Lecture 5: Power series

I. Continuous functions on compact sets

We begin with a general

Theorem Let $S \subseteq \mathbb{C}$ be compact, and let $f : S \to \mathbb{C}$ be continuous. Then

(i) $|f|$ attains a maximum on S

(ii) f is uniformly continuous on S

(iii) $f(S)$ is compact.

Before starting the proof, which will also work for $\mathrm{Re}(f)$, $\mathrm{Im}(f)$, etc., we need the

Definition f is uniformly continuous if for each $\epsilon > 0$ there exists $\delta > 0$ s.t.

$$z, w \in S \text{ AND } |z - w| < \delta \implies |f(z) - f(w)| < \epsilon.$$

Proof: (iii). We use a Theorem from lecture 4.

$$\underbrace{\{w_n\} \subseteq f(S)}_{\text{any given sequence}} \implies w = f(z_n) \text{ for } \{z_n\} \subseteq S$$

$$\underset{S \text{ compact}}{\implies} \exists \{n_k\} \subseteq \mathbb{N} \text{ s.t. } z_{n_k} \to v \in S$$

$$\underset{f \text{ continuous}}{\implies} w_{n_k} = f(z_{n_k}) \to f(v) \in f(S)$$

This gives a point of accumulation for w_n in $f(S)$, and so $f(S)$ is compact.

(i) (iii) holds, by the same proof, for continuous real-valued functions on S, in particular the composition

$$S \xrightarrow{f} \mathbb{C} \xrightarrow{|\cdot|} \mathbb{R}_{\geq 0}.$$

Let $b := \underbrace{\operatorname{lub}(|f(S)|)}_{\substack{\text{exists since} \\ |f(S)| \text{ compact} \\ \Downarrow \\ |f(S)| \text{ bounded}}} \in \overline{|f(S)|} \underset{\substack{\text{since} \\ |f(S)| \text{ compact} \\ \Downarrow \\ |f(S)| \text{ closed}}}{=\!=} |f(S)|$

$\implies \exists\, v \in S$ with $f(v) = b$,

and $b \geq |f(z)| \;\; \forall z \in S$.

(ii) Assume that f is NOT uniformly continuous, i.e.

$$\begin{cases} \exists \{z_n\}, \{w_n\} \subseteq S \quad \text{s.t.} \\ |z_n - w_n| < \frac{1}{n} \quad \text{AND} \quad |f(z_n) - f(w_n)| > \epsilon \end{cases}$$

[no δ is sufficiently small for all pairs of elements of S, in particular these]

S compact $\Rightarrow$ $\exists$ subsequences $\begin{cases} z_{n_k} \to v \\ w_{n_k} \to u \end{cases} \in S$

and then

$$|v-u| \le |v - z_{n_k}| + |z_{n_k} - w_{n_k}| + |w_{n_k} - u| \to 0$$

$\Rightarrow \quad v = u$

$\Rightarrow \quad f(v) = f(u)$

$\Rightarrow \quad |f(z_n) - f(w_n)| \le |f(z_n) - f(v)| + \cancel{|f(v) - f(u)|} + |f(v) - f(w_n)|$

$\to 0$ (since f continuous)

$\Rightarrow \quad f(z_n) - f(w_n) \to 0$

contradicts "$> \epsilon$" above

$\square$

Here is a quick application of this Theorem:

let $A, B \subseteq \mathbb{C}$ be subsets, and define

$$d(A,B) := \operatorname{glb}\{|z-w| \mid z \in A, w \in B\}.$$

$\boxed{\text{Corollary}}$

(i) $\underline{S \text{ closed AND } v \in \mathbb{C} \setminus S}$

$\underline{\Rightarrow \exists\, w_0 \in S \text{ s.t. } d(S, v) = |w_0 - v|.}$

(ii) $\underline{S \text{ closed AND } K \text{ compact}}$

$\underline{\Rightarrow \exists\, z_0 \in K,\ w_0 \in S \text{ s.t. } d(K, S) = |z_0 - w_0|.}$

