Lecture 8: Analytic functions

I. Definition & basic properties

Let $U \subseteq \mathbb{C}$ be open, $f \in \mathcal{F}(U)$.

- f is analytic at $z_0 \in U$ $\iff$

  $\exists \{a_n\} \subseteq \mathbb{C}$ and $r \in \mathbb{R}_{>0}$ s.t.

  $\sum_{n\geq 0} a_n (z - z_0)^n$ converges absolutely to $f(z)$

  for all $z \in D(z_0, r)$.

That is, "f is described by a power series in a neighborhood of z_0", or "has a power series expansion at z_0".

written "$f \in \mathrm{An}(U)$"

- f is analytic on U $\iff$

  f is analytic at every $z_0 \in U$.

We can also define analyticity on an arbitrary set $S \subseteq \mathbb{C}$:

- a given $f \in \mathcal{F}(S)$ is "analytic on S" $\iff$

  $\exists\, U \subseteq \mathbb{C}$ open containing S and

  $F \in \mathrm{An}(U)$ s.t. $f = F|_S$.

As you would expect, $\mathrm{An}(U)$ is a $\mathbb{C}$-algebra:

- $f, g \in \mathrm{An}(U)$, $\alpha \in \mathbb{C}$ $\Rightarrow$ $f+g,\ fg,\ \alpha f \in \mathrm{An}(U)$

  $f/g \in \mathrm{An}(U \setminus \{g=0\})$.

Proof: Given $z_0 \in U$, f & g are represented by $\mathfrak{L}_r$ of formal power series, for some suff. small $r>0$.
The formal $\underbrace{\text{sum / product}}_{}$ / $\underbrace{\text{quotient}}_{}$ (provided $g(z_0) \neq 0 \iff \mathrm{ord}(g(T)) = 0$) converges on the same / poss. smaller disk, to $f+g$ resp. fg resp. f/g (cf. Lecture 6). Do at each z_0. $\square$

- $f \in \mathrm{An}(U)$, $g \in \mathrm{An}(V)$ with $g(V) \subset U$

  $\Rightarrow f \circ g \in \mathrm{An}(V)$.

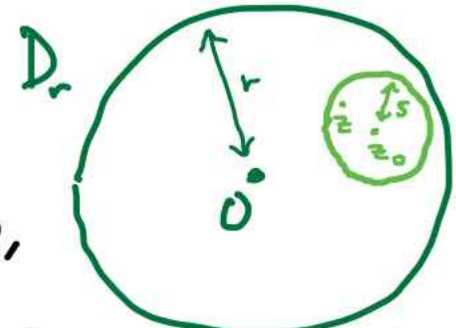
Proof: Similar argument, using "(B)" at end of lecture 6. $\square$

II. Relation to power series

- $f(T) \in \mathcal{L}_r \implies f(z) \in \mathrm{An}(D_r)$.

(There is a corresponding statement for origin replaced by any z_0.)

Proof: Goal: show f analytic at $z_0 \in D_r$.



Take $s < r - |z_0|$. For $z \in D(z_0, s)$,

$$f(z) \overset{\text{by}}{\underset{\text{assumption}}{=\!=}} \sum_{n\ge 0} a_n \underbrace{z^n}_{= (z_0+(z-z_0))^n} = \overbrace{\sum_{n\ge 0} a_n \sum_{k=0}^{n}}^{n\ge k} \binom{n}{k} z_0^{n-k}(z-z_0)^k$$

converges absolutely since

$$\sum_{n\ge 0} |a_n| \underbrace{\sum_{k=0}^{n} \binom{n}{k} |z_0|^{n-k} |z-z_0|^k}_{= (|z_0|+|z-z_0|)^n} \le \sum_{n\ge 0} |a_n| (\underbrace{|z_0|+s}_{<r})^n$$

converges (by hypothesis). Switching order of summation is therefore permissible, and yields

$$f(z) = \sum_{k\ge 0}\left(\sum_{n\ge k} a_n \binom{n}{k} z_0^{n-k}\right)(z-z_0)^k.$$

$\square$

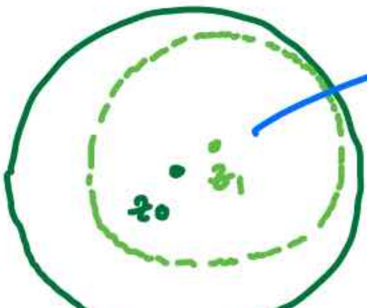
So we now have

$$\eta_r : \mathcal{L}_r \longrightarrow \mathrm{An}(D_r).$$

III. Zeroes of analytic functions

- Assume $f, g \in \mathrm{An}(U)$ with U connected.
  Then $\mathrm{acc}\{z \in U \mid f(z) = g(z)\} \neq \emptyset \Rightarrow f \equiv g$ on U.

