Lecture 35: Functions with prescribed zeroes

Now that we know how to construct functions in $Mer(U)$ with prescribed principal parts (hence poles), what about functions in $\mathcal{H}ol(U)$ with prescribed zeroes?

Of course, summing "doesn't preserve zeroes" of functions in the sum, so we'll take products instead. Also, note that the two problems are "independent" in the sense that taking the reciprocal of the function solving one of the problems will not solve the other. In particular, you can get "prescribed poles" by taking the reciprocal of the function with prescribed zeroes, but this won't give the principal parts.

I. Products of numbers

An infinite product $\prod_{n\geq 1} \alpha_n$ of complex numbers is said to converge iff (a) only a finite number of the $\{\alpha_n\}$ are 0

& (b) given N s.t. $n \geq N \Rightarrow \alpha_n \neq 0$,

$$\lim_{M\to\infty} \prod_{n=N}^{M} \alpha_n \text{ exists and is nonzero.}$$

(later we'll want to be able to control where products of functions are 0)

In this case, we say the product has value

$$\prod_{n\geq 1} \alpha_n := \left\{\prod_{n=1}^{N-1} \alpha_n\right\} \cdot \lim_{M\to\infty} \prod_{n=N}^{M} \alpha_n .$$

Now assume all $\alpha_n \neq 0$.

Lemma 1: $\prod_{n\geq 1} \alpha_n$ converges $\Longleftrightarrow$ $\sum_{n\geq 1} \overbrace{\log}^{\text{principal branch : } \arg\in(-\pi,\pi]} \alpha_n$ converges

Proof: ($\Longleftarrow$) Set $\mathcal{P}_N := \prod_{n=1}^{N} \alpha_n$, $\mathcal{S}_N := \sum_{n=1}^{N} \log \alpha_n$. Clearly

$$\mathcal{P}_N = \exp \mathcal{S}_N, \text{ so } \mathcal{S}_N \to \mathcal{S} \Rightarrow \mathcal{P}_N \to \exp \mathcal{S}.$$

($\Longrightarrow$) $\mathcal{P}_N \to \mathcal{P} \neq 0 \longrightarrow \frac{\mathcal{P}_N}{\mathcal{P}} \to 1 \Rightarrow \begin{cases} \alpha_N = \frac{\mathcal{P}_{N+1}}{\mathcal{P}_N} = \frac{\left(\frac{\mathcal{P}_{N+1}}{\mathcal{P}}\right)}{\left(\frac{\mathcal{P}_N}{\mathcal{P}}\right)} \\ \longrightarrow 1. \end{cases}$

Moreover, $\log\left(\frac{\mathcal{P}_N}{\mathcal{P}}\right) \to 0$

$$\log\left(\frac{\mathcal{P}_N}{\mathcal{P}}\right) = \mathcal{S}_N - \log \mathcal{P} + 2\pi i \mu_N \text{ for some } \mu_N \in \mathbb{Z},$$

and

$$2\pi i(\mu_{N+1} - \mu_N) = \log\left(\frac{\mathcal{P}_{N+1}}{\mathcal{P}}\right) - \log\left(\frac{\mathcal{P}_N}{\mathcal{P}}\right) - \log \alpha_{N+1}$$

$$= i\arg\left(\frac{\mathcal{P}_{N+1}}{\mathcal{P}}\right) - i\arg\left(\frac{\mathcal{P}_N}{\mathcal{P}}\right) - i\arg \alpha_{N+1} \longrightarrow 0$$

$\Longrightarrow$ for $N \geq N_0$, $\mu_N = \mu$ (fixed) $\Longrightarrow$ $\mathcal{S}_N \to \log \mathcal{P} - 2\pi i \mu$. $\square$

Next, we say that $\prod \alpha_n$ converges absolutely iff some tail of[†] $\sum |\log \alpha_n|$ converges. (It would not do to say "iff $\prod |\alpha_n|$ converges"; for instance, consider the product $\prod (-1)^n$.)

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† otherwise we'd have to assume that all $\alpha_n \neq 0$.

Lemma 2: $\prod \alpha_n$ conv. absolutely $\Leftrightarrow$ $\sum |1-\alpha_n|$ converges.

