Lecture 7: Variations on Abel's Theorem

I. Theme: Abel's theorem (and summation by parts)

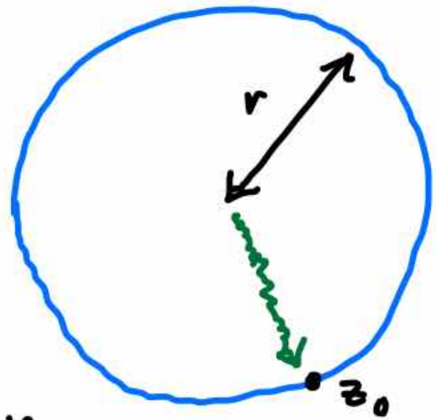
Given $\{a_k\} \subseteq \mathbb{C}$, we set (and assume > 0)

$$r := \text{lub}\left\{ s \in \mathbb{R}_{\geq 0} \;\middle|\; \sum |a_k| s^k \text{ converges} \right\}.$$

Then $\sum a_k z^k$ converges absolutely for $|z| < r$, and uniformly for $|z| \leq s < r$ (to a continuous function), hence defining a continuous function on D_r. Call this $f(z)$.

To what extent is an extension to $\bar{D}_r$ defined and continuous? We'll start by proving a result to the effect that for any

z_0 on ∂D_r (i.e. $|z_0| = r$) with $\sum a_n z_0^n$ convergent, the limit of $f(z)$ along the radial path out to z_0 equals $\sum a_n z_0^n$. (We will refine this result later.)

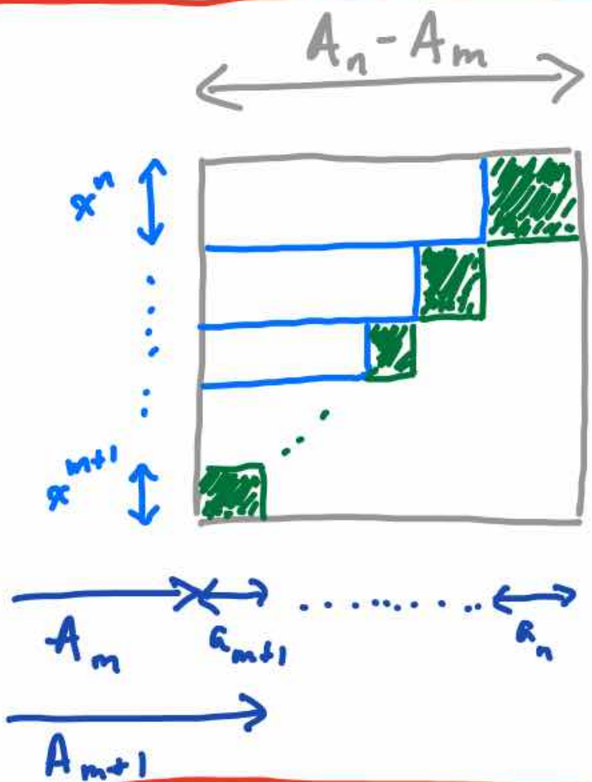


It suffices to consider the case $r = 1$ and $z_0 = 1$, and work in $\mathbb{R}$. Set $A := \sum_0^\infty a_k$, which we assume converges, and define $f(x) := \sum_0^\infty a_k x^k$, $x \in [0,1]$. From the above, we know that f is continuous on $[0,1)$.

$\boxed{\text{Abel's Theorem (1826)}}$ $\lim_{x \to 1^-} f(x) = A$, i.e. f is continuous on $[0,1]$.

Proof: Set $A_n := \sum_0^n a_k$ $\in \mathbb{C}$. This is Cauchy since it converges. The sequence of functions

$$f_n(x) := \sum_0^n a_k x^k \xrightarrow[\text{pointwise}]{\text{converges}} f(x) \text{ for } x \in [0,1].$$



Now (for $x \in [0,1]$)

$$|f_n(x) - f_m(x)| = \left| \sum_{m+1}^{n} a_k x^k \right|$$

$$= \left| (A_n - A_m) x^n + \sum_{(k=)m+1}^{n-1} (A_k - A_m) \underbrace{(x^k - x^{k+1})}_{\geq 0} \right|$$

$$\leq \epsilon + \epsilon \underbrace{\sum_{m+1}^{n-1} (x^k - x^{k+1})}_{\text{collapsing sum}}$$

