Lecture 4 : Topology of the Complex Plane

I. Topological spaces

We begin with some generalities.

$\boxed{\text{Definition}}$ A topological space is a pair (X, Σ), where X is a set (of "points"), and Σ is a collection of subsets of X including:

(a) X itself and the empty set $\emptyset$

(b) finite intersections $\bigcap_{j=1}^{n} U_j$ if each $U_j \in \Sigma$

(c) arbitrary unions $\bigcup_{j \in J} U_j$ if each $U_j \in \Sigma$.

Σ is called a topology on X; members of Σ are called open sets ($\Leftrightarrow$ complement is closed).

Remarks:
- notation for complement of $\mathcal{U} \in \Sigma$ is $\mathcal{U}^c$ or $\mathbb{X} \setminus \mathcal{U}$
- We will frequently write "$\mathbb{X}$" instead of "$(\mathbb{X}, \Sigma)$" for topological spaces. //

Some more terminology:

- $\mathbb{X}$ is Hausdorff $\iff$

  $\forall$ distinct $x, y \in \mathbb{X}$, $\exists$ disjoint open $\mathcal{U}, \mathcal{V} \in \Sigma$ with $\mathcal{U} \ni x$ and $\mathcal{V} \ni y$

- A basis (or base) for the topology is a "generating set" $\mathcal{B} \subseteq \Sigma$; i.e. every $\mathcal{U} \in \Sigma$ is a union of elements of $\mathcal{B}$.

- Let $\mathcal{S} \subseteq \mathbb{X}$ be a subset (not necessarily belonging to Σ!). We denote

$$(\mathcal{S} \supseteq)\ \mathring{\mathcal{S}} := \{x \in \mathbb{X} \mid \exists\, \mathcal{U} \overset{\ni x}{\in} \Sigma \text{ s.t. } \mathcal{U} \subseteq \mathcal{S}\}$$ interior of $\mathcal{S}$

$$\partial \mathcal{S} := \{x \in \mathbb{X} \mid \forall\, \mathcal{U} \overset{\ni x}{\in} \Sigma,\ \mathcal{U} \cap \mathcal{S} \neq \emptyset \neq \mathcal{U} \cap \mathcal{S}^c\}$$ boundary of $\mathcal{S}$

$$\overline{\mathcal{S}} := \{x \in \mathbb{X} \mid \forall\, \mathcal{U} \overset{\ni x}{\in} \Sigma,\ \mathcal{U} \cap \mathcal{S} \neq \emptyset\} = \mathring{\mathcal{S}} \amalg \partial \mathcal{S}$$ closure of $\mathcal{S}$

$$\cup\!|$$

$$\mathrm{acc}(\mathcal{S}) := \{x \in \mathbb{X} \mid \forall\, \mathcal{U} \overset{\ni x}{\in} \Sigma,\ \mathcal{U} \setminus \{x\} \cap \mathcal{S} \neq \emptyset\}$$ accumulation/limit points of $\mathcal{S}$

- subspace topology: $\Sigma_S := \{ \mathcal{U} \cap S \mid \mathcal{U} \in \Sigma \}$.
- S is connected $\Longleftrightarrow$ S cannot be written as the disjoint union $\mathcal{U} \amalg \mathcal{V}$ of nonempty $\mathcal{U}, \mathcal{V} \in \Sigma_S$

Remark: Any S has a unique decomposition into connected components (=: maximal clopen connected subsets).

- S is compact $\Longleftrightarrow$ every open cover has a finite subcover: i.e.

$$\left(\begin{array}{c} S \subseteq \bigcup_{j \in J} \mathcal{U}_j \quad (\text{each } \mathcal{U}_j \in \Sigma) \\ \Downarrow \\ \exists \{j_1, \dots, j_n\} \subseteq J \text{ s.t. } S \subseteq \bigcup_{i=1}^{n} \mathcal{U}_{j_i} \end{array} \right)$$

Distance.

Given a distance function

$$d : \mathbb{X} \times \mathbb{X} \to \mathbb{R},$$

PROPERTIES:

- symmetric
- nonnegative
- $0 \Longleftrightarrow x = y$
- Δ inequality

it makes sense to take as basis for Σ the "open disks"

$$D(x_0, r) := \{ x \in \mathbb{X} \mid d(x, x_0) < r \}.$$

If Σ is constructed in this way, the triple $(\mathbb{X}, \Sigma, d)$ is called a metric space.