Proof: In either case, we must have $\alpha_n \to 1$, so that

$$\lim_{z \to 1} \frac{\log z}{z-1} = \frac{1/z}{1} = 1 \implies \lim_{n \to \infty} \frac{\log \alpha_n}{\alpha_n - 1} = 1.$$ Thus for any $\epsilon > 0$, $\exists\, N$ s.t. $n \geq N \to 1-\epsilon < \left|\frac{\log \alpha_n}{1-\alpha_n}\right| < 1+\epsilon$

$$\Rightarrow (1-\epsilon)|1-\alpha_n| < |\log \alpha_n| < (1+\epsilon)|1-\alpha_n|$$

$\Rightarrow \sum_{n \geq N} |1-\alpha_n|$ and $\sum_{n \geq N} |\log \alpha_n|$ dominate each other. $\square$

Example// $\prod_{n \geq 2} \left(1 - \frac{1}{n^2}\right) = \frac{1}{2}$ (will prove later). //

II. Products of functions

(no fn (z) identically 0)

Let U be a subset of $\mathbb{C}$ and $\{f_n\} \subset C^0(U)^{\setminus\{0\}}$. Then

$\prod f_n$ converges uniformly on $\mathcal{E} \subset U$ $\Leftrightarrow$

- $\prod f_n(z_0)\ (=: F(z_0))$ converges $(\forall z_0 \in \mathcal{E})$
- $F_N(z) := \prod_{n=1}^{N} f_n(z)$ converges uniformly on $\mathcal{E}$, to $F(z)$.

Lemma 3: Let $K \subseteq \mathbb{C}$ be compact, and let $\{g_n\} \subset C^0(K)^{\setminus\{0\}}$ be such that $\sum |1-g_n|$ converges uniformly on K. Then

(i) $(F(z) :=) \prod_{n \geq 1} g_n(z)$ converges absolutely & uniformly on K,

and

(ii) $\exists\, N \in \mathbb{N}$ s.t. $(\forall z_0 \in K)$ $F(z_0) = 0 \Leftrightarrow g_n(z_0) = 0$ for some $n \leq N-1$.

Proof: Let $\epsilon > 0$. Then

$$\|1-g_n\|_K \to 0 \implies \left\| \frac{\log g_n}{1-g_n} - 1 \right\|_K \to 0$$

$$\implies |\log g_n(z)| < (1+\epsilon)\,|1-g_n(z)| \quad \forall \begin{cases} z \in K \\ n \geq N \end{cases}$$

$$\implies G_M := \sum_{n=N}^{M} \log g_n \text{ converges } (M \to \infty) \text{ absolutely } \& \text{ uniformly on } K,$$

with limit function $G \in C^0(K)$.

Let $A := \|G\|_K$; then $\|e^G\|_K \leq e^A$, and we pick

- $\delta > 0$ s.t. $|z| < \delta \implies |e^z - 1| < \epsilon / e^A$
- M_0 s.t. $M \geq M_0 \implies \|G_M - G\|_K < \delta$.

Then $\epsilon > \|e^G\|_K \|e^{G_M - G} - 1\|_K \geq \|e^{G_M} - e^G\|_K$, and so

$\lim_{M \to \infty} \prod_{n=N}^{M} g_n = \lim_{M \to \infty} e^{G_M}$ converges uniformly on K, to the

nowhere zero function e^G, and

$F = \left\{ \prod_{n=1}^{N-1} g_n \right\} \cdot e^G$ clearly implies (ii). $\square$

Proposition 1 Let $U \subset \mathbb{C}$ be a region, and $\{f_n\} \subset \mathrm{Hol}(U) \setminus \{0\}$ a sequence with $\sum |1 - f_n|$ converging uniformly on compact subsets. Then

(a) $F := \prod f_n$ converges uniformly on compact subsets ($\Rightarrow F \in \mathrm{Hol}(U)$), and

(b) For each $z_0 \in U$, $F(z_0) = 0 \iff f_n(z_0) = 0$ for some n (with $\mathrm{ord}_{z_0}(F) = \sum \mathrm{ord}_{z_0}(f_n)$).

