Lecture 36 : Jensen's formula

I. Counting zeroes of entire functions

Let $f \in \mathcal{H}ol(\mathbb{C})$, with zeroes $\{z_n\}$ ordered by increasing absolute value and repeated according to multiplicity (a convention we'll stick to tacitly from now on). Define†

(counted w/ multiplicity!)

$N(r) :=$ # of zeroes with absolute value $\leq r$,

and assume $f(0) \neq 0$. We'd like to understand the asymptotic growth of $N(r)$.

Consider $r > 0$ such that no $|a_n|$ equals r; the first step is to "divide out the zeroes of f inside D_r". To do this, recall

$$\varphi_\alpha(z) := \frac{z - \alpha}{1 - \bar{\alpha} z} \quad ;$$

† e.g., if $\mathrm{ord}_{i/2}(f) = 3$ then $z_1 = z_2 = z_3 = \frac{i}{2}$, and if $|z_n| > 1$ for $n \geq 4$, then $N(1) = 3$.

if we divide by this, then we introduce a zero at $1/\bar{\alpha}$, and we don't want to introduce zeroes in D_r.

So we rescale to

since $\left|\frac{r^2}{\bar{\alpha}}\right| > \frac{r^2}{r} = r$

$$\varphi_{\alpha/r}(z/r) = \frac{z/r - \alpha/r}{1 - \bar{\alpha} z/r^2} = \frac{r(z-\alpha)}{r^2 - \bar{\alpha} z} \in \mathcal{H}ol(\overline{D_r})$$

$\alpha \in D_r$

and define

$$F(z) := \frac{f(z)}{\prod_{n=1}^{N(r)} \varphi_{\frac{z_n}{r}}\left(\frac{z}{r}\right)} \in \mathcal{H}ol(\overline{D_r}).$$

Clearly, F is free of zeroes in $\overline{D_r}$, and the MMP yields

$$(*) \qquad |F(0)| \leq \|F\|_{\partial D_r}.$$

Now

$$F(0) = \frac{f(0)}{\prod_{n=1}^{N(r)} \varphi_{\frac{z_n}{r}}(0)} = \pm f(0) \cdot \prod_{n=1}^{N(r)} \frac{r}{z_n},$$

and for $z = re^{i\theta}$

$$F(z) = \frac{f(z)}{\prod_{n=1}^{N(r)} \varphi_{z_n/r}(e^{i\theta})} = \frac{f(z)}{\prod_{n=1}^{N(r)} \left(\frac{e^{i\theta} - z_n/r}{1 - (\bar{z}_n/r)e^{i\theta}}\right)} = \frac{f(z)}{\prod_{n=1}^{N(r)} e^{-i\theta} \frac{e^{i\theta} - \frac{z_n}{r}}{\left(\overline{e^{i\theta} - \frac{z_n}{r}}\right)}}$$

$$\Longrightarrow |F(z)| = |f(z)| \text{ on } \partial D_r.$$

So $(*)$ becomes

$$|f(0)| \le \|f\|_{\partial D_r} \left(\prod_{n=1}^{N(r)} \frac{r}{|z_n|} \right)^{-1}$$

or just $\overline{D_r}$ (by MMP)

$$\Longrightarrow \log|f(0)| \le \log\|f\|_{D_r} - \sum_{n=1}^{N(r)} \log\frac{r}{|z_n|}$$

$$\log\frac{r}{|z_n|} = \int_{|z_n|}^{r} \frac{d\rho}{\rho}$$

$$= \log\|f\|_{D_r} - \int_0^r \frac{N(\rho)}{\rho}\, d\rho$$

$$\Longrightarrow \boxed{\int_0^r \frac{N(\rho)}{\rho}\, d\rho \le \log\|f\|_{D_r} - \log|f(0)|}$$

Jensen's inequality.

Example // Let $\lambda \ge 0$, and suppose $f \in \mathrm{Hol}(\mathbb{C})$ satisfies $\|f\|_{D_R} \le C^{R^\lambda}$ for $R \ge R_0$, where $C > 1$ is a constant. (For instance, $\lambda = 0$ corresponds to f constant; while sin, cos, exp all have $\lambda = 1$.)

