Lecture 37: Rank, genus, and order

I. Functions of finite order

As in the last lecture, we begin with $f \in \mathcal{H}ol(\mathbb{C})$, with zeros $\{z_n\}$ arranged according to increasing absolute value and repeated according to multiplicity.

Definition 1: f is of finite rank $\iff$

$$\exists\ p \in \mathbb{Z} \text{ s.t. } \sum \frac{1}{|z_n|^{p+1}} < \infty ,$$

and then rank(f) := smallest such $p \in \mathbb{Z}$. //

If rank$(f) = p$, then

$$P(z) := \prod_{n \geq 1} E_p(z/z_n)$$

converges, with $\dfrac{f}{P} = z^m e^{h(z)}$ where $\begin{cases} m = \mathrm{ord}_0(f) \\ h \in \mathcal{H}ol(\mathbb{C}) . \end{cases}$

Definition 2: f is of finite genus $\iff$

f is of finite rank AND h is a polynomial.

The genus of f is then $g_f := \max(\mathrm{rank}(f), \deg(h))$. //

As we pointed out in Lecture 35, for f of finite genus, with P & h as above, the factorization

$$f(z) = z^m e^{h(z)} P(z)$$

is unique, and this will be called f's standard factorization.

The next definition is motivated by the first example from Lecture 36.

Definition 3: f is of order $\leq \lambda$ $\iff$

$$\forall \epsilon > 0,\ \exists C_\epsilon > 1 \text{ s.t. } \boxed{\|f\|_{D_R} \leq C_\epsilon^{R^{\lambda+\epsilon}} \text{ for } R \gg 0.} \quad (*)$$

If this holds for no λ, then f is of infinite order. Otherwise f is of finite order, with order $\lambda_f :=$ smallest λ for which the above holds. Further, f is said to be of strict order λ_f $\iff$ $\exists C_0 > 1$ s.t.

$$\|f\|_{D_R} \leq C_0^{R^\lambda} \text{ for } R \gg 0. //$$

Example

(i) nonconstant polynomials are of order 0, since for any $\epsilon > 0$, $R^{\epsilon} > \deg(\text{polynomial})$ for $R \gg 0$. They are <u>not</u> of strict order 0.

(ii) what functions are of infinite order? Look at

$$f(z) = \exp(\exp(z)).$$

If $(*)$ holds, then $e^{e^R} \leq C^{R^{\lambda+\epsilon}}$ implies $e^R \leq (\log C) R^{\lambda+\epsilon}$, which is absurd.

(iii)
$$(*) \Rightarrow \log\log \|f\|_{D_R} \leq \log\left[(\log C_\epsilon) R^{\lambda+\epsilon}\right] = \log\log C_\epsilon + (\lambda+\epsilon)\log R$$

$$\Rightarrow \frac{\log\log\|f\|_{D_R}}{\log R} \leq \frac{\log\log C_\epsilon}{\log R} + \lambda + \epsilon \quad (\text{for } R \gg 0)$$

$$\Rightarrow \limsup_{R\to\infty} \frac{\log\log\|f\|_{D_R}}{\log R} \leq \lambda + \epsilon$$

$$\underset{\substack{\epsilon>0 \\ \text{arbitrary}}}{\Rightarrow} \limsup_{R\to\infty} \frac{\log\log\|f\|_{D_R}}{\log R} \leq \lambda.$$

We conclude that

$$\boxed{\lambda_f = \limsup_{R\to\infty} \frac{\log\log\|f\|_{D_R}}{\log R}}.$$

Both notions (genus & order) are obviously tied to the rate of growth of $N(R)$, i.e. the distribution of zeroes of f. So how are they connected? Here is a first result:

$\boxed{\text{Proposition}}$ For f of finite order λ_f,

$$\text{rank}(f) \leq \lambda_f.$$

Proof: We will show $\sum \frac{1}{|z_n|^{\lfloor \lambda_f \rfloor + 1}} < \infty$, for which we may assume $f(0) = 1$. (With the formula for λ_f, one easily checks that $\lambda_{Cz^m f(z)} = \lambda_{f(z)}$: basically the point is that $\frac{\log \log R}{\log R} \to 0$.) From Example 1 of Lect. 36, we have the estimate (for $\epsilon > 0$)

$$N(R) \leq \left\{(1+e)^{\lambda + \frac{\epsilon}{2}} \log C_{\epsilon/2}\right\} R^{\lambda + \frac{\epsilon}{2}} - \cancel{\log|f(0)|}$$

$$\Rightarrow R^{-(\lambda+\epsilon)} N(R) \leq (1+e)^{\lambda + \epsilon/2} (\log C_{\epsilon/2}) R^{-\epsilon/2} \xrightarrow[(R \to \infty)]{} 0$$

$$\Rightarrow N(R) \leq R^{\lambda + \epsilon} \text{ for } R \gg 0$$

$$\Rightarrow n \leq N(|z_n|) \leq |z_n|^{\lambda + \epsilon}$$

$$\Rightarrow \frac{1}{n} \geq \frac{1}{|z_n|^{\lambda + \epsilon}}.$$

Now, taking the $\left(\frac{\lfloor\lambda\rfloor+1}{\lambda+\epsilon}\right)^{\text{th}}$ power, and assuming

$$\epsilon < 1 - \underbrace{(\lambda - \lfloor\lambda\rfloor)}_{\in [0,1)} \quad \text{so that } \lfloor\lambda\rfloor + 1 > \lambda + \epsilon$$

$$\left(\rightarrow \frac{\lfloor\lambda\rfloor+1}{\lambda+\epsilon} > 1 \right),$$

the last inequality becomes

$$\frac{1}{n^{\frac{\lfloor\lambda\rfloor+1}{\lambda+\epsilon}}} \geq \frac{1}{|z_n|^{\lfloor\lambda\rfloor+1}}.$$

Since $\sum_n$ of the LHS converges, so does $\sum_n$ of RHS. $\square$

II. Hadamard factorization theorem

We are now ready to prove the relationship between genus and order for $f \in \text{Hol}(\mathbb{C})$:

Theorem $\quad g_f \leq \lambda_f \leq g_f + 1$

Remark: The first inequality is Hadamard's Factorization Theorem: it says you can decompose a function of finite order (a growth condition) as $z^m e^{h(z)} P(z)$ where h is a polynomial and P a canonical product — a magnificent result. //

II (a). Proof of $\lambda_f \leq g_f + 1$

The assumption here is that f is of finite genus g_f.

We must show that f is of finite order $\leq g_f + 1$.

<u>Lemma</u>: $\log |E_g(z)| \leq (2g+1)|z|^{g+1}$ $(\forall z \in \mathbb{C})$.

Proof: $z \in D_1 \Rightarrow$
$$\log|E_g(z)| \leq \mathrm{Re}\left\{\log(1-z) + z + \frac{z^2}{2} + \dots + \frac{z^g}{g}\right\}$$
$$\leq \mathrm{Re}\left\{-\frac{z^{g+1}}{g+1} - \frac{z^{g+2}}{g+2} - \dots\right\}$$
$$\leq \frac{|z|^{g+1}}{g+1} + \frac{|z|^{g+2}}{g+2} + \dots$$
$$\leq \frac{1}{g+1}\left(|z|^{g+1} + |z|^{g+2} + \dots\right)$$
$$= \frac{|z|^{g+1}}{(g+1)(1-|z|)} \leq \frac{|z|^{g+1}}{1-|z|}$$

$z \in \mathbb{C} \Rightarrow$
$$\log|E_g(z)| = \mathrm{Re}\left\{\log(1-z) + z + \dots + \frac{z^{g-1}}{g-1}\right\} + \mathrm{Re}\left\{\frac{z^g}{g}\right\}$$
$$= \log|E_{g-1}(z)| + \mathrm{Re}\left\{\frac{z^g}{g}\right\}$$
$$\leq \log|E_{g-1}(z)| + |z|^g$$

Now assume inductively that
$$\log|E_{g-1}(z)| \leq (2g-1)|z|^g \quad (\forall z \in \mathbb{C}).$$

Then

$$\log|E_g(z)| \le \log|E_{g-1}(z)| + |z|^g$$
$$\le (2g-1)|z|^g + |z|^g$$
$$= 2g|z|^g,$$

which for $|z| \ge 1$ is $\le (2g+1)|z|^{g+1}$.

