Lecture 6: The analyst's nightmare

... is having an algebraist teach your complex analysis class? But in all seriousness, algebra can be very helpful as an organizing principle; and presently we will use it to clarify the manipulation of power series.

I. Formal power series

These are elements of the $\mathbb{C}$-algebra

$$\mathbb{C}[[T]] := \left\{ \underbrace{\sum_{n\geq 0} a_n T^n}_{} \;\middle|\; a_n \in \mathbb{C} \ (\forall n) \right\}$$

frequently written "f"; can also think of as $\{a_n\}$

in which I have to tell you how to add and multiply† (scalar multiplication is easy). They aren't functions, but we will MAP them to functions.

† see the next page

Definition Write $\mathrm{ord}(f) = r$ if $T^r \mid f$ but $T^{r+1} \nmid f$ (formally).

e.g. $f = 3T^2 + 4T^3 + T^4 + \dots$

has $\mathrm{ord}(f) = 2$, while

$$f = 1 + \text{h.o.t.}$$

has $\mathrm{ord}(f) = 0$; "0" has no order, and

$$\mathrm{ord}(fg) = \mathrm{ord}(f) + \mathrm{ord}(g).$$

Theorem f is invertible in $\mathbb{C}[[T]]$ $\iff$ $\mathrm{ord}(f) = 0$.

Remark: To define the algebra structure, we set

- $\sum_n a_n T^n + \sum_n b_n T^n := \sum_n (a_n + b_n) T^n$
- $\gamma \sum_n a_n T^n = \sum_n (\gamma a_n) T^n$
- $\left(\sum_n a_n T^n\right)\left(\sum_n b_n T^n\right) := \sum_n \left(\sum_{0 \le k \le n} a_k b_{n-k}\right) T^n$

Note that f is invertible $\iff$

$\exists\, g$ s.t. $f \cdot g = 1$, i.e. $\begin{cases} a_0 b_0 = 1 \\ a_1 b_0 + a_0 b_1 = 0 \\ a_2 b_0 + a_1 b_1 + a_0 b_2 = 0 \\ \vdots \end{cases}$

Proof of Theorem:

($\Rightarrow$) $fg = 1$, so $\mathrm{ord}(f) + \mathrm{ord}(g) = \mathrm{ord}(fg) = \mathrm{ord}(1) = 0$.

But $\mathrm{ord} \geq 0$, so <u>both</u> $\mathrm{ord}(f)$ and $\mathrm{ord}(g)$ must $= 0$.

($\Leftarrow$) Write $f = 1 - h$, $h := -a_1 T - a_2 T^2 - a_3 T^3 - \dots$

(wolog)

Observe that $g := 1 + h + h^2 + h^3 + \dots$ makes sense (in $\mathbb{C}[[T]]$), because the coefficient of T^n involves only $1, h, h^2, \dots, h^n$. (So no infinite series are involved in computing coefficients.) Clearly,

$$fg = (1-h)(1+h+h^2+\dots)$$

gives a collapsing sum which $= 1 \in \mathbb{C}[[T]]$. $\square$

Ex//

$$\left(1 + z + z^2/2 + z^3/6 + \dots\right)^{-1} = 1 + \left(-z - z^2/2 - z^3/6 - \dots\right) + \left(-z - z^2/2 - \dots\right)^2 + \left(-z - \dots\right)^3 + \dots$$

$$= 1 - z + z^2/2 - z^3/6 + \dots$$

(To get the 1^{st} 4 terms of the inverse, we only need the 1^{st} 4 terms of the formal power series.)

This "formally" shows $(e^z)^{-1} = e^{-z}$. //

If $\mathrm{ord}(h) > 0$, we can also define a formal composition $\underbrace{f}_{\sum a_n T^n} \circ h := \sum a_n h^n$ (not part of the $\mathbb{C}$-algebra structure).

Definition $f \equiv g \mod T^N$ iff their 0^{th} through $(n-1)^{st}$ coefficients are the same.

We can think of this as saying that two elements have the same image under the natural map

$$\mathbb{C}[[T]] \twoheadrightarrow \mathbb{C}[[T]]/(T^N) \cong \mathbb{C}[T]/(T^N).$$

That this is an algebra homomorphism is encapsulated in the following

Proposition $f_1 \equiv f_2$ & $g_1 \equiv g_2$ & $\overbrace{h_1 \equiv h_2}^{\text{ord} > 0}$ (all mod T^N)
$\Longrightarrow f_1 + g_1 \equiv f_2 + g_2, \; f_1 g_1 \equiv f_2 g_2, \; f_1 \circ h_1 \equiv f_2 \circ h_2 \pmod{T^N}$.

