Lecture 9: Continuation and multivaluedness

I. Exp and log

Recall that we have holomorphic functions

$$\exp(\underset{\underset{x+iy}{\|}}{z}) := e^{x}\cos(y) + i e^{x}\sin(y) \in \mathcal{H}ol(\mathbb{C})$$

$$\log(z) := \log|r| + i\arg(z) \in \mathcal{H}ol(\mathbb{C}\setminus\mathbb{R}_{\leq 0}),$$

which was proved by checking the C-R equations

Now write

$$\widetilde{\exp}(z) := \eta_{\infty}\Big(\sum_{n\geq 0}\frac{T^n}{n!}\Big) = \eta(\exp(T))$$

$$\widetilde{\log}(1-z) := \eta_1\Big(-\sum_{n\geq 1}\frac{T^n}{n}\Big) = \eta(\underbrace{\log(1-T)}_{f(T)}) \;;$$

we also had

$$\cos(z) := \eta\Big(\overbrace{\sum_{k\geq 0}\frac{(-1)^k T^{2k}}{(2k)!}}^{C(T)}\Big), \quad \sin(z) := \eta\Big(\overbrace{\sum_{\ell\geq 0}\frac{(-1)^{\ell} T^{2\ell+1}}{(2\ell+1)!}}^{S(T)}\Big).$$

The formal relation

$$\exp(iT) = C(T) + iS(T)$$

$$\Rightarrow \quad \widetilde{\exp}(iy) = \cos(y) + i\sin(y) = \exp(iy).$$

By Calculus, we also have $\quad (x, y \in \mathbb{R})$

$$\widetilde{\exp}(x) = \exp(x).$$

Now in $\mathbb{C}[[T]]$

$$\begin{aligned}
\exp((\gamma_1+\gamma_2)T) &= \sum_{n\geq 0} \frac{(\gamma_1 T + \gamma_2 T)^n}{n!} \\
&= \sum_{n\geq 0} \sum_{k=0}^{n} \frac{(\gamma_1 T)^k (\gamma_2 T)^{n-k}}{k!(n-k)!} \\
&= \left(\sum_{k\geq 0} \frac{(\gamma_1 T)^k}{k!}\right)\left(\sum_{l\geq 0} \frac{(\gamma_2 T)^l}{l!}\right) \\
&= \exp(\gamma_1 T)\exp(\gamma_2 T),
\end{aligned}$$

So applying η

$$\Longrightarrow \quad \widetilde{\exp}((\gamma_1+\gamma_2)z) = \widetilde{\exp}(\gamma_1 z)\widetilde{\exp}(\gamma_2 z)$$

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$z=1$

$$\Longrightarrow \quad \widetilde{\exp}(z) = \widetilde{\exp}(x)\widetilde{\exp}(iy) = \exp(x)\exp(iy) = \exp(z),$$

proving

$$\boxed{\exp(z) \in \mathrm{An}(\mathbb{C}).}$$

To do the same for log, by Calculus we know that

$$\widetilde{\log}(1-z) = \log(1-z) \quad \text{for } \underset{\text{real.}}{z = x} \in (-1,1)$$