II. Convergence of function series

Let $S \subseteq \mathbb{C}$ be a subset, and denote

$$\mathcal{F}(S) := \mathbb{C}\text{-valued functions on } S$$

$$\cup\!\!\!\!\mid$$

$$C^0(S) := \text{continuous } \mathbb{C}\text{-valued fcns.}$$

Convergence of series

- $\sum_{k=0}^{\infty} a_k = L \overset{\text{def.}}{\iff} \left\{ \sum_{k=0}^{n} a_k \right\}$ has limit L. ($a_k \in \mathbb{C}$)

Similarly, for $f_i, F \in \mathcal{F}(S)$,

- $\sum_{k=0}^{\infty} f_k = F \overset{\text{def.}}{\iff} \forall z \in S, \left\{ \sum_{k=0}^{n} f_k(z) \right\}$ has limit $F(z)$.

  (pointwise concept)

- AC **Absolute convergence** of $\sum_k a_k$ resp. $\sum f_k$ means convergence of $\sum_k |a_k|$ resp. $\sum |f_k|$.

$\boxed{\text{Proposition}}$ AC $\Rightarrow$ convergence.

Proof: Given $\epsilon > 0$. Since $z_n := \sum_{k=0}^{n} |a_k| \to z$, we can choose N sufficiently large that

$$\begin{cases} n > m \geq N \Longrightarrow \\ \qquad \epsilon > |z - z_m| = z - z_m = \sum_{k=m+1}^{\infty} |a_k| \geq \sum_{k=m+1}^{n} |a_k| . \end{cases}$$

Set $s_n := \sum_{k=0}^{n} a_k$; then

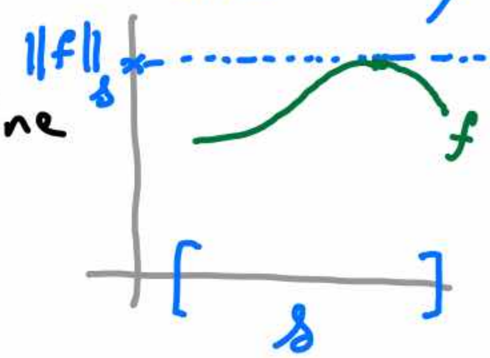
$$\begin{cases} n > m \geq N \Longrightarrow \\ \qquad |s_n - s_m| = |a_{m+1} + \cdots + a_n| \leq \sum_{k=m+1}^{n} |a_k| < \epsilon \end{cases}$$

$\Longrightarrow \{s_n\}$ Cauchy

$\xRightarrow[\text{completeness of } \mathbb{C}]{} \{s_n\}$ convergent. $\square$

Uniform convergence (= Convergence in sup/L^∞ norm)

For $f \in \mathcal{F}(S)$ bounded, define



$$\|f\|_S := \operatorname{lub}\{\, |f(z)| \mid z \in S \,\}.$$

(sup norm)

(This is a distance function on $\mathcal{F}(S)$.)

We say a sequence $\{f_n\} \subseteq \mathcal{F}(S)$ converges uniformly to $f \in \mathcal{F}(S)$ $\overset{\text{def.}}{\iff}$

$$\begin{cases} \forall \epsilon > 0, \ \exists N \in \mathbb{Z}_{>0} \text{ s.t.} \\ n \geq N \Rightarrow \|f_n - f\|_S < \epsilon. \end{cases}$$

This implies (pointwise) convergence since

$$\|f_n - f\|_S \geq |f_n(z_0) - f(z_0)| \quad (\forall z_0 \in S).$$

Theorem

(i) $\{f_n\}$ Cauchy in sup norm $\Rightarrow$ $\{f_n\}$ converges uniformly (i.e. in sup norm).

(ii) If $\{f_n\} \subseteq C^0(S)$ converges uniformly to $f \in \mathcal{F}(S)$, then $f \in C^0(S)$.

