Lecture 39 : Blaschke products

In Lecture 36, we proved (cf. §III)

(*) $f \in \underbrace{\mathcal{H}ol(D_1) \text{ bounded}}_{\text{i.e. } f \in H^\infty}$ & nonconstant $\Rightarrow \sum_{n \geq 1} (1 - |z_n|) < \infty$

where (as per our convention) $\{z_n\}$ are the zeros of f arranged in order of increasing modulus & repeated according to multiplicity.

We now prove a converse:

$\boxed{\text{Theorem}}$ If a sequence[†] $\{\alpha_n\} \subset D_1^*$ satisfies $\sum (1 - |\alpha_n|) < \infty$, then $\exists\, f \in H^\infty$ with zeros α_n.

Proof : Writing $\alpha_n = a_n e^{i\theta_n}$, define the Blaschke product[††]

$$\mathcal{B}(z) := \prod_{n \geq 1} \left(-e^{-i\theta_n}\right) \varphi_{\alpha_n}(z).$$

† arranged according to our convention

†† more generally, a factor of the form z^m is allowed (but no constant factor)

To check convergence, let $r \in (0,1)$ and consider

$$(**) \qquad \sum_{n \geq 1} \left| 1 + e^{-i\theta_n} \varphi_{\alpha_n}(z) \right|$$

on $\overline{D_r}$: for $n \gg 0$,

$$\left| 1 + e^{-i\theta_n} \varphi_{\alpha_n}(z) \right| = \left| 1 + \frac{\overline{\alpha}_n}{|\alpha_n|} \cdot \frac{z - \alpha_n}{1 - \overline{\alpha}_n z} \right|$$

$$= \left| \frac{|\alpha_n| - |\alpha_n| \overline{\alpha}_n z + \overline{\alpha}_n z - |\alpha_n|^2}{|\alpha_n| (1 - \overline{\alpha}_n z)} \right|$$

$$= \left| \frac{(|\alpha_n| + \overline{\alpha}_n z)(1 - |\alpha_n|)}{|\alpha_n| (1 - \overline{\alpha}_n z)} \right|$$

$z \in \overline{D_r}$, $\alpha_n \in D_1$

$$\leq \frac{(1+r)(1 - |\alpha_n|)}{|\alpha_n| (1-r)}$$

$n \gg 0 \Rightarrow |\alpha_n| \geq \frac{1}{2}$ (since $\sum (1 - |\alpha_n|) < \infty$)

$$\leq 2 \left(\frac{1+r}{1-r} \right) (1 - |\alpha_n|)$$

Since the sum of the $(1 - |\alpha_n|)$ converges, so therefore does $(**)$, uniformly on $\overline{D_r}$.

So, $(**)$ converges uniformly on compact sets; by Lect. 35 Prop. 1, the product defining B does also.

Thus $B \in \mathrm{Hol}(D_1)$.

Further, $\varphi_{\alpha_n}(z)$ has a simple zero at α_n, and there are finitely many zeroes in each compact subset of D_1. Hence $\mathcal{B}$ has the desired zeroes (and no more), courtesy of the same Proposition.

Finally, to show boundedness:

$$\|\varphi_{\alpha_n}\|_{D_1} \leq 1 \implies \|\mathcal{B}\|_{D_1} \leq 1. \qquad \square$$

The following is an analogue for D_1 of the Weierstraß/Hadamard product theorem (but much easier):

Corollary $f \in H^\infty$ with zeroes $\{z_n\}$ $\left(z_n = r_n e^{i\theta_n}\right)$ $\implies$

$$f(z) = z^{\operatorname{ord}_0(f) =: m} \cdot F(z) \cdot \prod_{n \geq 1} \left(-e^{i\theta_n} \varphi_{z_n}(z)\right) =: F(z) \cdot \mathcal{B}_f(z),$$

where $F \in H^\infty$ is zero-free, with $\|F\|_{D_1} = \|f\|_{D_1}$.

Proof: Set $F := \dfrac{f}{\mathcal{B}_f}$, where

- by $(*)$, $\sum(1-|z_n|) < \infty$
- hence, by the Theorem, $\mathcal{B}_f$ converges to a function in H^∞ with the same zeroes as f

$\Longrightarrow$ F has removable singularities at 0 and the $\{z_n\}$, so by Riemann extends to $\mathcal{H}ol(D_1)$ (and is nowhere zero).