Furthermore, $\exp(\log(1-x)) = 1-x$ there, and so the formal power series

$$(*) \qquad (\exp \circ f)(T) - 1 + T$$

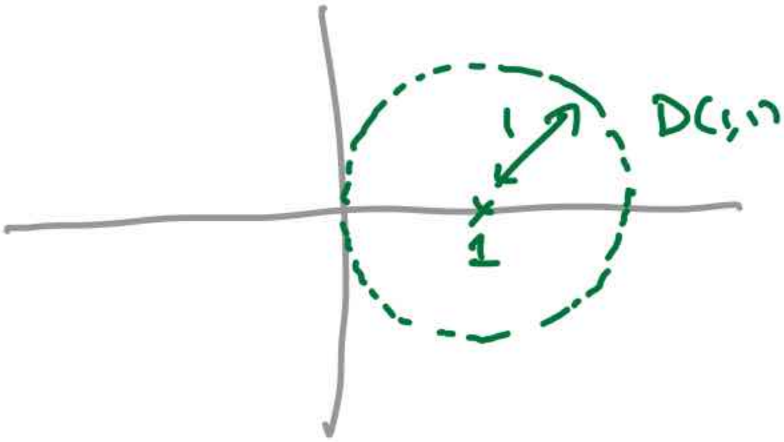
is sent by η to a function on D_1 which is identically zero on $(-1,1)$. Since $\{0\} \in \operatorname{acc}(-1,1)$, by Lecture 6 $(*)$ is the zero power series, and so

$$\exp(\widetilde{\log}(1-z)) = 1-z \quad \text{on } |z|<1.$$

We can repeat this for $\widetilde{\log} \circ \exp$ on some sufficiently small disk, and so

$$\widetilde{\log}(z) = \widetilde{\log}\big(|z| \exp(i \arg z)\big) \quad \exp(\log|z|)$$
$$= \widetilde{\log}\big(\exp(\log|z| + i\arg z)\big)$$
$$= \log|z| + i\arg(z)$$
$$= \log(z),$$

on some neighborhood of $z=1$ (and hence on all of $D(1,1)$).



We can do the same thing for $\log(z)$ on any disk in $\mathbb{C}\setminus\mathbb{R}_{\le 0}$; if the center is z_0, write

$$\frac{1}{z} = \frac{1/z_0}{1+\frac{z-z_0}{z_0}} = \frac{1}{z_0}\left(1+\frac{1}{z_0}(z-z_0)+\frac{1}{z_0^2}(z-z_0)^2+\cdots\right),$$

integrate, and add $\log(z_0)$ as constant. If $z_0 \in D(1,1)$, for example, we have that the resulting $\widetilde{\widetilde{\log}}(z)$ agrees with $\widetilde{\log}(z)$ at z_0 and has the same derivative $1/z$ hence by the above properties of analytic functions must agree on $D(1,1) \cap D(z_0, |z_0|)$. In this way one shows that

$$\boxed{\log(z) \in \mathcal{A}n(\mathbb{C} \setminus \mathbb{R}_{\leq 0}).}$$

Finally, from

$$\exp(\log \alpha\beta - \log\alpha - \log\beta) = \frac{\exp(\log \alpha\beta)}{(\exp\log\alpha)(\exp\log\beta)} = \frac{\alpha\beta}{\alpha\cdot\beta} = 1,$$

we have

$$\log(\alpha\beta) \equiv \log\alpha + \log\beta \pmod{2\pi i \mathbb{Z}}.$$

Out of this, we can get even more analytic functions:

Example By composing analytic functions,

$$\boxed{z^{\alpha} := \exp(\alpha \log(z)) \in \mathcal{A}n(\mathbb{C} \setminus \mathbb{R}_{\leq 0})}$$

Here, $\mathbb{R}_{\leq 0}$ can be replaced by any slit going from 0 to ∞.

Alternatively, instead of deleting the slit, you can think of z^α as being ambiguous by $\{\exp(2\pi i \alpha \mathbb{Z})\}$, which consists of finitely many values if and only if $\alpha \in \mathbb{Q}$. //

Example // Since $\cos(z)$ and $\sin(z)$ belong to $\mathrm{An}(\mathbb{C})$ by construction, any identity which is true on the level of power series holds also for the functions.