If $|z|<1$, we have $\begin{cases} \log|E_g(z)| \le 2g|z|^g \\ (1-|z|)\log|E_g(z)| \le |z|^{g+1} \end{cases}$,

so $$\log|E_g(z)| \le |z|\log|E_g(z)| + |z|^{g+1}$$
$$\le 2g|z|^{g+1} + |z|^{g+1}$$
$$= (2g+1)|z|^{g+1}.$$

To complete the induction, note that

$$e^{|z|} = 1 + |z| + \frac{|z|^2}{2!} + \dots \ge 1+|z| \ge |1-z|$$

$$\underset{\text{take log}}{\Longrightarrow} |z| \ge \log|1-z| = \log|E_0(z)|. \qquad \square$$

Now f of genus $g \Rightarrow A := \sum_{n\ge1} \frac{1}{|z_n|^{g+1}} < \infty$.

We have $f(z) = z^m e^{h(z)} P(z)$ with $\deg(h) \le g$, and

$$\log|P(z)| = \sum_{n\ge1} \log|E_g(z/z_n)| \underset{\text{lemma}}{\le} (2g+1)\sum_{n\ge1}\frac{|z|^{g+1}}{|z_n|^{g+1}}$$

$(\forall z \in \mathbb{C})$ $$= (2g+1)|z|^{g+1}A.$$

Moreover, $\deg(h) \leq g \Rightarrow \frac{h(z)}{z^g} \xrightarrow[(|z|\to\infty)]{} \text{constant} \in \mathbb{C}$

$\Rightarrow \exists\ B$ s.t. for $R \gg 0$,
$\|h\|_{\partial D_R} \leq B \cdot R^g$.

So $\log\|P(z)\|_{\partial D_R} \leq (2g+1) A R^{g+1}$, and

$\therefore\ \|f\|_{\partial D_R} \leq R^m \times e^{BR^g} \times e^{(2g+1)AR^{g+1}} \leftarrow (R \gg 0)$

$\Rightarrow \log\|f\|_{\partial D_R} \leq m \log R + BR^g + (2g+1)AR^{g+1} \leq a \cdot R^{g+1}$

$$\Rightarrow \underbrace{\limsup_{R\to\infty} \frac{\log\log\|f\|_{\partial D_R}}{\log R}}_{=\lambda_f} \leq \lim_{R\to\infty} \frac{\log a + (g+1)\log R}{\log R} = g+1,$$

finishing the proof of this part.

$\boxed{\text{II b. Proof of } g_f \leq \lambda_f}$

Here the assumption is that f is of finite <u>order</u> λ_f, and we try to prove it's of finite genus $\leq \lambda_f$.

<u>Lemma</u>: Assume $f(0)=1$, $p \in \mathbb{Z}$ with $p > \lambda_f - 1$. Then $\frac{d^p}{dz^p}\left(\frac{f'(z)}{f(z)}\right) = -p! \sum_{n\geq 1} \frac{1}{(z_n - z)^{p+1}}$ for $z \neq z_1, z_2, \ldots$

Obviously the lemma is true if there are finitely many $\{z_n\}$, so we'll assume there are infinitely many.

Sublemma A: $\displaystyle\lim_{R\to\infty} \sum_{n=1}^{N(R)} \frac{\bar{z}_n^{\,p+1}}{(R^2-\bar{z}_n z)^{p+1}} = 0$.

Proof: $z \in D_{R/2}$, $n \le N(R) \implies$

$$|R^2 - \bar{z}_n z| \ge R^2 - R\cdot R/2 = R^2/2 \implies$$

$$\frac{|\bar{z}_n|^{p+1}}{|R^2-\bar{z}_n z|^{p+1}} \le R^{p+1}\left(\frac{2}{R^2}\right)^{p+1} = \left(\frac{2}{R}\right)^{p+1}.$$

For $R \gg 0$, f of order $\lambda_f \implies$

$$\|f\|_{\partial D_R} \le C_{\epsilon/2} \cdot e^{(R^{\lambda_f+\epsilon/2})} \implies$$