Idea of Pf.: You only need the first N terms of the formal power series to obtain the first N terms of their sum, product, or composition. ∎

Finally, if $f = \sum a_n T^n$ and $F = \sum A_n T^n$ with $|a_n| \le A_n \in \mathbb{R}_+ \; (\forall n)$, we say that F dominates f and write $f \prec F$.

The field of fractions. Write $\mathcal{L}$ $(\supseteq \mathbb{C}[[T]])$ for the set of all $\sum_{n \geq m} a_n T^n$ where $m \in \mathbb{Z}$ (possibly < 0); "ord" now takes values in $\mathbb{Z}$. This allows us to invert <u>everything</u> (except 0): for $f = \sum_{n \geq m} a_n T^n$ of order m, $a_m \neq 0$, write

$$f = a_m T^m \underbrace{\left(\sum_{n \geq 0} b_n T^n \right)}_{1 + \text{h.o.t.}} =: a_m T^m \cdot F,$$

$$\frac{1}{f} = \frac{1}{a_m T^m} \cdot \underbrace{F^{-1}}_{\text{(as defined above)}}.$$

So now we get $\frac{f}{g}$ for any 2 formal power series $(g \neq 0)$, and $\text{ord}\left(\frac{f}{g}\right) = \text{ord}(f) - \text{ord}(g)$.

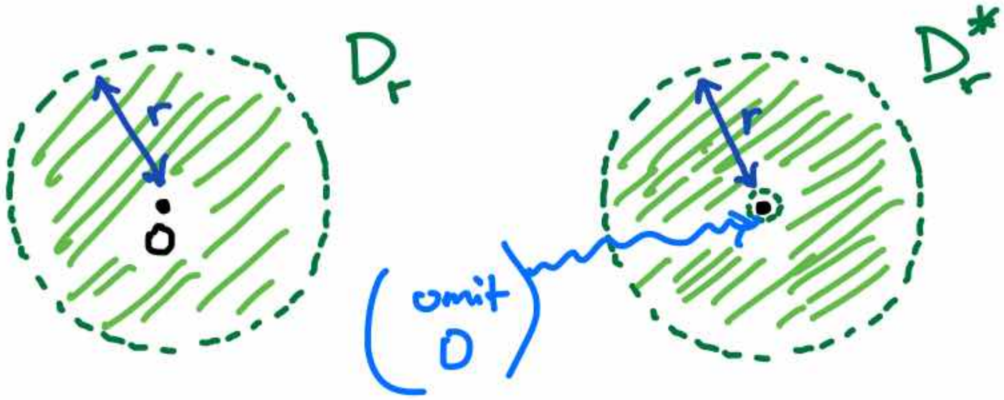
This "$\mathcal{L}$" is sometimes called $\mathbb{C}((T))$.

II. From formal to convergent

Set $\mathcal{Q}_r := \left\{ \sum_{k\geq 0} a_k T^k \in \mathbb{C}[[T]] \;\middle|\; \limsup |a_n|^{1/n} \leq \frac{1}{r} \right\}$

$\cap\!\!|$

$\mathbb{C}[[T]]$ ← a priori, just a subset



Define a map from formal power series to continuous functions on D_r by†

$$\eta_r : \mathcal{Q}_r \longrightarrow \mathcal{C}^0(D_r)$$

$$\sum_{k\geq 0} a_k T^k = f(T) \longmapsto f(z) := \lim_{n\to\infty} \sum_{k=0}^{n} a_k z^k$$

where $f(z)$ is the function defined by the pointwise limit of the partial sums.

---

† note: $\mathcal{C}^0(D_r)$ is a $\mathbb{C}$-algebra

Theorem
(i) $\mathcal{P}_r$ is a subalgebra† of $\mathbb{C}[[T]]$

(ii) η is a $\mathbb{C}$-algebra homomorphism

(iii) η is injective — in fact, given $S \subseteq D_r$ with $\mathrm{acc}(S) \ni \{0\}$, the composition $\mathcal{P}_r \xrightarrow{\eta} C^0(D_r) \xrightarrow{\text{restrict}} C^0(S)$ is injective.

