Lecture 40: Little Picard

(already done for functions of finite order)

A corollary of the Hadamard factorization theorem† is that if $f \in \mathcal{H}ol(\mathbb{C})$ is zero-free, and there exist $C \geq 1$ & $\rho \in \mathbb{R}_+$ s.t. $\|f\|_{D_R} \leq C^{R^\rho}$ for $R \gg 0$, then $f = e^h$ for h a polynomial of degree $\leq \rho$. Our aim here will be to get some more refined results involving upper & lower bounds for the (almost) average of $|f|$ around large circles.

For (more generally) $f \in \mathcal{M}er(\mathbb{C})$, an interesting object of study perhaps suggested by our brief tour of Jensen's formula & Mahler measures is

$$m_f(r) := \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\varphi})| \, d\varphi,$$

where for $\alpha \in \mathbb{R}_+$

$$\log^+ \alpha := \max\{0, \log \alpha\}.$$

† There is also a direct argument, similar to the proof of Liouville.

Lemma 1: (a) $\log^+ \alpha_1\alpha_2 \leq \log^+ \alpha_1 + \log^+ \alpha_2$

(b) $\log^+(\alpha_1 + \cdots + \alpha_n) \leq \sum_{j=1}^{n} \log^+ \alpha_j + \log n$

(c) $\log^+ \alpha - \log^+\left(\frac{1}{\alpha}\right) = \log \alpha$

(d) $\log^+ \alpha + \log^+\left(\frac{1}{\alpha}\right) = |\log \alpha|$.

These are all pretty easy to check; e.g.

Proof of (b): $\log^+(\alpha_1 + \cdots + \alpha_n) \leq \log^+(n \cdot \max\{\alpha_j\})$

$$\leq \max_{j=1,\dots,n}\{\log^+ \alpha_j\} + \log n$$

$$\leq \sum_{j=1}^{n} \log^+ \alpha_j + \log n.$$

$\square$

Lemma 2: (a) $m_{fg} \leq m_f + m_g$

(b) $m_{f_1 + \cdots + f_n} \leq \sum_i m_{f_i} + \log n$

(c) for $f \in Hol(\mathbb{C})$ zero-free, $m_f - m_{\frac{1}{f}} = \log |f(0)|$

Proof: All three use Lemma 1; (c) also uses MVT. $\square$

Let $N_f(0,R) :=$ number of zeroes of f in D_R.

The following should look familiar:

If $f(0) \neq 0$ set

$$n_f(0,R) := \int_0^R N_f(0,r)\, d\log r$$

$$= \sum_{\alpha \in D_r^*} \mathrm{ord}_\alpha(f) \cdot \log\left|\frac{R}{\alpha}\right|$$

which is clearly an increasing, continuous function of r.

If $f(0) \neq b$ set

$$n_f(b,R) := n_{f-b}(0,R).$$

<u>Lemma 3</u>: $f \in \mathrm{Hol}(\mathbb{C}) \Longrightarrow$

$$m_f(r) = \frac{1}{2\pi}\int_0^{2\pi} n_f(e^{i\theta}, r)\, d\theta + \log^+|f(0)|.$$

<u>Proof</u>: Jensen for $f(z) - e^{i\theta} \Longrightarrow$

$$n_f(e^{i\theta}, r) + \log|f(0) - e^{i\theta}| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\varphi}) - e^{i\theta}|\, d\varphi$$

$\frac{1}{2\pi}\int_0^{2\pi} (\cdot)\, d\theta$

$$\frac{1}{2\pi}\int_0^{2\pi} n_f(e^{i\theta}, r)\, d\theta + \underbrace{\frac{1}{2\pi}\int_0^{2\pi} \log|f(0) - e^{i\theta}|\, d\theta}_{= \log^+|f(0)|} = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\varphi}) - e^{i\theta}|\, d\theta}_{= \log^+|f(re^{i\varphi})|}\, d\varphi$$

$$= m_f(r).$$

Jensen for $(\beta - z)$ on ∂D_1

$\square$

Henceforth, assume $f \in \mathrm{Hol}(\mathbb{C})$ zero-free. Put

$$M_f(r) := \log \|f\|_{\partial D_r}.$$

Lemma 4: (a) $m_f \leq \max\{M_f, 0\}$

(b) $M_f(r) \leq 3 m_f(2r)$.