Now $\|B_f\|_{D_1} \leq 1 \Longrightarrow \|F\|_{D_1} \geq \|f\|_{D_1}$. We need the reverse inequality, which will also give the boundedness (for F). Let $N \in \mathbb{Z}_+$ and set

$$\begin{cases} B_N(z) := \prod_{n=1}^{N} \left(-e^{i\theta_n} \varphi_{z_n}(z)\right) \in C^0(\overline{D_1}) \cap \mathcal{H}ol(D_1) \\ F_N(z) := \dfrac{f(z)}{z^m B_N(z)} \end{cases}.$$

Given $\epsilon > 0$,

hence uniformly continuous

$$\left.\begin{array}{l} B_N \in C^0(\overline{D_1}) \\ |B_N| = 1 \text{ on } \partial D_1 \end{array}\right\} \Longrightarrow \exists\, r_0 \in (0,1) \text{ s.t. } \underline{r \in (r_0, 1) \Rightarrow r^m |B_N(re^{i\theta})| > 1-\epsilon}.$$

By MMP, $\|F_N\|_{D_1} = \|F_N\|_{D_1 \setminus \overline{D_{r_0}}} \leq \dfrac{\|f\|_{D_1}}{1-\epsilon}$

$$\xRightarrow[\epsilon \to 0]{} \|F_N\|_{D_1} \leq \|f\|_{D_1}$$

$f \in H^\infty$

$$\xRightarrow[N \to \infty]{} \|F\|_{D_1} \leq \|f\|_{D_1} < \infty,$$

(do on compact subsets using uniform convergence of $F_N \to F$ there)

done. $\square$

Remark// If f actually $\underline{\text{is}}$ a Blaschke product in the above, then $\|F\|_{D}$, $= \|f\|_{D}$, becomes

$$1 = \|B\|_{D},$$

which is a general property of convergent Blaschke products. The question naturally arises, "what happens at the boundary?" To give a complete treatment of this would require more measure theory than I want to get into at present. Instead we shall assume the following

Lemma: $f \in H^{\infty} \Rightarrow \exists f^{*} \in L^{\infty}(\partial D)$ defined a.e. by $f^{*}(e^{i\theta}) = \lim_{r \to 1^{-}} f(re^{i\theta})$, and $\|f\|_{D} = \|f^{*}\|_{\partial D}$.

and use it to prove the

Proposition Given a Blaschke product B (assumed convergent, so that $B \in H^{\infty}$),

$$|B^{*}(e^{i\theta})| = 1 \text{ a.e. (on } \partial D).$$

Proof: Define

$$\mu(r) := \frac{1}{2\pi}\int_0^{2\pi} \log|B(re^{i\theta})|\,d\theta \qquad (r\in(0,1))$$

and observe that $\|B\|_{D_1} \le 1 \Rightarrow \log|B| \le 0$

$\Rightarrow \mu(r) \le 0$ is bounded above. Wolog assume

that B has no z^m factor, so that $B(0)\neq 0$.

Jensen's formula says

$$\mu(r) = \log|B(0)| + \sum_{n=1}^{N(r)} \log\left|\frac{r}{a_n}\right|$$

$\Rightarrow \mu$ is a monotonically increasing function of r

$\underset{\text{bounded above}}{\Longrightarrow} \lim_{r\to1^-}\mu(r) =: \hat{\mu}$ exists. By Fatou and

the Lemma, we may write

as lim sup of measurable functions, this is measurable. Thus so is B^*.

$$\begin{aligned}\hat{\mu} &= \limsup_{r\to1^-} \frac{1}{2\pi}\int_0^{2\pi}\log|B(re^{i\theta})|\,d\theta \\ &\underset{\text{Fatou}}{\le} \frac{1}{2\pi}\int_0^{2\pi}\log\Big(\underbrace{\limsup_{r\to1^-}|B(re^{i\theta})|}_{\text{" a.e.}}\Big)\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi}\log|B^*(e^{i\theta})|\,d\theta \\ &\le 0.\end{aligned}$$

(Lemma $\Rightarrow \|B^*\|_{\partial D} \le \|B\|_D \le 1 \Rightarrow \log|B^*| \le 0$)

This shows that

$$\mu^* := \frac{1}{2\pi}\int_0^{2\pi} \log|B^*(e^{i\theta})|\,d\theta$$

is defined.

