PROBLEM SET 13

(Hand in all. This is the last problem set.)

(1) [Jacobson p. 188#1] Determine the structure of $\mathbb{Z}^3/K$, where K is generated by $f_1 = (2,1,-3)$, $f_2 = (1,-1,2)$. (This is a warm-up for the next problem.)

(2) Let G be the abelian group determined by generators V, W, X, Y, Z and relations

$$\begin{aligned} V - 7W + 14Y - 21Z &= 0 \\ 15V - 7W - 2X + 10Y - 5Z &= 0 \\ 7V - 3W - 2X + 6Y - 9Z &= 0 \\ V - 3W + 2Y - 9Z &= 0. \end{aligned}$$

Find the rank and torsion invariants (invariant factors) by putting an appropriate matrix in normal form. Write G in the form (IV.C.10) and IV.C.13.

(3) [Jacobson p. 181 #2] Find a base for the submodule of $\mathbb{Q}[\lambda]^3$ generated by $f_1 = (2\lambda - 1, \lambda, \lambda^2 + 3)$, $f_2 = (\lambda, \lambda, \lambda^2)$, and $f_3 = (\lambda + 1, 2\lambda, 2\lambda^2 - 3)$. [Hint: use the algorithm from the proof of IV.C.2.]

(4) [Jacobson p. 185 #2] Find a normal form for the matrix

$$A = \begin{pmatrix} \lambda - 17 & 8 & 12 & -14 \\ -46 & \lambda + 22 & 35 & -41 \\ 2 & -1 & \lambda - 4 & 4 \\ -4 & 2 & 2 & \lambda - 3 \end{pmatrix}$$

in $M_4(\mathbb{Q}[\lambda])$, λ an indeterminate. [Optional: also find invertible matrices P and Q such that PAQ is in normal form. This is a bit labor intensive to write up, but you should know how to do it.]

(5) [Jacobson p. 186 #3] Determine the invariant factors of

$$A = \begin{pmatrix} \lambda + 1 & 2 & -6 \\ 1 & \lambda & -3 \\ 1 & 1 & \lambda - 4 \end{pmatrix}$$

both by putting it in normal form (to get the d_i directly), and by using $d_1 = \Delta_1$, $d_2 = \frac{\Delta_2}{\Delta_1}$, $d_3 = \frac{\Delta_3}{\Delta_2}$ (without putting it in normal form).

(6) In problems (4) and (5) $A = \lambda \mathbb{1} - B$ for $B \in M_n(\mathbb{Q})$. In each case, determine the minimal polynomial of B, as well as its rational and Jordan canonical forms.

(7) [Jacobson p. 186 #10] Let R be a ring and define the elementary matrix $T_{ij}(a)$, $i \neq j$, $a \in R$, as in the notes. Verify the four *Steinberg relations*
  - (i) $(T_{ij}(a))^{-1} = T_{ij}(-a)$.
  - (ii) $T_{ij}(a)T_{ij}(b) = T_{ij}(a+b)$.
  - (iii) $[T_{ij}(a), T_{jk}(b)] = T_{ik}(ab)$ if $k \neq i$ (where the commutator is $[x,y] := x^{-1}y^{-1}xy$).
  - (iv) $[T_{ij}(a), T_{k\ell}(b)] = 1$ if $j \neq k$, $i \neq \ell$.

(8) [Jacobson p. 188 #2] Determine the structure of $M = \mathbb{Z}[\mathbf{i}]^3/K$ where K is generated by $f_1 = (1,3,6)$, $f_2 = (2+3\mathbf{i}, -3\mathbf{i}, 12-18\mathbf{i})$, and $f_3 = (2-3\mathbf{i}, 6+9\mathbf{i}, -18\mathbf{i})$. In particular, show that M is finite (of order 352512). [Hint: this is similar to problem (2) in that you need to write down a matrix A and reduce it to normal form. Note that $\mathbb{Z}[\mathbf{i}]$ is Euclidean (you can just use the absolute value).]

(9) [Jacobson p. 193 #1] Let $R = \mathbb{R}[\lambda]$ and suppose M is a direct sum of cyclic R-modules whose order ideals are the ideals generated by the polynomials $(\lambda-1)^3$, $(\lambda^2+1)^2$, $(\lambda-1)(\lambda^2+1)^4$, and $(\lambda+2)(\lambda^2+1)^2$. Determine the elementary divisors and invariant factors of M.

(10) Determine the number of non-isomorphic abelian groups of order 900, and find the invariant factors d_i (in the structure theorem) for each group. [Hint: use IV.C.13-14, then convert to the form (IV.C.10).]

(11) [Jacobson p. 202 #4] Verify that the characteristic polynomial of

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & -2 & 0 & 1 \\ -2 & 0 & -1 & -2 \end{pmatrix}$$

is a product of linear factors in $\mathbb{Q}[\lambda]$. Determine the rational and Jordan canonical forms for A in $M_4(\mathbb{Q})$. Also find matrices which show that A is similar to these canonical forms.

(12) [Jacobson p. 202 #8] Prove that any nilpotent matrix in $M_n(\mathbb{F})$ is similar to a matrix of the form

$$\begin{pmatrix} N_1 & & \\ & \ddots & \\ & & N_s \end{pmatrix}$$

where the N_i are blocks of the form

$$\begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}.$$

(13) Consider the $n \times n$ cyclic permutation matrix

$$P_n := \begin{pmatrix} 0 & 1 & 0 & & \\ & 0 & 1 & \ddots & \\ & & 0 & \ddots & 0 \\ & & & \ddots & 1 \\ 1 & & & & 0 \end{pmatrix}$$

(where entries not shown are zero) over any field $\mathbb{F}$. Find the normal form of $\lambda I_n - P_n$. Use this to do Jacobson p. 202 #12, in particular for P_7 over $\mathbb{Z}_7$. In contrast, what happens for P_6 over $\mathbb{Z}_7$?