n,m sufficiently large

$$\leq 2\epsilon$$

$\Longrightarrow$ $\{f_n\}$ Cauchy in $\|\cdot\|_{[0,1]}$

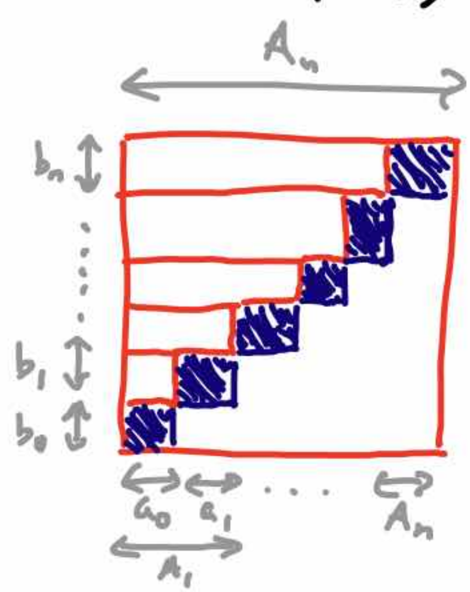
$\Longrightarrow$ $\{f_n\}$ converges uniformly there to its pointwise limit ($= f$)

$\Longrightarrow$ f is continuous. $\square$

We draw attention to the principle (used in the proof, and to be used in HW) of

$\boxed{\text{Summation by parts}}$ Given sequences $\{a_k\}, \{b_k\}$ with $A_n := \sum_{k=0}^{n} a_k$, we have

$$\sum_{k=0}^{n} a_k b_k = A_n b_n + \sum_{k=0}^{n-1} A_k (b_k - b_{k+1}).$$



II. Variation 1: Cauchy's (other) Theorem

Let

- $\{a_n\}_{n\ge 0}$ be a sequence of complex numbers with $\limsup_{n\to\infty} |a_n|^{1/n} = 1$
- $s_n := \sum_{k=0}^{n} a_k$ the sequence of n^{th} partial sums
- $\sigma_n := \dfrac{s_0 + s_1 + \cdots + s_{n-1}}{n}$ the sequence of "Césaro sums"
- $f(x) := \sum_{n\ge 0} a_n x^n$ the function on $(-1,1) \subseteq \mathbb{R}$.

Definition (3 conditions)

OS (ordinary summability) $\iff$ $\lim_{n\to\infty} s_n =: a$ exists

CS (Césaro summability) $\iff$ $\lim_{n\to\infty} \sigma_n =: \sigma$ exists

AS (Abel summability) $\iff$ $\lim_{x\to 1^-} f(x) =: A$ exists

Note: I won't use "$f(1)$" here for $\sum a_n$ $(=a)$, but Ahlfors does this.

What is the relation between these conditions and the resulting numbers a, σ, and A?

Theorem (Cauchy, 1821) OS $\Rightarrow$ CS, with $\sigma = a$.

Proof: Let $\epsilon > 0$ be given.

Pick M s.t.

$$k \geq M \Rightarrow |s_k - a| < \epsilon/2 .$$

Let $B := \sup_{k \in \{0, \dots, M-1\}} |s_k - a|$, and pick $N \geq M$ s.t. $\frac{MB}{N} < \epsilon/2$.

Then $n \geq N \Rightarrow$

$$\begin{aligned} |\sigma_n - a| &= \frac{1}{n} |s_0 + \cdots + s_{n-1} - na| \\ &\leq \frac{1}{n} \sum_{k=0}^{n-1} |s_k - a| \\ &\leq \frac{MB}{N} + \frac{1}{n} \sum_{k=M}^{n-1} |s_k - a| \\ &< \epsilon/2 + \epsilon/2 = \epsilon . \end{aligned}$$

$\square$

III. Variation 2: Abel in the disk

We begin by giving another proof of Abel's theorem.

Theorem (Abel) OS $\Rightarrow$ AS, with $A = a$.

Proof: On $|x| < 1$ we can rearrange

$$\sum_{k \geq 0} s_k x^k = a_0 + (a_0 + a_1) x + (a_0 + a_1 + a_2) x^2 + \cdots$$

$$= a_0 \underbrace{\sum_{k \geq 0} x^k}_{(1-x)^{-1}} + a_1 x \sum_{k \geq 0} x^k + a_2 x^2 \sum_{k \geq 0} x^k$$

$$\Longrightarrow (1-x) \sum_{k \geq 0} s_k x^k = \sum_{n \geq 0} a_n x^n = f(x).$$

Since $\lim_{k \to \infty} s_k \, (= a)$ exists, given $\epsilon > 0$ we can take:

- M to be such that $k \geq M \Rightarrow |s_k - a| < \epsilon/2$;
- $B := \sup_{k \in \{0, \dots, M-1\}} |s_k - a|$; and
- $\delta > 0$ such that $1 - \delta < x < 1 \Rightarrow B(1 - x^M) < \epsilon/2$.