Proof: From the last proof, if the (unique) power series representing a function F at z_0 has radius of convergence $r(z_0) < \infty$, then the radius $r(z_1)$ for z_1 nearby cannot be very different:



power series for z_1 must converge on the dotted disk.

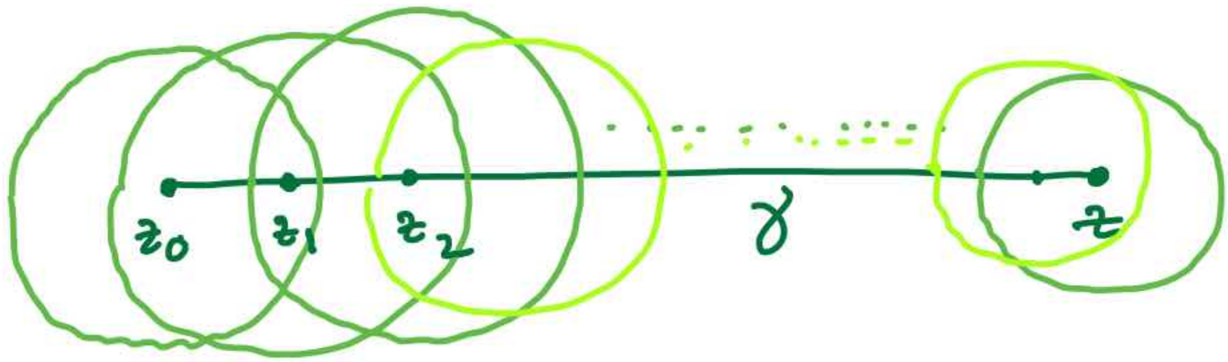
If it converges on a much larger disk, then the one at z_0 does too for the same reason.

$\Rightarrow$ "radius of convergence function" $r_f : U \to \mathbb{R}_{>0}$ either continuous or ∞ everywhere.

(Picture shows $|r_f(z_0) - r_f(z_1)| \leq |z_0 - z_1|$.)

Let $z_0 \in \operatorname{acc}\{z \in U \mid f(z) = g(z)\}$, write $F = f - g$, and let $z \in U$. We want to show $F(z) = 0$.

Let γ be a path (since U is connected) from z_0 to z, which for simplicity (and without loss of generality) we can assume linear:



Now

γ compact, r_F continuous

$$\Longrightarrow \quad r_F \geq b \in \mathbb{R}_{>0} \text{ on } \gamma,$$

i.e. the minimum value must actually be attained. (See the beginning of Lecture 5.) Let $r < \frac{b}{2}$ and take $z_1, z_2, \ldots$ at least within r of each other. The neighborhoods of convergence about each z_i therefore contain z_{i+1}, z_{i-1}.

Recall that in the situation where F is represented by power series at z_0 with $z_0 \in \operatorname{acc}\{z \mid F(z) = 0\}$, that power series is $\equiv 0$ (see Lecture 6); this implies

$$F = 0 \quad \text{on} \quad D(z_0, b).$$

Since $z_1 \in D(z_0, b)$, $z_1 \in \text{acc}\{F=0\}$ implies

$$F = 0 \quad \text{on} \quad D(z_1, b).$$

This contains z_2; continuing the argument we find $F(z) = 0$, done. $\square$

$\boxed{\text{Corollary}}$ $f \in \mathrm{An}(U) \setminus \{0\}$ and $K \subset U$ compact

$\Longrightarrow$ f has only finitely many zeroes on K.

IV. Relation with complex differentiability

- $f \in \mathrm{An}(U)$ region with $|f|$ constant, $f' \equiv 0$, $\mathrm{Re}(f)$ constant or $\mathrm{Im}(f)$ constant $\Longrightarrow$ f constant.

Proof: In a moment we will show that analytic $\Rightarrow$ holomorphic, for which we have the statement in the first two cases. If $\mathrm{Re}(f)$ is constant, then

$$2f' \underset{\uparrow\, f \text{ holomorphic}}{=} 2\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y} \underset{\uparrow\, u \text{ constant}}{=} i\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \underset{\uparrow\, \text{C-R equations}}{=} -i\cancelto{0}{\frac{\partial u}{\partial y}} + \cancelto{0}{\frac{\partial u}{\partial x}} = 0. \quad \square$$

- Analytic functions are holomorphic (in fact, infinitely differentiable).