In particular,

$$\underbrace{\cos(z)}_{w} = \frac{1}{2}\left(\underbrace{\exp(iz)}_{\varepsilon} + \exp(-iz)\right)$$

$$\Rightarrow \quad 2w = \varepsilon + \varepsilon^{-1}$$

$$\Rightarrow \quad 0 = \varepsilon^2 - 2w\varepsilon + 1$$

$$\Rightarrow \quad \varepsilon = \frac{2w \pm \sqrt{4w^2 - 4}}{2} = w \pm \sqrt{w^2 - 1}$$

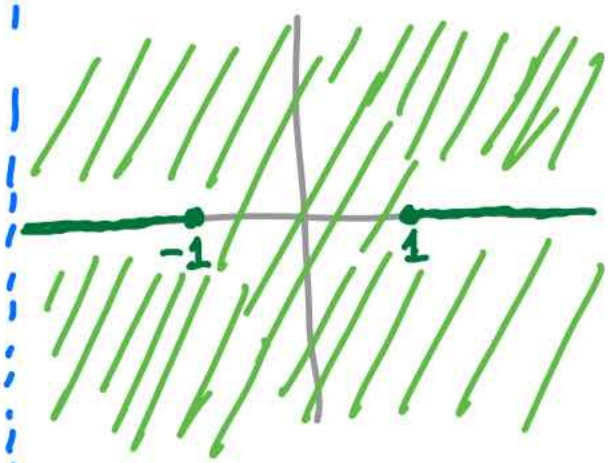
$$\Longrightarrow \quad \arccos(w) = z = -i\log(\varepsilon)$$

$$= -i\log\left(w \pm \sqrt{w^2 - 1}\right).$$

Now, $w \pm \sqrt{w^2 - 1}$ is an analytic function on

(choose one: we'll do "+")

$\mathbb{C} \setminus (\mathbb{R}_{\leq -1} \cup \mathbb{R}_{\geq 1})$ because $\sqrt{\mu} = \mu^{1/2}$ is of the



form in the last example, and $w^2 - 1$ maps this region into $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$. Furthermore, considering the equation

$$\begin{aligned} w + \sqrt{w^2 - 1} &= r \in \mathbb{R}_{>0} \\ \sqrt{w^2-1} &= r - w \\ \cancel{w^2} - 1 &= r^2 - 2rw + \cancel{w^2} \\ w &= \frac{r^2+1}{2r} = \frac{1}{2}\left\{\frac{1}{r} + r\right\} \in \mathbb{R}_{\geq 1} \end{aligned}$$

we see that $w + \sqrt{w^2-1}$ maps the region $\mathbb{C} \setminus \{\mathbb{R}_{\leq -1} \cup \mathbb{R}_{\geq 1}\}$ into $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$, where we can also define $\log(\cdot)$ as an analytic function; it follows that the composition is analytic:

$$\boxed{\arccos(w) \in \mathcal{A}n(\mathbb{C} \setminus \{\mathbb{R}_{\leq -1} \cup \mathbb{R}_{\geq 1}\}).}$$

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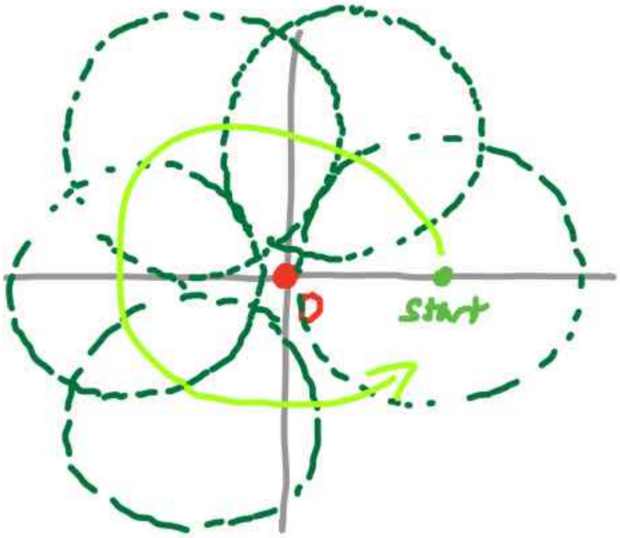
Remark: We have to do this all by hand, in the absence of something like the inverse mapping theorem (which wouldn't give us analyticity of arccos on more than a disk anyway!).