Now assume $\lambda > 0$: then by Jensen's inequality

$$\int_0^R \frac{N(\rho)}{\rho}\, d\rho \le R^\lambda \log C - \log|f(0)|$$

for $R \ge R_0$. This is compatible with $N(\rho) \le \text{const.} \times R^\lambda$ (for $R \gg 0$). To actually prove this, put

$$g_R := \frac{f}{\prod_{n=1}^{N(R)} (z - z_n)} \cdot \prod_{n=1}^{N(R)} z_n \quad \left(\in \mathcal{Hol}(\mathbb{C}) \right)$$

and note $g_R(0) = \pm f(0)$. Then the MMP $\Longrightarrow$

$$|f(0)| = |g_R(0)| \le \|g_R\|_{\partial D_{(1+e)R}} \le \frac{\|f\|_{D_{(1+e)R}}}{(eR)^{N(R)}} \cdot R^{N(R)} = \frac{\|f\|_{D_{(1+e)R}}}{e^{N(R)}}$$

$$\Longrightarrow \quad e^{N(R)} \le \frac{\|f\|_{D_{(1+e)R}}}{|f(0)|} \le \frac{C^{((1+e)R)^\lambda}}{|f(0)|}$$

$$\Longrightarrow \quad N(R) \le \{(1+e)^\lambda \log C\} \times R^\lambda - \log |f(0)|$$

take log

$$\le \text{const} \times R^\lambda \quad \text{for } R \gg 0.$$

We say $N(R) = O(R^\lambda)$. (Note that this is indeed true for sin, cos, exp and $\lambda = 1$!) //

II. Jensen's formula and Mahler measure

Given $f \in \mathcal{Hol}(\mathbb{C})$ and $r > 0$, define $F \in \mathcal{Hol}(\bar{D}_r)$ as above, and recall that F is zero-free on D_r.

Since

$$\log|F| = \mathrm{Re}(\log F)$$

is harmonic, we can apply the MVT to get

$$\log|F(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|F(re^{i\theta})|\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|\,d\theta - \underbrace{\frac{1}{2\pi}\sum_{n=1}^{N(r)}\int_0^{2\pi}\log\underbrace{\left|\frac{e^{i\theta} - z_n/r}{1-(\bar{z}_n/r)e^{i\theta}}\right|}_{=1}d\theta}_{=0}$$

$$\Longrightarrow \boxed{\frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|\,d\theta = \log|f(0)| + \sum_{n=1}^{N(r)}\log\left|\frac{r}{z_n}\right|}$$

(Jensen's formula) .

Example // The Mahler measure of a polynomial $P(x) \in \mathbb{Z}[x]$ is defined by

$$M(P) := \exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\log|P(e^{i\theta})|\,d\theta\right\}.$$

If $P = a\prod_{i=1}^{m}(x-\alpha_i)$, then by Jensen's formula

$$M(P) = \exp\left\{\log|a| + \sum_{i=1}^{n}\log|\alpha_i| - \sum_{i=1}^{n}\log^{-}|\alpha_i|\right\}$$

$$= |a|\prod_{i=1}^{m}\max\{1,|\alpha_i|\}.$$

takes value 0 if $\alpha_i \notin D_1$, $\log(\alpha_i)$ if $\alpha_i \in D_1$

For (products of) cyclotomic polynomials, $M(P)$ is clearly 1.

A major unsolved problem in number theory is the

Conjecture (Lehmer): For noncyclotomic $P \in \mathbb{Z}[x]$,

$$M(P) \geq M(\underbrace{1-x+x^3-x^4+x^5-x^6+x^7-x^9+x^{10}}_{=:\, P_0(x)}) \approx 1.1762.$$

(Note that all but 2 of P_0's roots lie on ∂D_1, so only one root contributes to the value of $M(P_0)$!) //

We actually cheated a bit in the last example, because we have not yet addressed what happens if some roots of f lie on ∂D_r. First, this is not a problem for convergence of the LHS of Jensen's formula, because