I want to put this in a broader context involving formal differentiation $D : \mathbb{C}[[T]] \to \mathbb{C}[[T]]$

$$\sum_{n\geq 0} a_n T^n \mapsto \sum_{n\geq 1} n a_n T^{n-1}$$

$\boxed{\text{Theorem}}$ Let $f(T) \in \mathbb{C}[[T]]$, with

$\eta(f(T)) =: f(z) \in \mathrm{An}(D_r)$,

where r is the radius of convergence of f at $z=0$.

(a) Ω_r is closed under D (in fact, Df has radius of convergence r) (from HW!)

(b) $\frac{df}{dz}(z) \stackrel{\text{"exists and equals"}}{=} (Df)(z)$

(c) $\mathrm{An}(U) \subseteq \mathrm{Hol}(U)$, for any region U.

Proof: (a) $\limsup |na_n|^{1/n} = \underbrace{\lim n^{1/n}}_{=1} \cdot \underbrace{\limsup |a_n|^{1/n}}_{r} = r$.

instead of $\frac{1}{n-1}$: multiply Df by T

(i) take log & apply L'Hôpital,

OR (to avoid such things)

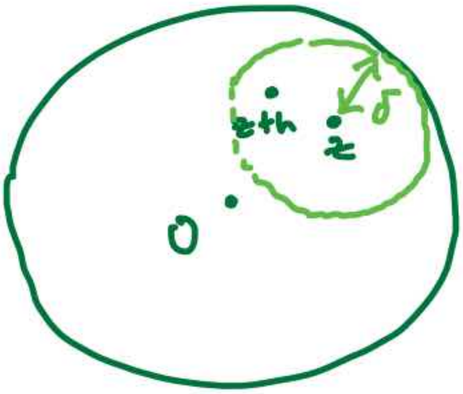
(ii) write $n = \left(1+(n^{1/n}-1)\right)^n > 1 + \binom{n}{2}(n^{1/n}-1)^2$

$$\cancel{n-1} > \frac{n\cancel{(n-1)}}{2}(n^{1/n}-1)^2$$

$$0 \leftarrow \sqrt{\frac{2}{n}} > n^{1/n}-1 > 0.$$

(b) Recall "f complex-differentiable at z with derivative α"

$$\iff \lim_{h\to 0}\left|\frac{f(z+h)-f(z)}{h} - \alpha\right| = 0.$$



Take $\delta < r-|z|$ and $|h|<\delta$:

then $f(z+h)-f(z) = \sum a_n\left((z+h)^n - z^n\right)$

$$= \left(\sum n a_n z^{n-1}\right)h + \left(\sum a_n P_n(z,h)\right)h^2$$

where $$|P_n(z,h)| = \left|\sum_{k=2}^{n}\binom{n}{k} h^{k-2} z^{n-k}\right|$$

$$\leq \sum_{k=2}^{n}\binom{n}{k}|h|^{k-2}|z|^{n-k}$$

$$\leq n(n-1)\sum_{l=0}^{n-2}\binom{n-2}{l}\delta^{l}|z|^{n-l-2}$$

$$= n(n-1)\big(\underbrace{|z|+\delta}_{=:r_0<r}\big)^{n-2}$$

$$= n(n-1)\, r_0^{\,n-2}$$

$l = k-2$.

$$\frac{n!}{k!(n-k)!} = n(n-1)\frac{(n-2)!}{(l+2)!(n-2-l)!} \leq n(n-1)\frac{(n-2)!}{l!(n-2-l)!}$$

Hence, $$\left|\frac{f(z+h)-f(z)}{h} - \underbrace{\sum n a_n z^{n-1}}_{(Df)(z)}\right| = \left|h\sum a_n P_n(z,h)\right|$$

$$\le |h| \underbrace{\sum |a_n|\, n(n-1)\, r_0^{\,n-2}}_{\substack{\text{know convergent} \\ \text{as } r_0<r \text{ and } (n(n-1))^{1/n}\to 1}} \xrightarrow{h\to 0} 0,$$

done.

(c) In a neighborhood of every $z_0 \in U$, f is represented by a power series and (by (b)) f' exists (and is represented by $\eta((Df)(T))$. $\square$

$\boxed{\text{Corollary}}$ (a) Let $f \in \mathcal{A}n(U)$; then f is infinitely complex-differentiable on U.