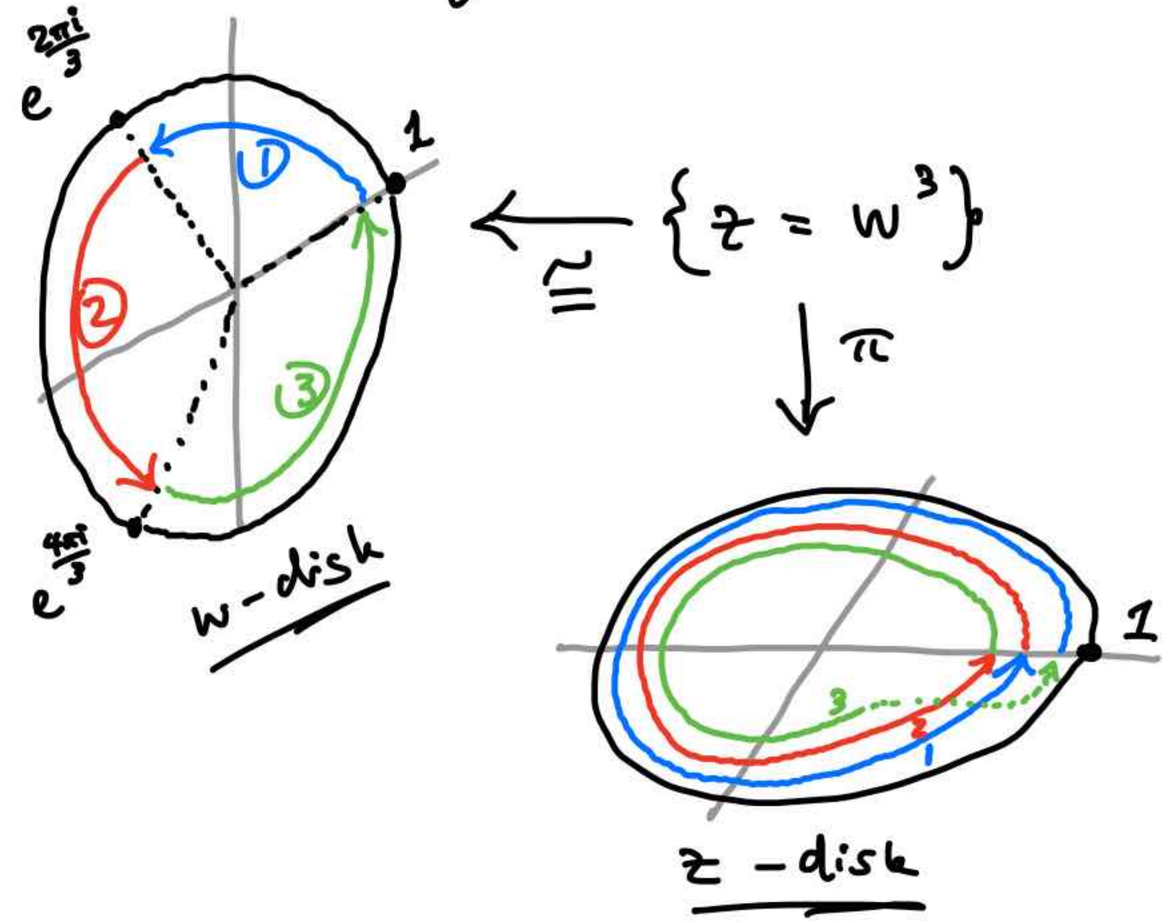
II. Riemann surfaces

Let's have another look at the z^{α} example — taking (say) $\alpha = 1/3$. If we decided to keep pasting power-series together along overlaps of neighborhoods, we would quickly get "multivalued" behavior in the functions constructed from log. So the question arises as to what the "existence domain" of a multivalued function over an open set in $\mathbb{C}$ (i.e. the minimal "cover" of $\mathbb{C}$ making the function single-valued) looks like. The resulting "complex 1-manifolds" are called Riemann surfaces.



