Lecture 38: Applications of product theorems

I. A remark on "interpolation"

We have seen how to construct functions on a region U with prescribed zeroes at a set of points in U (that has no limit point in U). Can we prescribe arbitrary values? In fact, we can do much better!

Theorem Let $U \subset \mathbb{C}$ be open/connected, $\mathcal{a} \subset U$ a set with no limit point in U, $m: \mathcal{a} \to \mathbb{Z}_{\geq 0}$ a function, and $\{\omega_{n,\alpha}\} \in \prod_{\alpha \in \mathcal{a}} \mathbb{C}^{m(\alpha)+1}$ (i.e., for each α, a choice of complex numbers $\omega_{0,\alpha}, \ldots, \omega_{m(\alpha),\alpha}$). Then there exists an $f \in \mathcal{Hol}(U)$ such that

$$\frac{f^{(n)}(\alpha)}{n!} = \omega_{n,\alpha} \quad (\forall \alpha \in \mathcal{a},\ n \in [0, m(\alpha)] \cap \mathbb{Z}).$$

(i.e., local power series coeffs.)

Proof: By the Weierstraß product theorem,

(*) $\exists\, g \in \mathrm{Hol}(U)$ s.t. $\mathrm{ord}_\alpha(g) = m(\alpha)+1$ $(\forall \alpha \in A)$,

with no other zeroes. We need to do a local calculation, for which we may take

$$\alpha = 0, \quad m := m(\alpha), \quad \omega_n := \omega_{n,\alpha}.$$

Near 0, by (*)

$$g(z) = \sum_{j \geq 1} b_j z^{m+j} \quad (b_1 \neq 0).$$

If $P(z) := \frac{c_1}{z} + \dots + \frac{c_{m+1}}{z^{m+1}}$, then

$$g(z)\,P(z) = (b_1 + b_2 z + b_3 z^2 + \dots)(c_1 z^m + \dots + c_{m+1})$$

we want $\|$

$$(\omega_0 + \omega_1 z + \dots + \omega_m z^m) + \text{higher-order terms}$$

Put
$$c_{m+1} = \omega_0 / b_1$$
$$c_m = (\omega_1 - b_2 c_{m+1})/b_1$$
$$c_{m-1} = (\omega_2 - b_3 c_{m+1} - b_2 c_m)/b_1$$
$$\vdots$$

$\Rightarrow$ gives desired local form to $f := g \cdot P$.

Now Mittag-Leffler's theorem $\Rightarrow \exists\, h \in \mathrm{Mer}(U)$ with principal parts P_α constructed in this manner, so that $f = g \cdot h$ gives the desired function. $\square$

II. Little Picard for functions of finite order

<u>Lemma</u>: Given $f \in \mathcal{H}ol(\mathbb{C})$ of finite order $\lambda_f \notin \mathbb{Z}$, and $\alpha \in f(\mathbb{C})$, $f^{-1}(\alpha)$ is an infinite set.

<u>Proof</u>: We may assume $\alpha = 0$. If $f^{-1}(\alpha) = \{z_1, \ldots, z_N\}$ then Hadamard product thm. $\Longrightarrow$ $f(z) = e^{h(z)} \prod_{n=1}^{N} (z - z_n)$ where h is a polynomial of degree $d \leq \lfloor \lambda_f \rfloor$.

Now (on the one hand)

$$\operatorname{ord}(e^h) = \limsup_{R \to \infty} \frac{\log \|h\|_{\partial D_R}}{\log R} = d \;;$$

while (on the other) for $R \gg 0$ †

$$\|e^h\|_{\partial D_R} = \left\| \frac{f}{\prod_{n=1}^{N}(z - z_n)} \right\|_{\partial D_R} \leq \frac{C_{\epsilon/2}\, e^{R^{\lambda_f + \epsilon/2}}}{(R/2)^N} \leq C_{\epsilon/2} e^{R^{\lambda_f + \epsilon}}$$

and

$$\|e^h\|_{\partial D_R} \geq \frac{\|f\|_{\partial D_R}}{(2R)^N} > \frac{C e^{R^{\lambda_f - \epsilon/2}}}{(2R)^N} > C e^{R^{\lambda_f - \epsilon}}.$$

So $\operatorname{ord}(e^h) = \lambda_f$, and $\left\{\begin{array}{l} \lambda_f \notin \mathbb{Z} \\ d \in \mathbb{Z} \end{array}\right. \Longrightarrow \lambda_f \neq d$.

This is a contradiction. $\square$

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† it may be necessary to take R in a particular sequence $\to \infty$ (b/c of lim sup vs. lim in formula for order).

$\boxed{\text{Theorem}}$ Let $f \in Hol(\mathbb{C})$ be nonconstant & of finite order. Then

(i) f assumes all values $\in \mathbb{C}$ except possibly one

(ii) if $\lambda_f \notin \mathbb{Z}$, f assumes each of these an infinite number of times, and in particular has infinitely many zeroes.

Proof: (ii) is done (by the Lemma) modulo the observation: if f omits 0, then $f = e^h$, h a polynomial by Hadamard, contradicting nonintegrality of the order λ_f.

(i) If $f(\mathbb{C}) \subset \mathbb{C} \setminus \{\alpha_1, \alpha_2\}$, then $f - \alpha_1$ is nowhere vanishing, so that $f(z) - \alpha_1 = e^{h(z)}$.