Remarks:

- (iii) $\Rightarrow$ a function given by power series is determined uniquely by its value on an arc or line segment (for example).
  — e.g. If $f(z)=g(z)$ on $[-\epsilon,\epsilon]=S$ then $(f-g)(z)\equiv 0$ there, and by (iii), $(f-g)(T)=0$. Hence $(f-g)(z)$ is identically zero on D_r.
- More generally,
  (ii) $\Rightarrow$ anything you can do on the level of formal power series (except, so far, composition) is true for the corresponding functions.

  while
  (iii) $\Rightarrow$ the converse.

---

† i.e. closed under $+$, $\cdot$, & scalar multiplication

Example // From Calculus / real analysis, you know that $\sin^2(x) + \cos^2(x) = 1$ for $x \in \mathbb{R}$. Writing

$$S(T) := \sum_{n\geq 0} \frac{(-1)^n}{(2n+1)!} T^n, \quad C(T) := \sum_{n\geq 0} \frac{(-1)^n}{(2n)!} T^n$$

and $S(z) := \eta(S(T))$, $C(z) := \eta(C(T))$, we have (for $x \in \mathbb{R}$) $S(x) = \sin(x)$, $C(x) = \cos(x)$. So

$$\eta\left[S(T)^2 + C(T)^2 - 1\right] \overset{(ii)}{=} (\eta(S(T)))^2 + (\eta(C(T)))^2 - \eta(1)$$
$$= S(z)^2 + C(z)^2 - 1$$

restricts to 0 on $\mathbb{R}$. By (iii), $S(T)^2 + C(T)^2 - 1$ is the zero power series! (This is unpleasant to do by hand.) So $S(z)^2 + C(z)^2 = 1$ for all $z \in \mathbb{C}$. //

Proof of (i): (closure under "·" only)

Assume $f, g \in \mathcal{P}_r$. Then

$$fg := \sum_{n\geq 0} \underbrace{\left(\sum_{k=0}^{n} a_k b_{n-k}\right)}_{=: \gamma_n} T^n \in \mathbb{C}[[T]].$$

Is it in $\mathcal{P}_r$?

Recall that for any $B > \frac{1}{r}$, $\exists\ C \in \mathbb{R}_{>0}$ such that $|a_n|, |b_n| \leq C \cdot B^n$ ($\forall n$). Hence

$$|\gamma_n| \leq \sum_{k=0}^{n} C^2 B^k B^{n-k} = (n+1) C^2 B^n,$$

and

$$|\gamma_n|^{1/n} \le (n+1)^{1/n} C^{2/n} B \xrightarrow[n\to\infty]{\lim} B.$$

So $\limsup |\gamma_n|^{1/n} \le B$, for any $B > \frac{1}{r}$

$\Rightarrow \limsup |\gamma_n|^{1/n} \le \frac{1}{r} \Rightarrow fg \in \mathcal{L}_r$. □

Proof of (ii): (again, "⊆" only)

Does $\underbrace{(fg)}_{\text{formal power series product}}(z) = f(z) \cdot g(z)$?

(↑ function product)

Fix $z \in D_r$: $f_n(z) \longrightarrow f(z)$ $(n \to \infty)$

$g_n(z) \longrightarrow g(z)$

(by (i)) $(fg)_n(z) \longrightarrow (fg)(z)$ $\Longrightarrow$

$$|f(z)g(z) - (fg)(z)| \le |f(z)g(z) - f_n(z)g_n(z)|$$
$$+ |f_n(z)g_n(z) - (fg)_n(z)| + |(fg)_n(z) - (fg)(z)|$$

(circled terms: easy to bound each $< \epsilon/3$ for $n \ge N$ suff. large)

$$|f_n(z)g_n(z) - (fg)_n(z)| \le \sum_{n \ge N+1} \left(\sum_{k=0}^{n} |a_k||b_{n-k}| \right) |z|^n,$$

which we can get $< \epsilon/3$ as

(i) applied to $\sum |a_n| T^n$ & $\sum |b_n| T^n$ $(\in \mathcal{L}_r)$

$\Longrightarrow \sum_n \left(\sum_{k=0}^{n} |a_k||b_{n-k}| \right) T^n \in \mathcal{L}_r$

$< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$; done. □

Alternately, we could have argued in terms of rearrangements of absolutely convergent series.

Proof of (iii): Take $f(T) \in \mathcal{L}_r$.

If $f(T) \neq 0$ then it has (say) order m, and

$$f(T) = a_m T^m (1 + h(T)), \quad \begin{cases} h(T) \in \mathcal{L}_r \Rightarrow h(z) \in C^0(D_r) \\ \mathrm{ord}(h) \geq 1 \Rightarrow h(0) = 0 \end{cases}$$

Hence, $\exists\, \delta > 0$ s.t. $z \in D_\delta \Rightarrow |h(z)| < \frac{1}{2}$
$\Rightarrow |1 + h(z)| > \frac{1}{2}$

$\therefore\ z \in D_\delta^* \Rightarrow f(z) \neq 0$.