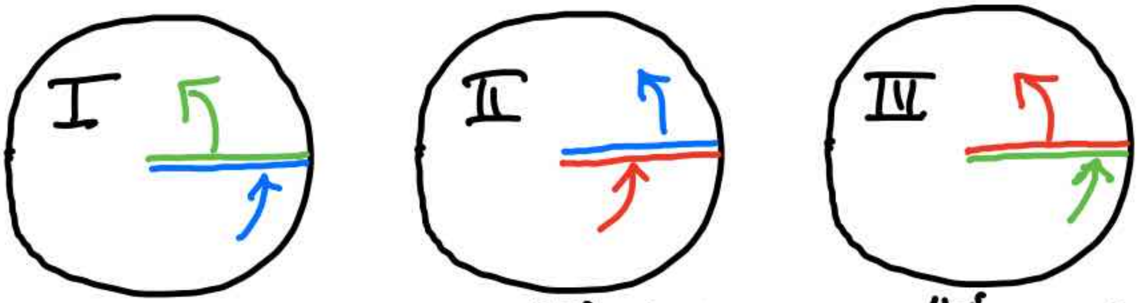
Example// Riemann surface of $\overset{w}{\overset{\shortparallel}{g(z)}} = z^{1/3}$ over D_1.

This is some object fitting (as "$\{z = w^3\}$") into the following picture:



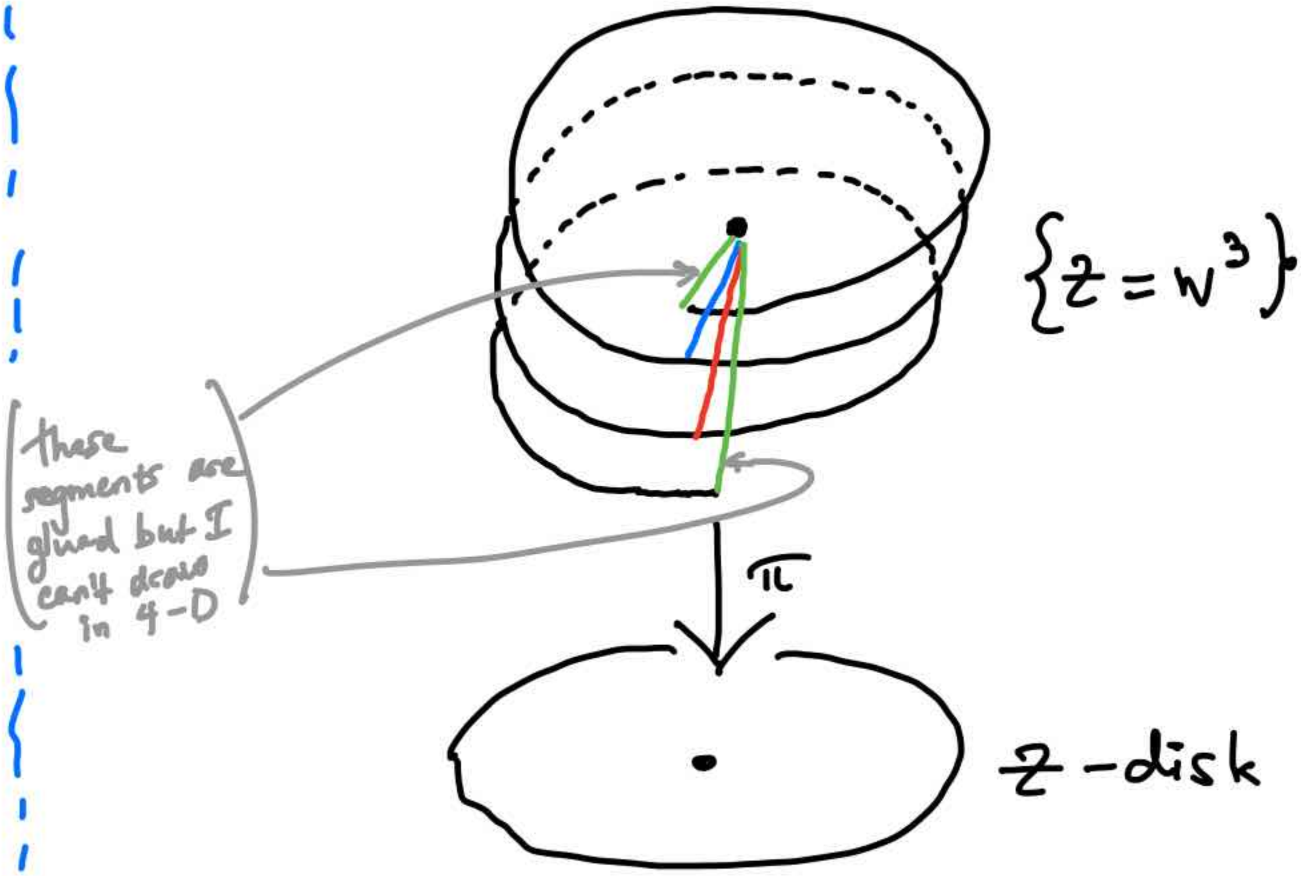
To construct it, think about following $z^{1/3}$ around the disk once counterclockwise: when you reach your starting point, the function has become $e^{2\pi i/3}$ times the branch of $z^{1/3}$ you started with; going around once more, you get $e^{4\pi i/3} z^{1/3}$; and one more time gets you back to your original branch.