Proof: (a) is a trivial consequence of Lemma 3(i).

(b) Suppose $f(z_0) = 0$, and consider $\overline{D}(z_0, r) \subset U$. Since $\sum |1 - f_n|$ converges uniformly on $\overline{D}$, Lemma 3(ii) $\Rightarrow \exists N$ s.t. $F = f_1 \cdots f_{N-1}\, g$ where g doesn't vanish on $\overline{D}$ (and, being a pole-free quotient of holomorphic functions, is clearly holomorphic). $\square$

III. Canonical products (informal motivation)

Suppose $\sum \frac{1}{|a_n|}$ converges. Then

$\sum \frac{z}{a_n}$ converges absolutely & uniformly on all $\overline{D}_R$ ($\subset \mathbb{C}$)
(hence on all compact K)

$\Longrightarrow \prod \left(1 - \frac{z}{a_n}\right)$ converges uniformly on compact subsets,

(Prop. 1 w/ $f_n = 1 - \frac{z}{a_n}$)

and clearly has zeroes exactly at the $\{a_n\}$ (with multiplicity equal to the # of times a_n occurs).

This is called a canonical product of genus zero.

Next, suppose $\sum \frac{1}{|a_n|^{g+1}}$ converges, and that g is the smallest integer for which this holds. Consider the product $\prod \left\{\left(1 - \frac{z}{a_n}\right) \underbrace{e^{P_n(z)}}\right\}$, P_n = polynomials.

nowhere-zero "correction term" (similar idea to Mittag-Leffler)

Log of this product is

$$\sum_n \left\{ \left(-\frac{z}{a_n} - \frac{1}{2}\left(\frac{z}{a_n}\right)^2 - \frac{1}{3}\left(\frac{z}{a_n}\right)^3 - \dots \right) + P_n(z) \right\}$$

(so let this be the first g terms of this series)

$$\Big\|$$

$$\sum_n \left\{ -\frac{1}{g+1}\left(\frac{z}{a_n}\right)^{g+1} - \frac{1}{g+2}\left(\frac{z}{a_n}\right)^{g+2} - \dots \right\},$$ with absolute value

$$\left| \qquad \text{"} \qquad \right| \le \sum_n \frac{1}{g+1}\left(\frac{R}{|a_n|}\right)^{g+1} \Big/ \left(1 - \frac{R}{|a_n|}\right) \quad (*)$$

on $\overline{D}_R$.

Now, $(*)$ converges if $\sum_n \frac{1}{(g+1)|a_n|^{g+1}}$ does (here the point is that $a_n \to \infty$, so $1 - \frac{R}{|a_n|} \to 1 \Rightarrow \frac{1}{1-\frac{R}{|a_n|}} \to 1$), and in this way we get uniform convergence of the original product on compact sets. The result is called a canonical product of genus g.

A function which is entire and of the (unique) form

$$z^m e^{G(z)} \underbrace{\prod \left(1 - \frac{z}{a_n}\right) e^{P_n(z)}}_{(**)}$$

where

- G is a polynomial of minimal degree
- $(**)$ is a canonical product of genus g_0 (also minimal)

is said to have genus

$$g := \max(g_0, \deg(G)).$$

IV. Weierstraß factorization Theorem

We want to generalize this approach to something which both works for an arbitrary sequence $\{a_n\}$ w/o accumulation points, and reproduces an arbitrary entire function. Also we want to know if the idea generalizes (like Mittag-Leffler) beyond $\mathbb{C}$ to arbitrary regions.

By an elementary factor, we shall mean one of the entire functions

$$\begin{cases} E_0(z) = 1-z \\ E_p(z) = (1-z)\,e^{\left\{z+\frac{z^2}{2}+\cdots+\frac{z^p}{p}\right\}} & (p \geq 1). \end{cases}$$

$$\left\{z+\frac{z^2}{2}+\cdots+\frac{z^p}{p}\right\} = \int (1+z+z^2+\cdots+z^{p-1})\,dz = \int \frac{1-z^p}{1-z}\,dz$$

Lemma 4: $|1-E_p(z)| \leq |z|^{p+1}$ on $\overline{D_1}$.