Proof: (i). $f_n(z)$ Cauchy $\forall z \in S$, so we can put

$$f(z) := \underbrace{\lim_{n \to \infty} f_n(z)}_{\text{uses completeness of } \mathbb{C}}.$$

Take $\epsilon > 0$ and N sufficiently large that

$$|f_m(z) - f_n(z)| < \frac{\epsilon}{2} \quad (\forall\ m, n \geq N \ \&\ z \in S).$$

For <u>any particular z_0</u>, we can take <u>m</u> suff. large that also

$$|f(z_0) - f_m(z_0)| < \frac{\epsilon}{2},$$

hence that

$$|f(z_0) - f_n(z_0)| < \epsilon.$$

But n <u>doesn't</u> depend on z_0.

(ii). Take $\epsilon > 0$ and $z_0 \in S$. There exist

- n suff. large that $\|f - f_n\|_S < \epsilon/3$
- $\delta > 0$ suff. small that $|f_n(z) - f_n(z_0)| < \epsilon/3$ for $|z - z_0| < \delta$.

Hence, $|f(z) - f(z_0)| \leq$

$$|f(z) - f_n(z)| + |f_n(z) - f_n(z_0)| + |f_n(z_0) - f(z_0)|$$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

for $|z - z_0| < \delta$. $\square$

We say that $\sum_i f_n$ converges uniformly $\iff$ the sequence of partial sums does.

$\boxed{\text{Theorem (Weierstraß M-test)}}$ Suppose $c \in \mathbb{R}_{\geq 0}$, $\sum_i c_n$ converges, c_n bounds $|f_n|$ on S, and $f_n \in C^0(S)$. Then $\sum_i f_n \in C^0(S)$.

Proof: Let $\epsilon > 0$. Then $n > m \geq N$ suff. large

$$\Rightarrow \epsilon > c_{m+1} + \cdots + c_n \geq \|f_{m+1}\|_S + \cdots + \|f_n\|_S \geq \|f_{m+1} + \cdots + f_n\|_S$$

$\Rightarrow \sum_{k=0}^{n} f_k =: s_n$ is Cauchy in $\|\cdot\|_S$, hence converges uniformly.

But $s_n \in C^0(S) \Rightarrow \lim_{n\to\infty} s_n = \sum_i f_n$ is too. $\square$

III. Convergent power series

The M-test has as an immediate consequence:

$\boxed{\text{Corollary}}$ $\{a_n\} \subseteq \mathbb{C}$, $s \in \mathbb{R}_{>0}$ s.t. $\sum |a_n| s^n$ converges

$\Longrightarrow$ $\sum a_n z^n$ $\left\{ \begin{array}{l} \bullet \text{ converges absolutely \& uniformly} \\ \bullet \text{ defines a continuous function} \end{array} \right\}$ on $|z| \leq s$.

Define the radius of convergence of $\sum a_n z^n$ by

$$r := \text{lub}\left\{ s \in \mathbb{R}_{\geq 0} \text{ s.t. } \sum |a_n| s^n \text{ converges} \right\}.$$

By the Corollary, $\sum |a_n||z|^n$ and $\sum a_n z^n$ are

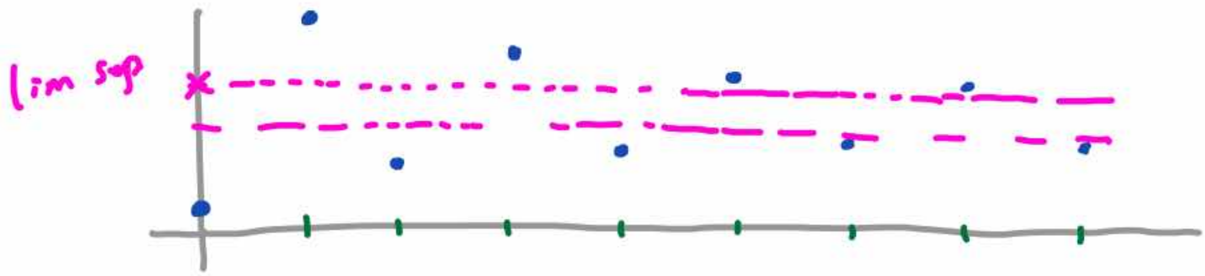
- convergent for $|z| < r$
- divergent for $|z| > r$.