Proof of (b): Let $z \in \partial D_r$, $r \in (0, R)$

$$\implies \log|f(z)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})| \underbrace{\mathrm{Re}\left(\frac{Re^{i\theta}+z}{Re^{i\theta}-z}\right)}_{\leq \frac{R+r}{R-r}} d\theta$$

Poisson applied to $\log|f|$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(Re^{i\theta})| \frac{R+r}{R-r} d\theta$$

$$= \frac{R+r}{R-r} m_f(R)$$

$$\implies M_f(r) \leq \frac{R+r}{R-r} m_f(R). \quad \text{Take } R = 2r. \qquad \square$$

Lemma 5: (a) m_f bounded for $r \to \infty$ $\implies$ f constant

(b) If $\exists k$ s.t. $m_f(R_j) \leq k \log(R_j)$ for a sequence $R_j \to \infty$, then f is a polynomial.

Proof: (a) by Lemma 4(b), M_f is bounded. Apply Liouville.

(b) $f = \sum a_n z^n \xrightarrow[\text{Cauchy estimate}]{} |a_n| \leq \dfrac{\|f\|_{\partial D_R}}{R^n}$

$$\Longrightarrow \log|a_n| \le M_f(R) - n\log R$$

$$\overset{\text{Lemma 4(b)}}{\le} 3m_f(2R) - n\log R$$

$$\overset{\text{assumption}}{\le} 3k\log(2R) - n\log R$$

$$\overset{\substack{n>3k \\ \epsilon := n-3k}}{=} 3k\log 2 - \epsilon\log R$$

$$\longrightarrow a_n = 0 \quad \text{for } n > 3k. \qquad \square$$

Lemma 6: $r \in [1, R) \Longrightarrow$

$$m_{f'/f}(r) \le \log^+(R) + 2\log^+\left(\frac{1}{R-r}\right) + 2\log^+ m_f(R) + \text{const.}$$

Proof: $\log f(z) \in \mathcal{H}ol(\mathbb{C})$

$\quad \| $ Poisson: $z \in D_R$

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})| \frac{Re^{i\theta}+z}{Re^{i\theta}-z}\, d\theta + iK$$

const.

$\frac{d}{dz}$

$$\frac{f'}{f} = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})| \frac{2Re^{i\theta}}{(Re^{i\theta}-z)^2}\, d\theta$$

$$\Rightarrow \left|\frac{f'}{f}\right| \le \frac{2R}{(R-r)^2}\left\{m_f(R) + m_{\frac{1}{f}}(R)\right\}$$

(using Lemma 1(d))

$$\Longrightarrow \quad m_{f'/f}(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|\frac{f'}{f}(re^{i\theta})\right| d\theta$$

$$\leq \log^+ R + 2\log^+\left(\frac{1}{R-r}\right) + \log^+ m_f(R) + \underbrace{\log^+ m_{\frac{1}{f}}(R)}_{= \log^+ m_f(R) - \log|f(0)|} + 2\log 2$$

(using Lemma 1(a) & Lemma 2(c))

$\square$

Lemma 7: $S \in C^0(\mathbb{R}_+)$ increasing, non-negative, nonconstant

$$\Longrightarrow \boxed{S\left(r + \frac{1}{S(r)}\right) < 2S(r)} \quad (*)$$

for all $r \in \mathbb{R}_+$ outside a set E of $\underbrace{\text{finite measure}}$.

(i.e., $\subseteq \bigcup_i [a_i, b_i]$ s.t. $\sum_i (b_i - a_i) < \infty$)

Proof: Let $E \subset \mathbb{R}_+$ be where $(*)$ fails, and let $r_1 \in E$ be such that $S(r_1) \neq 0$.

Inductively construct

$$r_{n+1} := \inf\left\{r \in E \;\middle|\; r \geq r_n + \frac{1}{S(r_n)}\right\};$$

then $S(r_{n+1}) \underset{\substack{\uparrow \\ S \text{ increasing}}}{\geq} S\left(r_n + \frac{1}{S(r_n)}\right) \underset{\substack{\uparrow \\ r_n \in E}}{\geq} 2S(r_n) \geq 2^n S(r_1)$

$$\Longrightarrow \sum_{n \geq 1} \frac{1}{S(r_n)} \leq \sum_{n \geq 0} \frac{1}{2^n S(r_1)} = \frac{2}{S(r_1)}. \quad (**)$$

If $\{r_n\}$ is an infinite sequence, then it is unbounded: otherwise, the $\{r_n\}$ bunch up and the $\left\{\frac{1}{S(r_n)}\right\}$ have to go to 0 for finite r, contradicting the definition of S (continuous on all of $\mathbb{R}_+$). So $E \subset \bigcup_n \left[r_n, r_n + \frac{1}{S(r_n)}\right]$ which has finite measure by $(**)$. $\square$