Remark: I will also use the notation

$$\overline{D}(x_0, r) := \{x \in X \mid d(x, x_0) \leq r\}$$

and $D^*(x_0, r) := \{x \in X \mid 0 < d(x, x_0) < r\}$. //

Examples //

X	name of metric	$d(z_1, z_2)$
$\mathbb{C}$	Euclidean	$\lvert z_1 - z_2 \rvert$
$\mathbb{C}$	stereographic	$\dfrac{2\lvert z_1 - z_2\rvert}{\sqrt{(1+\lvert z_1\rvert^2)(1+\lvert z_2\rvert^2)}}$
$\mathfrak{h}$ upper half-plane	Poincaré	$\log\left(\dfrac{\lvert z_1 - \overline{z_2}\rvert + \lvert z_1 - z_2\rvert}{\lvert z_1 - \overline{z_2}\rvert - \lvert z_1 - z_2\rvert}\right)$
Δ unit disk	Poincaré	$\tanh^{-1}\left(\dfrac{\lvert z_1 - z_2\rvert}{\lvert 1 - z_1\overline{z_2}\rvert}\right)$

//

II. The complex plane

Henceforth we work on $X = \mathbb{C}$ with the Euclidean metric $d(x,y) = |x-y|$. In particular, open sets contain disks around each of their points.

Examples of subsets of $\mathbb{C}$ //

① $\mathcal{S}$: 

$\partial \mathcal{S}$ = entire circle

$\mathring{\mathcal{S}}$ = shaded area

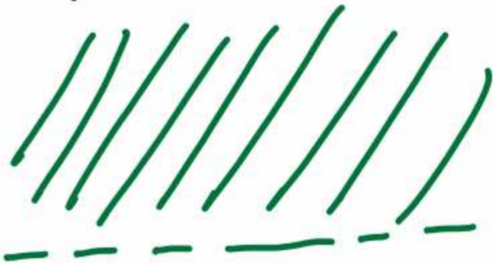
$\mathrm{acc}(\mathcal{S}) = \overline{\mathcal{S}} = \mathring{\mathcal{S}} \sqcup \partial \mathcal{S}$

② $\mathcal{S}$: 

here $\mathcal{S} = \overline{\mathcal{S}}$ ($\mathcal{S}$ is closed)

and $\mathrm{acc}(\mathcal{S}) = \emptyset$

③ (upper half plane)

$S = \mathfrak{h}$:  $S = \mathring{S}$ (open set)

④

S: 

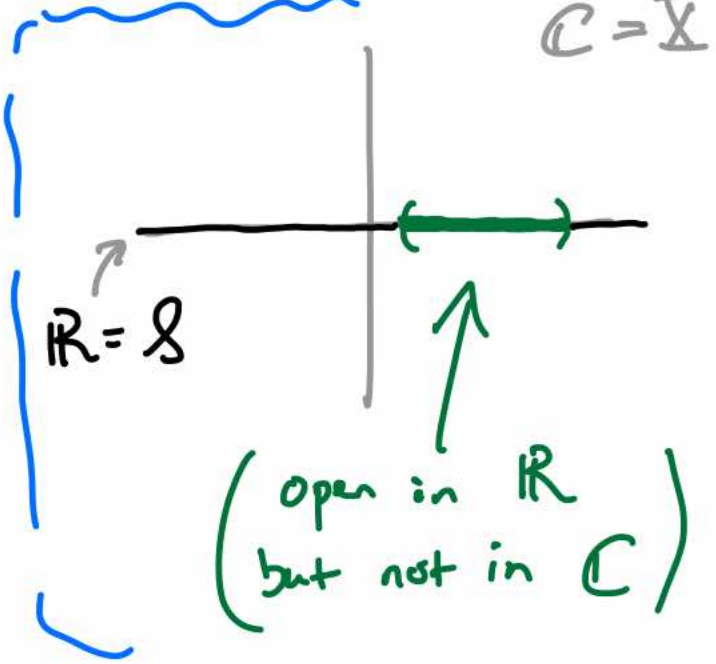
$\mathrm{acc}(S) = \{x\}$, whether or not $x \in S$.