Formulas for the radius of convergence.

Recall that for a bounded sequence $\{t_n\} \subseteq \mathbb{R}$,

$$\limsup t_n := \lim_{n \to \infty} \text{lub}\{t_k \mid k \geq n\} = \text{lub}\left\{ \begin{array}{c} \text{accumulation pts.} \\ \text{of } \{t_n\} \end{array} \right\}.$$

Since the accumulation points constitute a nonempty and bounded set, the lim sup exists. Picture:



For the radius, we then have:

- <u>root test</u>: $r = \left(\limsup |a_n|^{1/n}\right)^{-1}$
- <u>ratio test</u>: $r = \lim\limits_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ if it exists.

<u>Proof of ratio test</u>: glorified comparison-to-geometric-series.

<u>Proof of root test</u>: Set $A_n := |a_n|^{1/n}$,

$A := \limsup A_n \in [0,\infty)\cup\{\infty\}$.

- <u>CASE 1</u>: $\{A_n\}$ unbounded $\left(\begin{array}{l} \Leftrightarrow A = \infty \\ \Leftrightarrow \text{can extract } A_{n_k} \geq k \end{array}\right)$

In this case, $(|a_n| s^n)^{1/n} = A_n s$ is unbounded for any $s \in \mathbb{R}_{>0}$. Therefore $|a_n| s^n$ is unbounded $\Rightarrow \sum |a_n| s^n$ doesn't converge $(\forall s \in \mathbb{R}_{>0}) \Rightarrow r = 0$.

• CASE 2 : $\{A_n\}$ bounded $\left(\begin{array}{l} \Longleftrightarrow A \in [0, \infty) \\ \Longleftrightarrow \{A_n\} \subseteq [0, B] \text{ for some } B \in \mathbb{R}_{\geq 0} \end{array} \right)$

If infinitely many $A_n \geq A + \epsilon$, one would have a subsequence $\subseteq [A+\epsilon, B]$, with sub-subsequence converging to a number in $[A+\epsilon, B]$. (Bolzano-Weierstraß)

This is impossible as $A = \text{lub}\{\text{limits of subsequences of } \{A_n\}\}$.

Therefore, given $\epsilon > 0$, $\exists N$ s.t.

$$n \geq N \Rightarrow A_n < A + \epsilon \quad \left(\Rightarrow |a_n| < (A+\epsilon)^n\right).$$

For $s < \frac{1}{A+\epsilon}$,

$$(*) \quad \sum |a_n| s^n \leq \sum (A+\epsilon)^n s^n = \frac{1}{1-(A+\epsilon)s}.$$

Since $r = \text{lub}\{\text{those "} s \text{" for which } (*) \text{ converges}\}$,

$$r \geq \frac{1}{A+\epsilon}.$$

As $\epsilon > 0$ was arbitrary, we find

$$r \geq \frac{1}{A} \quad \text{if } A \in (0, \infty)$$

and

$$r = \infty \quad \underbrace{\text{if } A = 0}_{\text{done in this case}}.$$

Now if $A \in (0, \infty)$, then we must have infinitely many $A_n \geq A - \epsilon$ ($\Rightarrow |a_n| \geq (A-\epsilon)^n$).
Thus $\sum |a_n| s^n$ cannot converge for $s \geq \frac{1}{A-\epsilon}$, and so

$$r \leq \frac{1}{A-\epsilon}.$$

Again, since ϵ was arbitrary,

$$r \leq \frac{1}{A}. \qquad \square$$

Corollary to the root test: Let $b > \frac{1}{r}$.
Then $\limsup \frac{|a_n|}{b^n} = \left(\limsup \frac{|a_n|^{1/n}}{b}\right)^n$

$$= \lim_{n\to\infty} \left(\frac{1}{br}\right)^n \left(= 0\right) < 1$$

$\Rightarrow$ for $n \geq N$, $|a_n| \leq b^n$. Throw in a constant to get this to hold for $a_0, \dots, a_{N-1}$: for appropriate $C \in \mathbb{R}_{>0}$,

$$\boxed{|a_n| \leq b^n \cdot C.}$$

Remark: This is what all convergent power series have in common.