Then for $x \in (1-\delta, 1)$,

$$\left| \overbrace{(1-x)\sum_{k\geq 0} s_k x^k}^{f(x)} - \underbrace{a}_{(1-x)\sum_{k=0}^{\infty} a x^k} \right| = \left| (1-x) \sum_{k\geq 0} (s_k - a) x^k \right|$$

$$\leq \underbrace{(1-x) \sum_{k=0}^{M-1} |s_k - a| x^k}_{\leq B \frac{1-x^M}{1-x}} + \underbrace{(1-x) \sum_{k\geq M} |s_k - a| x^k}_{< \frac{\epsilon}{2} \frac{x^M}{1-x}}$$

$$\leq B(1-x^M) + x^M \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \quad \square$$

Discussion: What has to be done to make this proof viable inside the unit disk? Everything is fine up to

$$\left| (1-z) \sum_{k\geq 0} (s_k - a) z^k \right| \leq$$

$$|1-z| \underbrace{\sum_{k=0}^{M-1} |s_k - a| |z|^k}_{\leq B \cdot \frac{1-|z|^M}{1-|z|}} + |1-z| \underbrace{\sum_{k\geq M} |s_k - a| |z|^k}_{< \frac{\epsilon}{2} \frac{|z|^M}{1-|z|}},$$

where $\dfrac{|1-z|}{1-|z|}$ causes problems if you just take $|z|<1$ and $|1-z|<\delta$. We need to assume that

$$\boxed{\frac{|1-z|}{1-|z|} \le K}$$

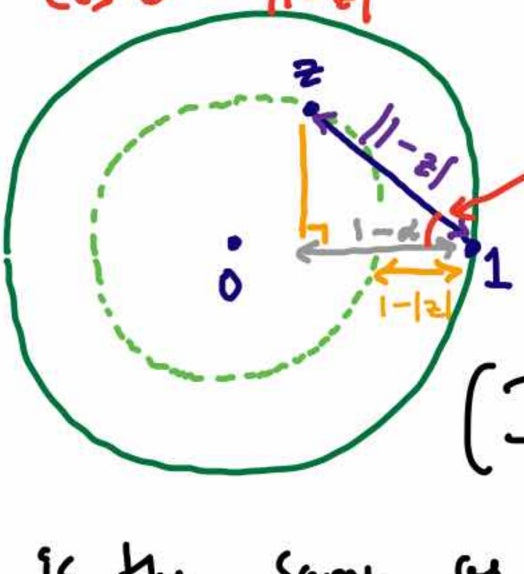
for some fixed $K \in \mathbb{R}_{>0}$.

Note that for $z = \alpha + \beta i$,

$$\frac{|1-z|}{1-|z|} \le K \implies \frac{|1-z|}{1-\alpha} \le K$$

$$\left(\alpha \le |z| \Rightarrow 1-\alpha \ge 1-|z| \Rightarrow \frac{1}{1-\alpha} \le \frac{1}{1-|z|} \right)$$

$\cos\theta = \dfrac{1-\alpha}{|1-z|}$



$$\implies \cos\theta \ge K^{-1}$$

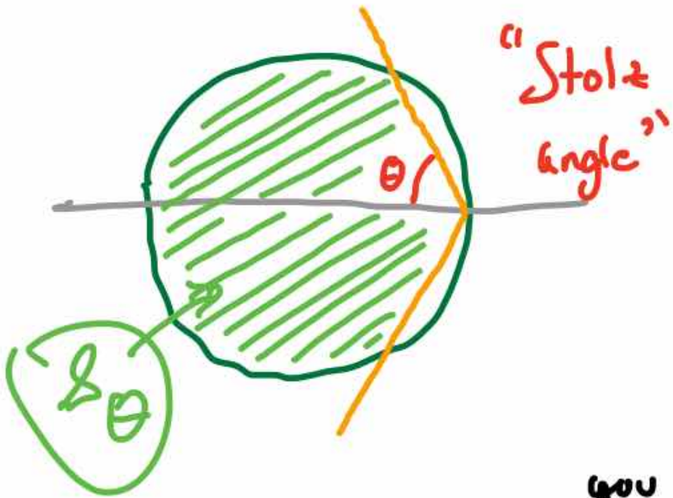
$$\implies \boxed{|\theta| \le \arccos(K^{-1}).}$$

(In the limit $z \to 1$, this condition is the same as $\dfrac{|1-z|}{1-|z|} \le K$.)