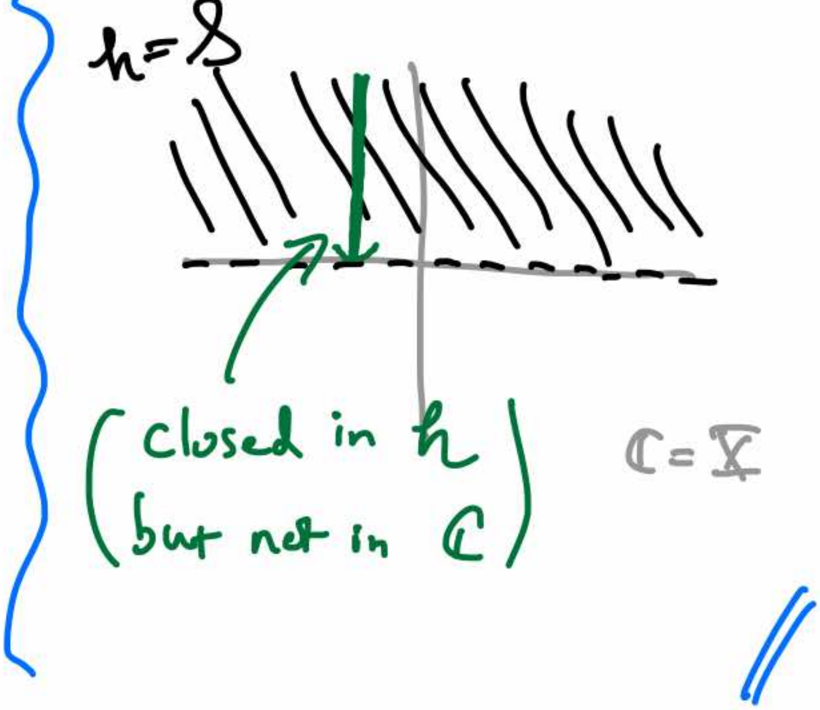
Remarks:

(i) The definition of $x \in \mathrm{acc}(S)$ says that for any $\epsilon > 0$, $D^*(x, \epsilon) \cap S$ is nonempty. In effect, this means that $D(x,\epsilon) \cap S$ contains infinitely many points. For $X = \mathbb{C}$, $\mathring{S} \subseteq \mathrm{acc}(S) \subseteq \bar{S}$ but we need not have $\mathrm{acc}(S) \subseteq S$ or $S \subseteq \mathrm{acc}(S)$ (cf. Examples ② & ④ above).

(ii) Given $\mathcal{S} \subseteq \mathbb{C}$, the open subsets of $\mathcal{S}$ are the $\{\mathcal{S} \cap \mathcal{U}\}$ for $\mathcal{U} \in \Sigma$ (i.e. $\mathcal{U}$ open in $\mathbb{C}$); the closed subsets are their complements (in $\mathcal{S}$). But $\mathcal{S} \cap \mathcal{U}$ (resp. $\mathcal{S} \setminus \mathcal{S} \cap \mathcal{U}$) may <u>not</u> be open (resp. closed) <u>in $\mathbb{C}$</u>. In this sense, openness & closedness are <u>relative</u> properties. (From the form of the definition of <u>compactness</u>, it is clear that <u>this</u> is an <u>absolute</u> property.)

Examples





III. Limits

Given $f : S \to \mathbb{C}$, $\alpha \in \overline{S}$ and $\beta \in \mathbb{C}$,

define

$$\underbrace{\lim_{\substack{z \to \alpha \\ \in S}} f(z) = \beta}_{} \overset{\text{def.}}{\iff} \forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } f(S \cap D(\alpha, \delta)) \subseteq D(\beta, \epsilon).$$

Now suppose $\boxed{\alpha \in S}$; then

so that this is defined!

$$f \text{ is continuous at } \alpha \overset{\text{def.}}{\iff} \lim_{\substack{z \to \alpha \\ \in S}} f(z) = \boxed{f(\alpha)}.$$

A sequence $\{z_n\}$ amounts to a (continuous) function

$$f : \underbrace{\left\{ \tfrac{1}{n} \;\middle|\; n \in \mathbb{Z}_+ \right\}}_{S} \to \mathbb{C},$$

and the limit of the sequence (if it exists) is given by

$$\lim_{n \to \infty} z_n = w \overset{\text{def.}}{\iff} \lim_{\substack{z \to 0 \\ \in S}} f(z) = w.$$

The accumulation points (or limit points) of the sequence are all the limits of subsequences $\{z_{k_i}\}$. (Note that the accumulation points $\mathrm{acc}(\{z_n\})$ of the set $\{z_n\}$ may be different — e.g., for constant sequences, empty!)

A sequence is Cauchy $\overset{\text{def.}}{\Longleftrightarrow}$ $\forall \epsilon > 0$, $\exists N \in \mathbb{Z}_+$ s.t. $n, m \geq N \Rightarrow |z_n - z_m| < \epsilon$.

$\Longrightarrow$ the sequence converges.
(since $\mathbb{C}$ is complete)

IV. Remarks on connectedness

Connectedness was defined in §I.