Proof: $E_p(z) = 1 + \sum_{k\geq 1} \varepsilon_k z^k$

$$\Longrightarrow E_p'(z) = \sum_{k\geq 1} k\,\varepsilon_k z^{k-1} \overset{d/dz}{=} \frac{1-z^p}{\cancel{1-z}}\cancel{(1-z)}e^{(\cdots)} - e^{(\cdots)}$$

$$= -z^p e^{(\cdots)}$$ has power series expansion with coeffs. all >0

$\Longrightarrow \varepsilon_1 = \cdots = \varepsilon_p = 0,\ \varepsilon_k \leq 0$ for $k \geq p+1$

$$\Longrightarrow 0 = E_p(1) = 1 + \sum_{k\geq p+1} \varepsilon_k = 1 - \sum_{k\geq p+1} |\varepsilon_k|$$

$$\Longrightarrow |E_p(z)-1| = \left|\sum_{k\geq p+1} \varepsilon_k z^k\right| \underset{z\in\overline{D_1}}{\leq} |z|^{p+1} \sum_{k\geq p+1} |\varepsilon_k| = |z|^{p+1}. \qquad \square$$

$\boxed{\text{Proposition 2}}$ Given sequences

- $\{a_n\} \subset \mathbb{C}^*$ with $|a_n| \to \infty$
- $\{p_n\} \subset \mathbb{Z}$ satisfying $\sum_{n\geq 1} \left(\frac{r}{|a_n|}\right)^{p_n+1} < \infty \ \forall r \in \mathbb{R}_{>0}$,

we have

(i) $f(z) = \prod_{n\geq 1} E_{p_n}\left(\frac{z}{a_n}\right)$ converges (uniformly on compact sets) to an entire function

(ii) $\mathrm{ord}_{z_0}(f) =$ multiplicity of z_0 in the sequence $\{a_n\}$.

(# of a_n's equaling z_0)

Proof: By Lemma 4, $\|1 - E_{p_n}(z)\|_{\overline{D}_r} \leq \left\|\frac{z}{a_n}\right\|_{\overline{D}_r}^{p_n+1} \leq \left(\frac{r}{|a_n|}\right)^{p_n+1}$ provided $r \leq |a_n|$. If we fix $\overline{D}_r$, then $\exists N$ s.t. $n \geq N \implies |a_n| \geq r$, so $\sum \left|1 - E_{p_n}(z/a_n)\right|$ is dominated by the sequence which converges by assumption $\xRightarrow{\text{Prop. 1}}$ done. $\square$

Remark: If we take $p_n = n-1$, then taking N s.t. $n \geq N \implies |a_n| \geq 2r$, $\sum_{n\geq N} \left(\frac{r}{|a_n|}\right)^{p_n+1} \leq \sum_{n \geq N} \left(\frac{1}{2}\right)^n$ on $\overline{D}_r$, so that we get the second condition of Prop. 2 for every sequence $\{a_n\} \to \infty$!!

$\boxed{\text{Theorem 1}}$ Given $f \in \mathcal{H}ol(\mathbb{C}) \setminus \{0\}$, let $\{a_n\}$ be the list of its nonzero zeroes, repeated according to multiplicity, and set $m := \mathrm{ord}_0(f)$. Then $\exists\, g \in \mathcal{H}ol(\mathbb{C})$ and $\{p_n\} \subset \mathbb{Z}$ s.t.

$$f(z) = z^m e^{g(z)} \prod_{n\geq 1} E_{p_n}\left(\frac{z}{a_n}\right).$$

Proof: Taking $p_n = n-1$, Prop. 2 implies that

$f(z) \Big/ z^m \prod_{n\geq 1} E_{p_n}\left(\frac{z}{a_n}\right)$ has removable singularities and extends to a nowhere vanishing holo. fcn. $G(z)$ on $\mathbb{C}$. Since $\mathbb{C}$ is simply-connected, $\exists\, F \in \mathcal{Hol}(\mathbb{C})$ with $F' = G'/G$

$$\Rightarrow \frac{d}{dz} G(z) e^{-F(z)} = G'(z) e^{-F(z)} - \cancel{G(z)\frac{G'(z)}{G(z)}} e^{-F(z)} = 0$$