$\Longrightarrow\ 0 \notin \mathrm{acc}\{z \mid f(z) = 0\}$.

So, $0 \in \mathrm{acc}(\mathcal{S})$ and $f(z)|_{\mathcal{S}} \equiv 0 \Rightarrow 0 \in \mathrm{acc}\{z \mid f(z) = 0\}$
$\Rightarrow f(T) = 0$. □

There are nice statements for inverse and composition, and we conclude with those:

(A) $f(T) \in \mathcal{L}_r$ with power series (formal) inverse $g(T) \in \mathbb{C}[[T]]$

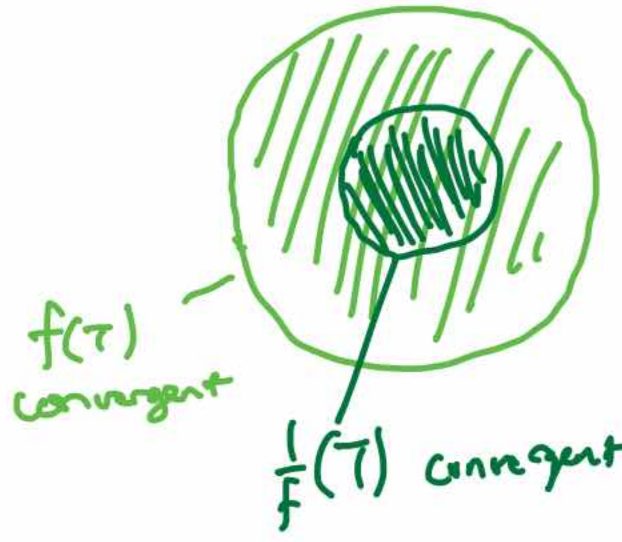
$\Longrightarrow$ $f(z)g(z) = 1$ and g is convergent (but possibly on a smaller D_r, i.e. $g(T)$ may not live in $\mathcal{L}_r$).

Proof: wolog $f = 1 - h$,

$$h = -a_1 T - a_2 T^2 - \cdots .$$

$$|a_n| \le B^n \ (n \ge 1) \implies$$

$$h(T) \prec \sum_{n \ge 1} B^n T^n = \frac{BT}{1 - BT}$$



$$\implies g := 1 + h + h^2 + \cdots \prec 1 + \frac{BT}{1-BT} + \frac{(BT)^2}{(1-BT)^2} + \cdots$$

$$= \frac{1}{1 - \frac{BT}{1-BT}}$$

$$= \frac{1-BT}{1-2BT}$$

$$= (1-BT)(1 + 2BT + 4B^2T^2 + \cdots)$$

$$\prec (1+BT)(1 + 2BT + 4B^2T^2 + \cdots)$$

each in $\mathcal{L}_{\frac{1}{2B}}$

$$\implies g(T) \in \mathcal{L}_{\frac{1}{2B}} .$$

□

(B) $f \in \mathcal{L}_r$ ($= \sum a_n T^n$), $\begin{cases} h \in \mathcal{L}_{s_0} \\ \text{with } \operatorname{ord}(h) > 0 \end{cases}$ ($= \sum c_n T^n$), and $\sum |c_n| s^n \le r$ for some $s \in [0, s_0]$

$$\implies (f \circ h)(T) \in \mathcal{L}_s, \quad (f \circ h)(z) \overset{(*)}{=} f(h(z)) \text{ on } D_s .$$

Proof: $\sum_{n\geq 0} |a_n| \left(\sum_{k\geq 1} |c_k| s_1^k\right)^n$ converges absolutely for any $s_1 \in (0,s)$ by assumption.

$\Longrightarrow$ the rearrangement $\sum_{l\geq 0} \beta_l T^l$ of

$\sum_{n\geq 0} |a_n| \left(\sum_{k\geq 1} |c_k| T^k\right)^n$ belongs to $\mathcal{L}_s$.

This dominates $f\circ h$; hence $f\circ h \in \mathcal{L}_s$.

Since absolutely convergent series can be rearranged without affecting the limit,

$$f(h(z)) = (f\circ h)(z)$$

also follows immediately on $|z| < s$. $\square$

Remark: If f is absolutely convergent on $\overline{D_r}$, then $(*)$ holds on $\overline{D_s}$.