(b) Let $f \in \mathcal{A}n(D_r)$ be given as $\eta_r(f(T))$.

Then (i) $f^{(k)}(z) = \eta(D^k f(T)) = k!\, a_k + h_k(z)$ $\in \mathcal{A}n(D_r)$, with $h_k(0) = 0$.

so $f^{(k)}(0) = k!\, a_k$.

(ii) f has a primitive on D_r (i.e. a function F satisfying $F' = f$),

$$F = \sum_{n\ge 0} \frac{a_n}{n+1} z^{n+1} \ (+\text{ constant}) \in \mathcal{A}n(D_r).$$

(b) then obvious

Pf. of (a): About each point f is represented by power series; and we just showed f' is represented by the formal derivative (in some neighborhood), hence is analytic itself (and we may repeat this process). $\square$

V. Tauber's theorem

We conclude by revisiting the topic of the last lecture, where we had shown that

$$(*) \quad \underset{\text{ordinary summability}}{OS} \underset{\cancel{\Leftarrow}}{\Rightarrow} \underset{\text{Césaro summability}}{CS} \underset{\cancel{\Leftarrow}}{\Rightarrow} \underset{\text{Abel summability}}{AS}$$

The problem with this story is that in doing mathematics (not just analysis!), one encounters situations where one needs to be able to deduce information (asymptotic behavior, ordinary summability) about the $\{a_k\}$ from information (asymptotic behavior at $|z|=1$) about $f(z)=\sum_{k\geq 0} a_k z^k$. [Frequently f may be a function one understands well, for which we decide to compute the $f^{(k)}(0)/k!$. What can we say about these numbers?]

The theory that provides this kind of information is Tauberian theory, and its first result is:

$\boxed{\text{Theorem (Tauber, 1897)}}$ AS + $na_n \to 0 \Rightarrow$ OS

That is, we can (conditionally) go "backwards" in (*)!

Proof: $0<|x|<1 \Rightarrow$

$$\left|\sum_{n=0}^{N} a_n - f(x)\right| = \left|\sum_{n=1}^{N} a_n(1-x^n) - \sum_{n\geq N+1} a_n x^n\right|$$

want to prove limit exists, assuming $\lim_{x\to 1^-} f(x)$ exists

$$\leq \sum_{n=1}^{N} n(1-x)|a_n| + \frac{1}{N}\sum_{n\geq N+1} |na_n| x^n$$

$$\leq (1-x)\sum_{n=1}^{N} |na_n| + \frac{1}{N(1-x)}\sup_{n>N} |na_n|.$$

$1-x^n = (1-x)(1+x+\cdots+x^{n-1}) \leq (1-x)n$

Therefore (taking $x = 1-\frac{1}{N}$)

$$\left| s_N - f\left(1-\tfrac{1}{N}\right)\right| \le \underbrace{\frac{1}{N}\sum_{n=1}^{N} |n a_n| + \sup_{n>N} |n a_n|}$$

Now $n a_n \to 0 \Rightarrow$ this $\to 0$ (easy),
$(n\to\infty)$ $(N\to\infty)$

while AS $\Rightarrow \lim_{N\to\infty} f\left(1-\frac{1}{N}\right)$ $(=: A)$ exists. $\square$

To complete our discussion of these results, we contrast what Abel & Tauber say for a function

$$f(x) := \sum a_n x^n \quad \text{on} \quad (-1,1).$$

<u>Abel</u>: $\sum a_n$ converges $\Rightarrow$ f has continuous extension to $(-1,1]$ (by setting $f(1) := \sum a_n$)

<u>Tauber</u>: f has continuous extension to $(-1,1]$ (and $n a_n \to 0$) $\Rightarrow$ $\sum a_n$ converges (to $f(1)$).

So Tauber's theorem is a conditional converse.

There's a stronger version due to Littlewood (1911), relaxing "$n a_n \to 0$" to $|n a_n| \le B$.

<u>Remark</u>: $a_n = \dfrac{1}{n\log(n)}$ is an example of a sequence satisfying $na_n \to 0$ but <u>not</u> ordinary summable:

$$\sum_{n\geq 2} \frac{1}{n\log n} = \infty \text{, essentially by the integral test}$$

$$\int_2^\infty \frac{dx}{x\log x} \underset{\uparrow\, u=\log x}{=} \int_{\log 2}^\infty \frac{du}{u} = \infty .$$

So we really do need both hypotheses in Tauber's theorem to conclude OS.