So taking 3 unit disks, slitting them along the positive reals, and gluing them as indicated



“$z^{1/3}$” “$e^{\frac{2\pi i}{3}} z^{1/3}$” “$e^{\frac{4\pi i}{3}} z^{1/3}$”

we get the “cyclic parking lot”



An easier way to visualize this RS is: it's just the w-disk. The difficulty is in seeing the w-disk “over” the z-disk.

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Example // Next we construct an existence domain for

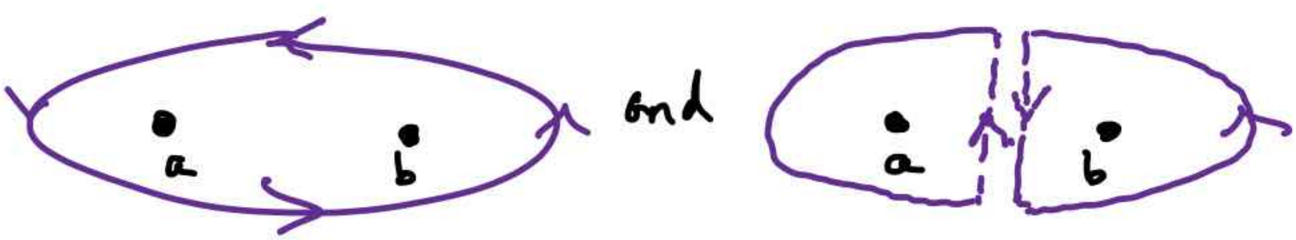
$$\mathcal{F}(z) = \sqrt{(z-a)(z-b)(z-c)} = \sqrt{f(z)}$$

over $\hat{\mathbb{C}}$ (or $\mathbb{C}\setminus\{a,b,c\}$ if you prefer).