Now $(f - \alpha_1)(\mathbb{C})$ omits $\alpha_2 - \alpha_1 \Rightarrow h$ omits $\log(\alpha_2 - \alpha_1) + 2\pi i \mathbb{Z} \Rightarrow h - \beta$ is nowhere vanishing for $\beta \in \log(\alpha_2 - \alpha_1) + 2\pi i \mathbb{Z}$. But Hadamard $\Rightarrow h$ polynomial $\Rightarrow h - \beta$ polynomial $\xRightarrow[\text{Thm of Algebra}]{\text{Fundamental}}$ $h - \beta$ constant $\Rightarrow h$ constant $\Rightarrow f$ constant. $\square$

III. Ugly functions on D_1

In Lecture 35, we discussed the existence of functions on any region $U \subsetneq \mathbb{C}$ which are <u>nowhere regular</u> on the boundary ∂U. This was an application of Weierstraß products; now we will discuss one very concrete way to produce such functions if $U = D_1$, involving power series (not product theorems).

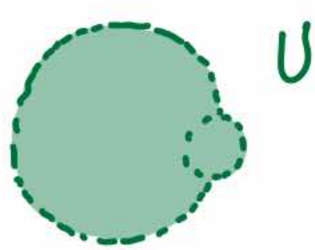
$\boxed{\text{Theorem (Hadamard)}}$ Given

- $\{p_n\} \subset \mathbb{Z}_+$ increasing sequence with $\frac{p_{n+1}}{p_n} > \lambda$ $(\forall n)$ for some $\lambda \in (1, \infty)$
- $\{\alpha_n\} \subset \mathbb{C}$ s.t. $f(z) := \sum \alpha_n z^{p_n}$ has radius of convergence 1.

"gap series"

Then no point of ∂D_1 is regular for f.

Proof: Suppose otherwise; wolog wma f extends to $F \in \mathcal{H}ol(\underbrace{D_1 \cup D(1,\epsilon)}_{=:U})$.

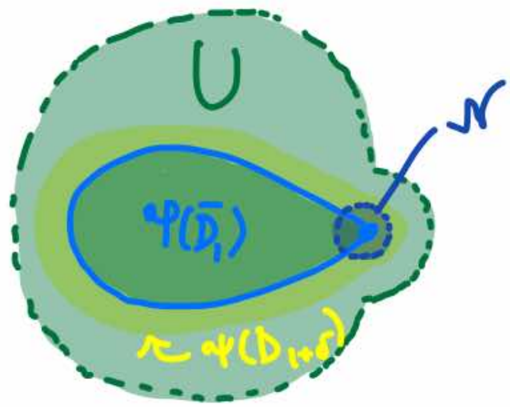


Pick $N \in \mathbb{Z}_+$ s.t. $\frac{N+1}{N} < \lambda$, and set

$$\psi(z) := \frac{z^N + z^{N+1}}{2}.$$

Note that
- $\psi(1) = 1$
- $z \in \overline{D_1} \setminus \{1\} \Rightarrow |\psi(z)| = \frac{1}{2}|z^N||z+1| < \frac{1}{2}|z^N| \cdot 2 \le 1$

$\Rightarrow \psi(\overline{D_1}) \subset U$



$\Rightarrow 1 \in \psi(D_{1+\delta}) \subset U$ (for some $\delta > 0$)

$\Rightarrow \psi(D_{1+\delta}) \supset \mathcal{N}$ = small nbhd. of 1.

Set $G := F \circ \psi \in \mathcal{H}ol(D_{1+\delta})$, and write

$$G(z) = \sum_{m \ge 0} \gamma_m z^m$$

on D_1 $\|$

$$f(\psi(z)) = \sum_{n \ge 1} \alpha_n \left(\frac{z^N + z^{N+1}}{2}\right)^{p_n}$$

$(*)$

The n^{th} term of the sum has powers of z from z^{Np_n} to $z^{(N+1)p_n}$, and the $(n+1)^{\text{st}}$ has powers from $z^{Np_{n+1}}$ to $z^{(N+1)p_{n+1}}$. Since $\frac{p_{n+1}}{p_n} > \lambda > \frac{N+1}{N}$, $(N+1)p_n < Np_{n+1}$ and so <u>there is no overlapping of powers of z</u>, and the two power series in $(*)$

are equal, in particular

$$\sum_{n=1}^{N} \alpha_n \Psi(z)^{p_n} = \sum_{m=0}^{(N+1)p_N} \gamma_m z^m$$

where convergence (assumed!) $(N \to \infty)$ of the RHS on $D_{1+\delta} \Longrightarrow$

convergence of the LHS $(N \to \infty)$ on $D_{1+\delta} \Longrightarrow$

$\sum_{n \geq 1} \alpha_n w^{p_n}$ converges for $w \in \Psi(D_{1+\delta}) \Longrightarrow$

$\sum_{n \geq 1} \alpha_n w^{p_n}$ converges for $w \in \mathcal{N} \Longrightarrow$

radius of convergence > 1 (a contradiction). $\square$

So here's the simplest example of "ugly function":

Example// Let $F(z) := \sum_{n \geq 1} z^{2^n}$. The radius of convergence is $\left(\limsup_{n \to \infty} |a_n|^{1/n}\right)^{-1} = \left(\lim_{n \to \infty} 1^{1/2^n}\right)^{-1} = 1$ ($\Rightarrow$ converges absolutely & uniformly on D_1). Further, $p_n = 2^n$ satisfies $\frac{p_{n+1}}{p_n} = 2 > \frac{3}{2} =: \lambda$. So the Theorem applies, and D_1 is a "maximal domain of holomorphicity" for F. //