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$$N(R) \le \underbrace{\left\{(1+\epsilon)^{\lambda_f+\epsilon/2} \log C_{\epsilon/2}\right\}}_{=:\,K_\epsilon} R^{\lambda_f+\epsilon/2} \implies$$

$$\frac{N(R)}{R^{p+1}} \le K_\epsilon R^{\lambda_f-1-p+\epsilon/2} \le K_\epsilon R^{-\epsilon/2}$$

($p > \lambda_f - 1$
i.e. $\lambda_f - 1 - p < 0$; we can therefore
choose $\epsilon > 0$ s.t. $\lambda_f - 1 - p + \epsilon \le 0$)

$$\Longrightarrow \left| \sum_{n=1}^{N(R)} \frac{\bar{z}_n^{\,p+1}}{(R^2 - \bar{z}_n z)^{p+1}} \right| \le N(R) \cdot \left(\frac{2}{R}\right)^{p+1}$$

$$\le 2^{p+1} K_\epsilon R^{-\epsilon/2} \longrightarrow 0 \quad (R \to \infty)$$

$\square$

Sublemma B: $\displaystyle \lim_{R\to\infty} \frac{1}{2\pi} \int_0^{2\pi} \frac{2Re^{i\theta}}{(Re^{i\theta} - z)^{p+2}} \log|f(Re^{i\theta})| \, d\theta = 0$.

Proof: $z \in D_{R/2} \Longrightarrow \operatorname{Res}_{w=z} \left(\frac{1}{(w-z)^{p+2}} \right) = 0$

$$\Longrightarrow \int_0^{2\pi} \frac{iRe^{i\theta} \, d\theta}{(Re^{i\theta} - z)^{p+2}} = 0$$

$$\Rightarrow \left| \frac{1}{2\pi} \int_0^{2\pi} \frac{2Re^{i\theta}}{(Re^{i\theta} - z)^{p+2}} \log|f(Re^{i\theta})| \, d\theta \right|$$

Can add this term for free

$$= \left| \frac{1}{2\pi} \int_0^{2\pi} \frac{2Re^{i\theta}}{(Re^{i\theta} - z)^{p+2}} \left(\log|f(Re^{i\theta})| - \log\|f\|_{\partial D_R} \right) d\theta \right|$$

$$\le \frac{1}{2\pi} \int_0^{2\pi} \frac{2R}{(R/2)^{p+2}} \left(\log\|f\|_{\partial D_R} - \log|f(Re^{i\theta})| \right) d\theta$$

$$\le \frac{1}{2\pi} \int_0^{2\pi} \frac{2R}{(R/2)^{p+2}} \log\|f\|_{\partial D_R} \, d\theta$$

$$= \frac{2^{p+3}}{R^{p+1}} \log\|f\|_{\partial D_R}$$

Recall Jensen's formula

$$\frac{1}{2\pi} \int_0^{2\pi} \log|f(Re^{i\theta})| \, d\theta = \log|f(0)| + \sum_{n=1}^{N(R)} \log\left|\frac{R}{z_n}\right| \ge \log|\underset{\substack{\parallel \\ 1}}{f(0)}| = 0.$$

$$\boxed{\le}\ \frac{2^{p+3}}{R^{p+1}} \underbrace{\log\left[C_{\epsilon/2}\, e^{(R^{\lambda_f+\epsilon/2})}\right]}_{\log C_{\epsilon/2} + R^{\lambda_f+\epsilon/2}}.$$

f of order λ_f, $R \gg 0$

Choosing ϵ again so that $\lambda_f - p - 1 + \epsilon \le 0$,

this expression $\longrightarrow 0$ as $R \to \infty$. $\square$

Proof of Lemma: For $z \in D_{2R}$, Poisson-Jensen

$$\Rightarrow \log|f(z)| = -\sum_{n=1}^{N(R)} \log\left|\frac{R^2 - \bar{z}_n z}{R(z-z_n)}\right| + \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left(\frac{Re^{i\theta}+z}{Re^{i\theta}-z}\right)\log|f(Re^{i\theta})|\, d\theta$$

$$\overset{\partial/\partial z}{\Longrightarrow} \frac{f'(z)}{f(z)} = -\sum_{n=1}^{N(R)} \frac{1}{z_n - z} + \sum_{n=1}^{N(R)} \frac{\bar{z}_n}{R^2 - \bar{z}_n z} + \frac{1}{2\pi}\int_0^{2\pi} \frac{2Re^{i\theta}}{(Re^{i\theta}-z)^2}\log|f(Re^{i\theta})|\, d\theta$$