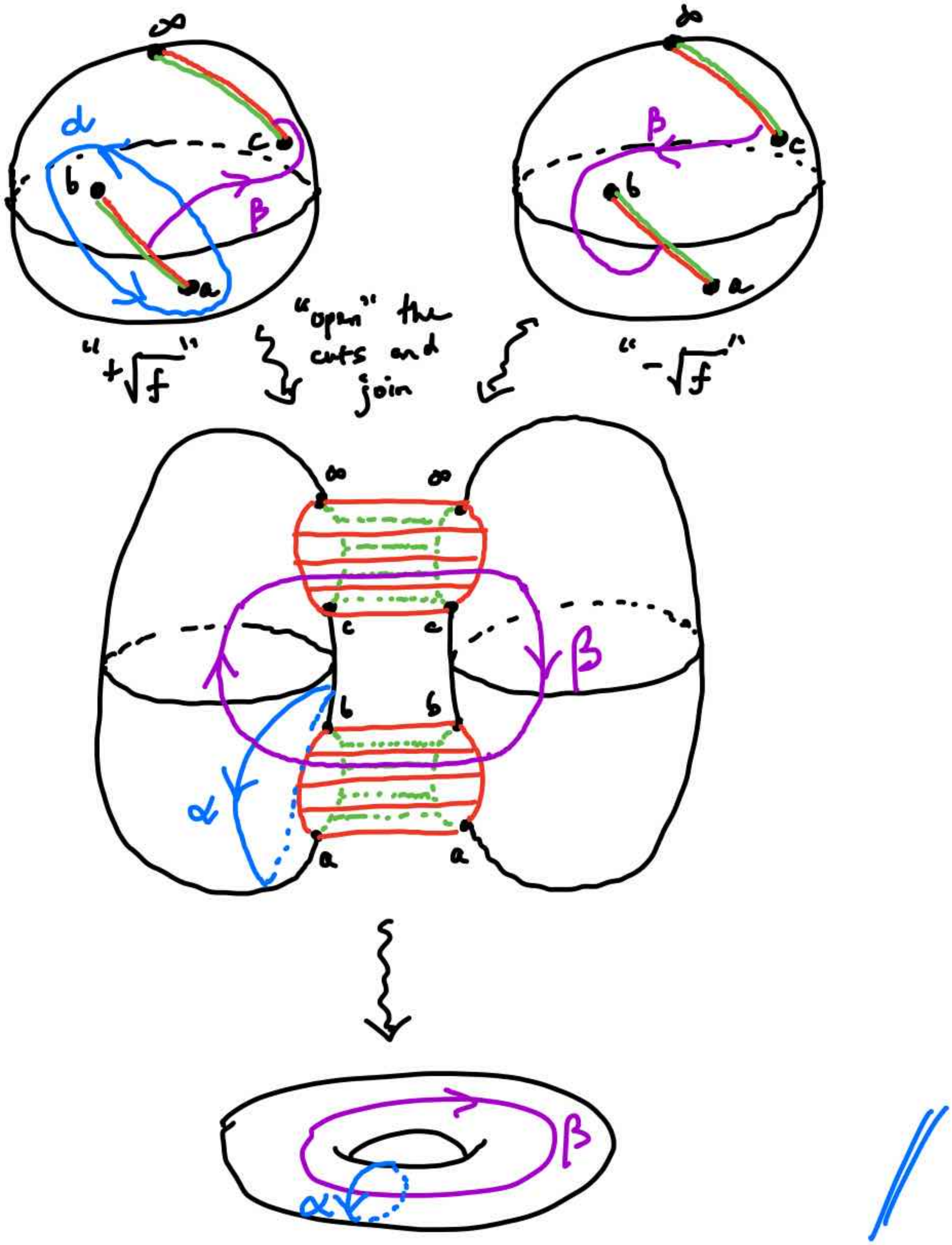
In a neighborhood of $z_0 = a, b, c$, this looks like the "RS of $(z-z_0)^{1/2}$ over a disk", which is the same as the construction we just did except with 2 disks replacing 3. Indeed, going once around $z = a, b,$ or c takes

$$\mathcal{F} \longmapsto -\mathcal{F}.$$

Furthermore, because the degree of the polynomial is odd, going once around ∞ does the same thing. Since



are equivalent, going around 2 points at once gives no change. So taking 2 $\hat{\mathbb{C}}$'s and cutting & pasting them as indicated, we end up with a donut-shaped surface on which $\tilde{\sigma}_H$ becomes well-defined:



<u>Remark</u>: Ahlfors also does RSs of

- $\log(z)$ (∞ level parking lot)
- $\arccos(z)$ (more complicated; please read it).