IV. Examples

① What does a NON convergent power series look like?

- $\sum n z^n$? no — that would ruin derivatives of power series!
- $\sum n^n z^n$? $A = \limsup |n^n|^{1/n} = \limsup n = \infty$
  $\Longrightarrow r = 0$.
- $\sum e^n z^n$? $A = \limsup |e^n|^{1/n} = e$
  $\Rightarrow r = \frac{1}{e}$. Nope.
- $\sum n! z^n$? Use

$$(2n)! \geq (2n)(2n-1) \cdots n \geq n^n$$

$$\Rightarrow ((2n)!)^{1/2n} \geq (n^n)^{1/2n} = \sqrt{n} \to \infty$$

$$\Rightarrow r = 0.$$

- $\sum \frac{n!}{n^n} z^n$??? we need a more powerful tool....

② Stirling's formula (easy version)

$$\sum_{k=2}^{n} \log k \geq \int_1^n \log(x)\, dx \geq \sum_{k=1}^{n-1} \log k$$

recall antiderivative
$x\log(x) - x$

$$\log(n!) \geq \underbrace{n\log(n)}_{=\log(n^n)} - n + 1 \geq \log((n-1)!)$$

$$n! \geq n^n \cdot e^{-n} \cdot e \geq (n-1)!$$

Set $u_n := \dfrac{n!}{n^n e^{-n}}$, so $n! = u_n n^n e^{-n}$ and

$$(ne)^{1/n} \geq u_n^{1/n} \geq e^{1/n}$$

$$n \to \infty$$

$$1$$

squeeze lemma

use: $\log n^{1/n} = \frac{1}{n}\log n \to 0$

Hence, for $\sum \underbrace{\dfrac{n!}{n^n}}_{= a_n} z^n$,

$$\frac{1}{r} = \limsup |a_n|^{1/n} = \lim_{n\to\infty} \left| \frac{u_n \cancel{n^n} e^{-n}}{\cancel{n^n}} \right|^{1/n} = \frac{1}{e}$$

$$\Longrightarrow \quad r = e.$$

③ Binomial & hypergeometric series.

ⓐ Define $(1+z)^{\alpha} := \sum_{n \geq 0} \binom{\alpha}{n} z^n$, where

$$\binom{\alpha}{n} := \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$$ for $\alpha \in \mathbb{C}$. (esp. useful for $\alpha = \frac{1}{m}$)

To check $r=1$, we ratio test:

$$\left| \binom{\alpha}{n} \Big/ \binom{\alpha}{n+1} \right| = \left| \frac{\alpha \cdots (\alpha-n+1)}{\alpha \cdots (\alpha-n)} \right| \frac{(n+1)!}{n!} = \frac{n+1}{|\alpha - n|} \xrightarrow[n\to\infty]{} 1$$

ⓑ For $\sum_{m \geq 0} \underbrace{\frac{(3m)!}{(m!)^3}}_{= a_{3m}} t^{3m}$ (a function that arises from integrals on the cubic curve $\underbrace{\{xy = t(x^3+y^3+1)\}}_{C_t} \subseteq \mathbb{C}^2$)

the ratio test gives (thinking of this as a power series in t^3)

$$r^3 = \lim_{m\to\infty} \frac{(m+1)^3}{(3m+3)(3m+2)(3m+1)} = \frac{1}{3^3}$$

$\Rightarrow$ $r = \frac{1}{3}$ (= smallest nonzero value of $|t|$ for which the curve C_t is singular).