Having fixed the bound K, we proceed as before and get $|f(z)-a| < K\epsilon$, yielding

$\boxed{\text{Theorem (Abel in the disk)}}$ $\sum a_n \, (=a)$ convergent

$\Rightarrow \lim_{\substack{z \to 1 \\ \in S_\Theta}} f(z) = a$ for any $\Theta < 90°$.



That is, you can fix $\Theta = 89.99°$ but then your path of approach must be in $S_{89.99°}$, or the limit you get may not $= a$ (or may not exist).

IV. Variation 3: A theorem of Frobenius

The Abel and Cauchy theorems say that OS is stronger than CS or AS.

But which one is weakest?

$\boxed{\text{Theorem (Frobenius, 1880)}}$ $\underline{CS \Rightarrow AS,}$

$\underline{\text{with } A = \sigma.}$

So $OS \Rightarrow CS \Rightarrow AS$.

Proof: $|x|<1 \Rightarrow$ we can write

$$\sum_{n\ge 0} a_n x^n = s_0 + \sum_{n\ge 1} (s_n - s_{n-1}) x^n$$

$$= \sum_{n\ge 0} s_n x^n - \sum_{n\ge 0} s_n x^{n+1}$$

$$= (1-x) \sum_{n\ge 0} s_n x^n$$

$$= (1-x)^2 \sum_{n\ge 0} \underbrace{(s_0 + \dots + s_n)}_{(n+1)\sigma_{n+1}} x^n$$

$\limsup |s_n|^{1/n} = 1$, so can rearrange in this fashion

$$= \frac{\sum (n+1)\sigma_{n+1} x^n}{\sum (n+1) x^n}$$

$\frac{1}{(1-x)^2} = \frac{d}{dx} \frac{1}{1-x} = \frac{d}{dx} \sum x^n = \sum_{n\ge 0} (n+1) x^n$

$$= \sigma + \frac{\sum (n+1)(\sigma_{n+1} - \sigma) x^n}{\sum (n+1) x^n}$$

$$= \sigma + (1-x)^2 \underbrace{\sum_{n\geq 0} (n+1)(\sigma_{n+1}-\sigma)x^n}$$

$x \in (1-\delta, 1) \implies$ bound modulus by (in same fashion as above)

$$B(1-x)^2 \sum_{n=0}^{M-1} (n+1)x^n + (1-x)^2 \sum_{n\geq M} (n+1)\frac{\epsilon}{2} x^n$$

$$< B\cancel{(1-x)^2} \frac{1-(M+1)x^M + Mx^{M+1}}{\cancel{(1-x)^2}} + \frac{\epsilon}{2} x^M$$

$$\overset{M \gg 0}{<} \epsilon/2 + \epsilon/2 .$$

$\square$

V. Coda: some Examples

1 $a_n = (-1)^n (n+1)$.

$$\lim_{x\to 1^-} \sum (-1)^n (n+1) x^n = \lim_{x\to(-1)^+} \sum (n+1)x^n$$

$$= \lim_{x\to(-1)^+} \frac{1}{(1-x)^2}$$

$$= \frac{1}{4}.$$

AS $\Leftarrow$

But $\sigma_{2n+1} \approx \frac{1}{2}$, $\sigma_{2n} = 0 \Rightarrow \cancel{CS}$.

② $a_n = (-1)^n$.

$$S_n = \begin{cases} 1, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

$$\rightsquigarrow \sigma_n = \begin{cases} 1/2, & n \text{ even} \\ 1/2 + \frac{1}{2n}, & n \text{ odd} \end{cases} \xrightarrow[n\to\infty]{} 1/2$$

$\Longrightarrow$ CS.

Check the Abel sum $= 1/2$:

$$\lim_{x\to 1^-} \sum (-1)^n x^n = \lim_{x \to (-1)^+} \sum x^n$$

$$= \lim_{x\to -1} \frac{1}{1-x}$$

$$= \frac{1}{2}. \quad \checkmark$$

③ $a_n = 1/n^k$. $(a_0 = 0,\ k \geq 2)$

In fact, $\displaystyle\sum_{n\geq 1} \frac{1}{n^k} =: \zeta(k) \implies$ OS.

Note that $\displaystyle\sum \frac{x^n}{n^k} =: \mathrm{Li}_k(x)$, the

kth polylogarithm, and (by Abel) this

must have $\displaystyle\lim_{x\to 1^-} \mathrm{Li}_k(x) = \zeta(k)$.

Altogether we get the table

	OS	CS	AS
$(-1)^n(n+1)$	✗	✗	✓
$(-1)^n$	✗	✓	✓
$1/n^k$	✓	✓	✓

which shows all our implications are "sharp".