \mathcal{S}$ family $\sqrt{\mathbb{Q}}$ of ab. vars. over Shimura variety of PEL type (= MFD for HS of level one cut out by 2-torsors)

Then $A_s/\bar{\mathbb{Q}}$ of $\varphi \in \check{D}(\bar{\mathbb{Q}}) \Rightarrow A_s$ has CM
 i.e. $s \in \bar{\mathbb{Q}}$, $s = p(\varphi)$

Trotteroff: gen. to fam. of CY / Shimura variety

What is behind all this?

Wüstholz analytic subgroup thm.

(the version I state is a corollary - but thm. is much more general)

Let $G =$ connected $\bar{\mathbb{Q}}$ -algebraic gp., $\mathfrak{h} \subset \mathfrak{a}_{G, \mathbb{C}}$ a proper subspace with $\begin{cases} \mathfrak{h} \text{ defined } / \bar{\mathbb{Q}} \\ 0 \neq v \in \mathfrak{h} \cap \ker(\exp) \end{cases}$ Then \exists closed connected alg.

Subgroup $G_0/\bar{\mathbb{Q}} \leq G$ s.t. $v \in \mathfrak{a}_{G_0, \mathbb{C}} \subset \mathfrak{h}$.

Wüstholz \Rightarrow Schneider: $E/\bar{\mathbb{Q}}$, $\omega \in H^0(E, \Omega_{E/\bar{\mathbb{Q}}}^1)$, $\Lambda := \mathbb{Z}\langle \tau_0, \tau_1 \rangle$, $\tau := \frac{\tau_1}{\tau_0} \in \bar{\mathbb{Q}}$.

Have short exact seq. $\Lambda^2 \rightarrow \mathbb{C}^2 \rightarrow E^2 =: G \rightarrow 0$, so $v := (\tau_0, \tau_1) \in \ker(\exp)$.

Put $\mathfrak{h} := \mathbb{C}\langle v \rangle \subset \mathfrak{a}_{G, \mathbb{C}} \xRightarrow{\text{Wüstholz}} \exists G_0/\bar{\mathbb{Q}} \subset \text{Ext} \text{ s.t. } v \in \mathfrak{a}_{G_0, \mathbb{C}} \subset \mathbb{C}\langle v \rangle \Rightarrow \mathfrak{a}_{G_0, \mathbb{C}} = \mathbb{C}\langle v \rangle$

$\Rightarrow G_0 = \exp(\mathfrak{h})$ is closed \Rightarrow mult by τ gives a correspondence $\Rightarrow E$ has CM. \square