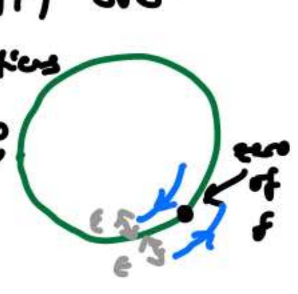
$$\left|\int_0^\epsilon \log x \, dx\right| = \left| \underbrace{\text{“}(x\log x - x)\Big|_0^\epsilon\text{”}}_{} \right| = \left|\epsilon \log \epsilon - \epsilon\right| < \infty.$$

(i.e. $\lim\limits_{\epsilon_0\to 0} (x\log x - x)_{\epsilon_0}^{\epsilon}$, using $\lim\limits_{\epsilon_0 \to 0} \epsilon_0 \log \epsilon_0 = 0$)

Further, the RHS & LHS are both continuous in r:

RHS: $\log\left|\frac{r}{z_n}\right| = 0$ when $|z_n| = r$, so that when the # of terms in the sum jumps with $N(r)$, the value of the sum does not.

LHS: need to show that the difference of $\int$'s of $\log|f|$ over small arcs (as shown) limits to 0, for which it suffices to check the same thing for a small circle about the zero, which is just $\lim\limits_{\epsilon \to 0} \int_{\partial D(z_0,\epsilon)} \log \epsilon \; d|z| = 2\pi\epsilon \log \epsilon \to 0$.



III. Two more applications

In fact, there is really nothing in the proof of Jensen which requires f to be entire: we need only that $\overline{D}_r \subset U$ and $f \in \mathcal{H}ol(U)$.

Example // Let $f \in \mathcal{H}ol(D_1)$ be nonconstant and bounded, with zeroes $\{z_n\}$ (conventions as above).

Claim: $\boxed{\sum_{n=1}^{\infty} (1 - |z_n|) < \infty, \text{ i.e. the } \{z_n\} \text{ must rapidly approach } 1}$.

Proof: From the proof of Jensen's inequality, for $r < 1$

$$\sum_{n=1}^{N(r)} \log \frac{r}{|z_n|} \leq \log \|f\|_{D_r} - \log |f(0)|$$

$$\leq \underbrace{\log \|f\|_{D_1}}_{\text{finite b/c } f \text{ bounded}} - \log |f(0)| =: M.$$

Hence $\forall\, r < \tilde{r} < 1$,

$$\sum_{n=1}^{N(r)} \log \frac{\tilde{r}}{|z_n|} \leq \sum_{n=1}^{N(\tilde{r})} \log \left| \frac{\tilde{r}}{z_n} \right| \leq M$$

$$\xRightarrow[\lim_{\tilde{r} \to 1^-}]{\text{take}} \sum_{n=1}^{N(r)} \log \frac{1}{|z_n|} \leq M \quad (\forall\, r < 1)$$

$$\Longrightarrow \sum_{n\geq 1}(1-|z_n|) < \sum_{n\geq 1} \underbrace{\log\frac{1}{|z_n|}}_{} \leq M$$

$$\log\frac{1}{|z_n|} = -\log|z_n| = -\log(1-(1-|z_n|)) = \sum_{k\geq 1}\frac{(1-|z_n|)^k}{k} > 1-|z_n| \qquad \square$$

Finally, if we apply Poisson's formula to (the harmonic function on $\overline{D_r}$) $\log|F|$, we get

$$\log|F(z)| = \frac{1}{2\pi}\int_0^{2\pi} \log \underbrace{|F(re^{i\theta})|}_{=|f(re^{i\theta})|} \cdot \frac{r^2-|z|^2}{|re^{i\theta}-z|^2}\, d\theta$$

$$\Longrightarrow \boxed{\log|f(z)| = -\sum_{n=1}^{N(r)} \log\left|\frac{r^2-\bar{z}_n z}{r(z-z_n)}\right| + \frac{1}{2\pi}\int_0^{2\pi}(\log|f(re^{i\theta})|)\frac{r^2-|z|^2}{|re^{i\theta}-z|^2}\, d\theta}$$

(Poisson–Jensen formula),

where we have used

$$\begin{aligned}\log|F| &= \log|f| - \sum_{n=1}^{N(r)}\log|\varphi_{z_n/r}(z/r)| \\ &= \log|f| + \sum_{n=1}^{N(r)}\log\left|\frac{r^2-\bar{z}_n z}{r(z-z_n)}\right|.\end{aligned}$$

This will be applied later.