Real numbers

$I \subset \mathbb{R}$ is connected $\Longleftrightarrow$ I is an interval.†

Consequently, any bounded-above [resp. below], nonempty subset $S \subseteq \mathbb{R}$ has a least upper bound [resp. greatest lower bound], since the set

† see Ahlfors

of all upper bounds for $\mathcal{S}$ is easily shown to be connected and closed. More precisely:

$\underbrace{\mathrm{lub}(\mathcal{S})}$:= $s \in \mathbb{R}$ s.t. (i) $s \geq$ every element of $\mathcal{S}$

(always belongs to $\overline{\mathcal{S}}$)

(ii) if $t \geq$ every element of $\mathcal{S}$, then $t \geq s$ as well.

Complex numbers

For $\mathcal{S} \subset \mathbb{C}$, a path in $\mathcal{S}$ is a continuous function $f : [0,1] \to \mathcal{S}$, and the important thing to remember is:

Proposition Assume $\mathcal{S} \subseteq \mathbb{C}$ open. Then $\mathcal{S}$ is connected $\iff$ $\mathcal{S}$ is pathwise connected.

(i.e. $\forall \alpha, \beta \in \mathcal{S}$ $\exists$ path with $f(0) = \alpha$, $f(1) = \beta$)

Proof (idea):

($\Rightarrow$) $\mathcal{S}_0 \subset \mathcal{S}$ maximal pathwise connected
$\Rightarrow \mathcal{S}_0$ open & closed (use open balls)
$\Rightarrow \mathcal{S}_0$ maximal connected (so $\mathcal{S}_0 = \mathcal{S}$).

($\Leftarrow$) If $\mathcal{S} = \underset{\alpha}{\mathcal{U}} \amalg \underset{\beta}{\mathcal{V}}$ and $f(0) = \alpha$, $f(1) = \beta$,

we get $\underbrace{f^{-1}(\mathcal{U})}_{\text{open}} \amalg \underbrace{f^{-1}(\mathcal{V})}_{\text{open}} = [0,1]$, a contradiction. $\square$

A connected open set in $\mathbb{C}$ is called a region, and it is on such sets that we will generally want to study holomorphic functions.

V. Boundedness

We say that

$$\mathcal{S} \subseteq \mathbb{C} \text{ is bounded } \overset{\text{def.}}{\iff} |s| \leq \overset{\text{fixed}}{C} \in \mathbb{R}_+ \quad (\forall\, s \in \mathcal{S}).$$

Theorem (Bolzano-Weierstraß)

$$\mathcal{S} \subseteq \mathbb{C} \text{ bounded infinite} \implies \operatorname{acc}(\mathcal{S}) \neq \emptyset.$$

Proof: Let $\{r_n\} \subseteq \mathbb{R}$ be a bounded sequence.

We can extract either a nonincreasing or non-decreasing subsequence: if the subset of $\{r_n\}$ consisting of "elements $\geq$ all subsequent terms" is infinite, it gives a nonincreasing sequence.

Otherwise, one can construct a non-decreasing one. Now take glb or lub of that subsequence; this exists and must be its limit s.

Next let $\{s_n\} \subset \mathcal{S}$ be an arbitrary sequence of distinct elements (which exists as $|\mathcal{S}| = \infty$). We may extract a subsequence with convergent real part, then a sub-subsequence with convergent imaginary part (by the result for $\{r_n\}$ above). The latter $\{s_{k_m}\}$ must then not only converge to some $s \in \mathbb{C}$, but have $s \in \operatorname{acc}(\{s_{k_m}\})$ $(\subseteq \operatorname{acc}(\mathcal{S}))$ since all the s_{k_m} are distinct. $\square$

Theorem Let $K \subseteq \mathbb{C}$ be a subset. The following are equivalent:

(i) K is compact

(ii) Any $\{z_n\} \subseteq K$ has a point of accumulation in K (equivalently: a convergent subsequence with limits in K)

(iii) K is closed and bounded

$K = \overline{K}$

Proof: (ii) $\Rightarrow$ (i) : HW

(iii) $\Rightarrow$ (ii) : See proof of Bolzano-Weierstraß

(i) $\Rightarrow$ (iii): boundedness : take $\mathcal{U}_n = \{|z| < n\}$;

$\cup \mathcal{U}_n = \mathbb{C} \supset K$ ($\Rightarrow \{\mathcal{U}_n\}$ gives an open cover).

K compact $\Rightarrow K \subset \mathcal{U}_N$ for some N.

closedness: assume K not closed.