Now writing (for any N)

$$B_{\geq N}(z) := \prod_{n\geq N}(-e^{i\theta_n})\varphi_{\alpha_n}(z),$$

$\log|B/B_{\geq N}|$ is continuous on an open set containing ∂D, and is zero on ∂D. Hence

$$\hat{\mu} = \lim_{r\to 1^-}\mu(r) = \lim_{r\to 1^-}\frac{1}{2\pi}\int_0^{2\pi}\log|B_{\geq N}(re^{i\theta})|\,d\theta$$

$$\geq \log|B_{\geq N}(0)|.$$

(Jensen for $B_{\geq N}$)

Putting everything together,

$$\log|B_{\geq N}(0)| \leq \hat{\mu} \leq \mu^* \leq 0$$

$$\Big\downarrow {\scriptstyle N\to\infty}$$

$$0 \implies 0 = 2\pi\mu^* = \int_0^{2\pi}\underbrace{\log|B^*(e^{i\theta})|}_{\leq 0 \text{ a.e.}}\,d\theta$$

$$\implies \log|B^*| = 0 \text{ a.e.} \Rightarrow |B^*| = 1 \text{ a.e.} \qquad \square$$

Remark // There is a general picture involving the Hardy spaces $(p \geq 1)$

$$H^p := \left\{ f \in \mathrm{Hol}(D_1) \;\middle|\; \underbrace{\sup_{r \in (0,1)} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{1/p}}_{=: \|f\|_{H^p}} < \infty \right\}$$

and

$$H^\infty := \left\{ f \in \mathrm{Hol}(D_1) \;\middle|\; \underbrace{\|f\|_{D_1}}_{=: \|f\|_{H^\infty}} < \infty \right\} \quad \text{(as above)}.$$

The "map" sending

$$f \mapsto \tilde{f}(e^{i\theta}) := \lim_{r \to 1^-} f(re^{i\theta})$$

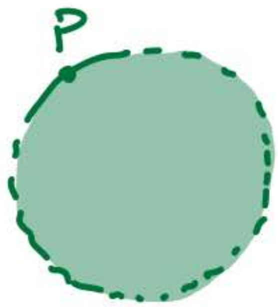
$\exists$ a.e.

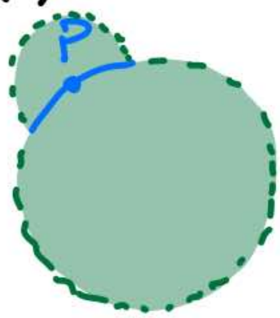
gives an embedding

$$H^p \hookrightarrow L^p(\partial D_1)$$

which "preserves norms": $\|\tilde{f}\|_{L^p(\partial D)} = \|f\|_{H^p}$ (i.e. gives a Banach subspace). A given $g \in L^p(\partial D)$ arises as $\tilde{f}$ iff $\hat{g}(n) := \frac{1}{2\pi} \int_0^{2\pi} g(e^{i\phi}) e^{-in\phi} \, d\phi = 0 \quad \forall n < 0$. //

An arbitrary Blaschke product has radial limits of modulus 1 at a.e. point of ∂D, as we have seen. The next step would seem to be the following: continuous extension to the union of D, & a small arc about some $e^{i\theta_0} =: P$, and an argument involving Schwarz reflection $f(z) \mapsto 1/\overline{f(1/\bar{z})}$, which should extend the function to something holomorphic on $D_1 \cup D(P, \epsilon)$ for some $\epsilon > 0$. So we would conclude that P is regular for our Blaschke product $\mathcal{B}$. Clearly P cannot then be an accumulation point for the zeroes of $\mathcal{B}$. So we have the





Proposition If $\mathcal{B}$ has zeroes $\{\alpha_n\}$ accumulating to $P \in \partial D$, then P is not regular for $\mathcal{B}$ (and in fact, $\mathcal{B}$ does not have a continuous extension to an arc $\subset \partial D$ containing P).

This gives us yet another way to construct "ugly functions on D_1":

Example// Take $\{\alpha_n\}$ to be $\{(1-\frac{1}{2^n})e^{i\log n}\}$.

This has every point of ∂D, as accumulation point! Further, $\sum(1-|1-\frac{1}{2^n}|) = \sum\frac{1}{2^n}$ converges, and $\therefore$ so does B. By the Proposition, no point of ∂D is regular for B, and $\tilde{B}$ is nowhere continuous. //