Lemma 8: For all r sufficiently large and outside a set of finite measure,

$$m_{f'/f}(r) < C \cdot \{\log r + \log m_f(r)\}.$$

Proof: Take $S(r) := \log^+ m_f(r)$ in Lemma 7 (which is increasing by Lemma 3), $R := r + \frac{1}{\log^+ m_f(r)}$ in Lemma 6:

$$m_{f'/f}(r) \leq \underbrace{\log^+\left(r + \frac{1}{\log^+ m_f(r)}\right)}_{\substack{< \log r^2 \\ \text{eventually} \\ \text{(again use Lemma 5)}}} + \underbrace{2\log^+\left(\log^+ m_f(r)\right)}_{\substack{< 2\log^+ m_f(r) \\ \text{as soon as } m_f(r) > 1 \\ \text{(and } m_f \text{ is unbounded} \\ \text{[Lem. 5] \& increasing} \\ \text{[Lem. 3] unless } f \text{ constant)}}} + \underbrace{2\log^+ m_f\left(r + \frac{1}{\log^+ m_f(r)}\right)}_{\substack{< 2\log^+ m_f(r) \\ \text{outside } E \text{ (by} \\ \text{Lemma 7)}}}$$

in which case the statement is trivially true. $\square$

Now suppose $f_1, f_2 \in Hol(\mathbb{C})$ are zero-free and satisfy $f_1 + f_2 = 1$. Then also $f_1' + f_2' = 0$, so

$$f_1 = \frac{1}{\left(\frac{f_1+f_2}{f_1 f_2}\right) f_2} = \frac{1/f_2}{1/f_1 + 1/f_2} \overset{\text{mult. numerator \& denom. by } -f_1}{=} \frac{-f_1'/f_2}{-f_1'/f_1 - f_1'/f_2} = \frac{f_2'/f_2}{f_2'/f_2 - f_1'/f_1}$$

unless $0 = f_2'/f_2 - f_1'/f_1 = d\log(f_2/f_1) \Longrightarrow \frac{1}{f_1} - 1 = \frac{f_2}{f_1}$ constant

$\Longrightarrow f_1, f_2$ constant.

By Lemma 2, $\begin{cases} m_{1/f} = m_f - \log|f(0)| \\ m_{f_1 f_2} \leq m_{f_1} + m_{f_2} \\ m_{f_1+f_2} \leq m_{f_1} + m_{f_2} + \log 2 \end{cases}$

$$\Longrightarrow m_{f_1}(r) \leq m_{f_2'/f_2}(r) + \underbrace{m_{\left(\frac{1}{f_2'/f_2 - f_1'/f_1}\right)}(r)}$$

$= m_{(f_2'/f_2 - f_1'/f_1)}(r) + \text{const.}$

$\leq m_{f_2'/f_2}(r) + m_{f_1'/f_1}(r) + \text{const.}$

$$\Longrightarrow m_{f_1}(r) < C' \cdot \{\log r + \log m_{f_1}(r) + \log m_{f_2}(r)\}$$

for all suff. large $r \notin E$ (by Lemma 8)

$$\Longrightarrow m_{f_1}(r) + m_{f_2}(r) < C'' \cdot \{\log r + \log m_{f_1}(r) + \log m_{f_2}(r)\}$$

$$< C'' \cdot \left\{\log r + \frac{1}{2C''} m_{f_1}(r) + \frac{1}{2C''} m_{f_2}(r)\right\}$$

for all suff. large $r \notin E$

$$\Longrightarrow \quad \underbrace{m_{f_1}(r) + m_{f_2}(r)}_{\text{(both} > 0)} < 2C'' \log r$$

subtract $\frac{1}{2}(m_{f_1}(r)+m_{f_2}(r))$ from both sides.

for all suff. large $r \notin E$

— in particular, $\exists$ sequence $\to \infty$ of such r's.

Applying Lemma 5(b), conclude f_1 & f_2 are polynomials. But they're also zero-free, so are constant. This gives

$\boxed{\text{Picard's Little Theorem}}$ Let $f \in Hol(\mathbb{C})$ omit 2 values. Then f is constant.

Proof: Suppose f omits α, β. Then

$$f_1 := \frac{f-\alpha}{\beta-\alpha} \quad \text{and} \quad f_2 := \frac{\beta - f}{\beta-\alpha}$$

are zero-free, and $f_1 + f_2 = 1$. Above argument now applies. $\square$