[Here I've used $2\frac{\partial}{\partial z}\underbrace{\mathrm{Re}\, \mathcal{F}}_{U} = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)U = \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y}$ ← holo. $= U+iV$; $\frac{\partial U}{\partial y} = -\partial V/\partial x$

$= \frac{\partial}{\partial x}(U+iV) = \frac{\partial}{\partial x}\mathcal{F} = \frac{d}{dz}\mathcal{F} = \mathcal{F}'$.]

$$\overset{(d/dz)^p}{\Longrightarrow} \left(\frac{d}{dz}\right)^p \frac{f'(z)}{f(z)} = -p!\sum_{n=1}^{N(R)} \frac{1}{(z_n - z)^{p+1}} + p!\sum_{n=1}^{N(R)} \frac{\bar{z}_n^{\,p+1}}{(R^2 - \bar{z}_n z)^{p+1}}$$

$$+ (p+1)!\,\frac{1}{2\pi}\int_0^{2\pi} \frac{2Re^{i\theta}}{(Re^{i\theta}-z)^{p+2}}\log|f(Re^{i\theta})|\, d\theta.$$

By Sublemmas A & B, the last 2 terms tend to 0 as $R \to \infty$. $\square$

Remark: Note that the Lemma contains within it the statement that the sum $\sum_n \frac{1}{(z_n - z)^{p+1}}$ converges. It does so precisely because of the hypothesis (on p and) that f have finite order λ_f. //

We are now ready to prove Hadamard.

Wolog assume $f(0) = 1$ (cf. the proof of the Proposition above), $p > \lambda_f - 1$. In fact, we can take $p = \lfloor \lambda_f \rfloor$ ($\leq \lambda_f$). From the Proposition, †

(#) $$f(z) = e^{h(z)} \times P(z)$$

with P a canonical product of rank / genus $\leq p$.

We must show that h is a polynomial of degree $\leq p$.

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† the point is that $\mathrm{rank}(f) = p_0$ is the smallest integer s.t. $\sum |z_n|^{-p_0 - 1} < \infty$, and then the canonical product of genus p_0 is defined. But Proposition $\Rightarrow p_0 \leq \lambda_f$ $\Rightarrow p_0 \leq \lfloor \lambda_f \rfloor =: p \Rightarrow \sum |z_n|^{-p-1} < \infty$, etc.

For this, note that $P(z)$ and $f(z)$ **BOTH** satisfy the conclusion of the lemma! For f, this is because it satisfies the hypothesis. For P, it is simply by writing

$$\left(\frac{d}{dz}\right)^{p+1} \log E_p(z) = \left(\frac{d}{dz}\right)^{p+1} \left\{ \log(1-z) + z + \frac{z^2}{2} + \cdots + \frac{z^p}{p} \right\} = \frac{-p!}{(1-z)^{p+1}}$$

$$\Rightarrow \left(\frac{d}{dz}\right)^{p+1} \log P(z) = \left(\frac{d}{dz}\right)^{p+1} \sum_{n\geq 1} E_p\left(\frac{z}{z_n}\right) = -p! \sum_{n\geq 1} \frac{1}{(z_n-z)^{p+1}}. \quad (**)$$

$$\underset{\left(\frac{d}{dz}\right)^{p} \frac{P'(z)}{P(z)}}{\parallel}$$

Hence, applying dlog to $(*)$ gives

$$\frac{f'}{f} = h' + \frac{P'}{P}$$

$$\Longrightarrow \underbrace{\left(\frac{d}{dz}\right)^{p} \frac{f'}{f}}_{\overset{\parallel \text{ Lemma}}{-p! \sum_{n\geq 1} \frac{1}{(z_n-z)^{p+1}}}} = h^{(p+1)} + \underbrace{\left(\frac{d}{dz}\right)^{p} \frac{P'}{P}}_{\overset{\parallel\ (**)}{-p! \sum_{n\geq 1} \frac{1}{(z_n-z)^{p+1}}}} \qquad \left(\frac{d}{dz}\right)^{p}$$

$$\Longrightarrow h^{(p+1)} = 0,$$ done.