Take $p \in \partial K$, $\notin K$, and

$\mathcal{U}_n = \left(\overline{D}(p, \frac{1}{n})\right)^c$.

Then $\cup \mathcal{U}_n = \mathbb{C}\setminus\{p\} \supseteq K$.

K compact $\Rightarrow K \subseteq \mathcal{U}_N$ which is false as $\mathcal{U}_N^c$ must (as $p \in \partial K$) contain a point of K. $\square$

Remark: Any accumulation point of $\{z_n\} \subseteq K$ compact is contained in K, since K is closed.

Theorem Let $K \subseteq \mathbb{C}$ be compact, with nonempty closed nested subsets $(K \supseteq) \mathcal{S}_1 \supseteq \mathcal{S}_2 \supseteq \cdots$. Then $\cap \mathcal{S}_n \neq \emptyset$.

Proof: The $\mathcal{S}_n$ are compact (since closed, & inherit boundedness from K). For each m, choose $z_m \in \mathcal{S}_m$; K contains an accumulation point of $\{z_m\}$, say γ. This γ is an accumulation point of tails $\{z_m\}_{m \geq n} \subseteq \mathcal{S}_n$, hence $\mathcal{S}_n \ni \gamma$ ($\forall n$). Hence $\cap \mathcal{S}_n$ contains γ. $\square$

APPENDIX: An interesting non-Hausdorff space

A more general definition of the limit of a sequence, for arbitrary (X, Σ), is

$$\lim_{n\to\infty} x_n = x \overset{\text{def.}}{\iff} \forall\, \mathcal{U} \overset{\ni x}{\in} \Sigma,\ \mathcal{U} \cap \{x_n\} = \{x_n\} \setminus \text{finite set}$$

HW: Uniqueness of $\lim_{n\to\infty} x_n$ is implied by Hausdorffness of (X, Σ).

(Hence, we can show that a topological space is non-Hausdorff by exhibiting a "non-unique limit point".)

Example Writing

$$\Delta := D(0,1), \quad \Delta^* := D^*(0,1), \quad \ell(s) := \frac{\log(s)}{2\pi i},$$

define a lattice $\Gamma_s \subseteq \mathbb{C}^2$ for each $s \in \Delta$ by

- $s \in \Delta^*$: $\Gamma_s := \mathbb{Z}\left\langle \begin{pmatrix} i \\ i\ell(s) \end{pmatrix}, \begin{pmatrix} 0 \\ i \end{pmatrix}, \begin{pmatrix} 1 \\ \ell(s) \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$
- $s = 0$: $\Gamma_0 := \mathbb{Z}\left\langle \begin{pmatrix} 0 \\ i \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$.

Then $\bigcup_{s\in\Delta} \Gamma_s \times \{s\} =: \Gamma \subseteq \mathbb{C}^2 \times \Delta$ use Euclidean topology as a subset of $\mathbb{C}^3$

is reasonably nice (e.g. union of smooth submanifolds), but

$$\underline{X} := \frac{\mathbb{C}^2 \times \Delta}{\Gamma} \quad \text{is non-Hausdorff}$$

use quotient topology

Proof (sketch): Let $a \in \mathbb{Z}\setminus\{0\}$, $b \in \mathbb{C}$, and note that for $\Delta \ni s_n := e^{2\pi i (b+ni)/a}$,

$$\lim_{n\to\infty} s_n = 0 .$$

Consider

$$\underline{v_n} := \underbrace{a \begin{pmatrix} 1 \\ \ell(s_n) \end{pmatrix} - n \begin{pmatrix} 0 \\ i \end{pmatrix}}_{\in \Gamma_{s_n}} = \begin{pmatrix} a \\ b \end{pmatrix} .$$

Then

$$\begin{tikzcd}
(\underline{v_n}, s_n) \arrow[r, phantom, "\equiv \ \mathrm{mod}\ \Gamma_{s_n}"] \arrow[d] & (\underline{0}, s_n) \arrow[d] \\
\left(\begin{pmatrix} a \\ b \end{pmatrix}, s_n\right) \arrow[r, phantom, "\not\equiv \ \mathrm{mod}\ \Gamma_{s_0}"] & \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, s_n\right)
\end{tikzcd}$$

(in $\mathbb{C}^2 \times \Delta$)

← same sequence in $\underline{X}$

← limits distinct in $\underline{X}$!!!

□

Remark: There are easier examples of non-Hausdorff spaces, if one is willing to leave the realm of complex geometry: for instance, the Zariski topology.