$$\Rightarrow G(z) = \text{const.} \times e^{F(z)} =: e^{g(z)}. \qquad \square$$

Example// $\sin(\pi z)$ has zeroes at all $k \in \mathbb{Z}$, but $\sum \frac{1}{k}$ diverges. On the other hand, $\sum \frac{1}{k^2}$ converges $\Rightarrow$ can construct genus 1 product (all $p_n = 1$). Noting that $E_1(z) = (1-z)e^z$, we have that

$$\left(e^{g(z)} :=\right) \frac{\sin(\pi z)}{z \prod_{n\in\mathbb{Z}\setminus\{0\}} \left(1-\frac{z}{n}\right) e^{z/n}} \quad \text{is entire \& nowhere vanishing}$$

dlog $\downarrow$

"Mittag-Leffler" expansion of $\pi\cot(\pi z)$

$$g'(z) = \pi\cot(\pi z) - \frac{1}{z} - \sum_{n\in\mathbb{Z}\setminus\{0\}} \left(\frac{1}{z-n} + \frac{1}{n}\right) = 0$$

$$\Rightarrow g(z) \text{ constant. So } e^{g(z)} = \lim_{z\to 0} \frac{\sin(\pi z)}{z} = \pi$$

$$\Rightarrow \sin(\pi z) = \pi z \prod_{n\in\mathbb{Z}\setminus\{0\}} \left(1-\frac{z}{n}\right) e^{z/n} = \pi z \prod_{n\geq 1}\left(1-\frac{z^2}{n^2}\right).$$

Taking $\lim_{z\to 1} \prod_{n\geq 2}\left(1-\frac{z^2}{n^2}\right) = \lim_{z\to 1} \frac{\sin \pi z}{\pi z(1-z^2)} = \lim_{z\to 1} \frac{\pi\cos\pi z}{\pi - 3\pi z^2} = \frac{-\pi}{-2\pi} = \frac{1}{2}$,

we find (using uniform conv. on compact sets) that

$$\boxed{\prod_{n\geq 2}\left(1-\frac{1}{n^2}\right) = \frac{1}{2}.}$$ //

V. Arbitrary regions

Theorem 2 Given $U \subseteq \mathbb{C}$, $\{a_n\} \subset U$ w/o limit point in U, $\{m_n\} \subset \mathbb{Z}$. Then $\exists\, f \in \mathcal{H}ol(U)$ whose only zeroes are at the points a_n, with multiplicity m_n.

distinct now

Let's first look at the (striking) consequences:

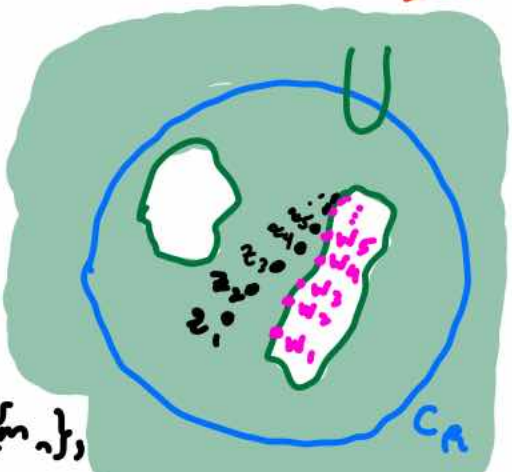
Corollary 1 For any $F \in \mathcal{M}er(U)$, $\exists\, g, f \in \mathcal{H}ol(U)$ s.t. $F = g/f$.

Proof: We know the poles $\{a_n\}$ (w/ multiplicities $\{m_n\}$) of f must not have accumulation points in U. Let f be the holomorphic function provided by Theorem 2. Then fF has removable singularities at each a_n (as $\mathrm{ord}_{a_n} fF = \mathrm{ord}_{a_n} f + \mathrm{ord}_{a_n} F = 0$), so extends to a holo. fcn. g on U. $\square$

Now assume $U \subsetneq \mathbb{C}$ is a region, $f \in \mathcal{H}ol(U)$, $P \in \partial U$.
We shall say that P is regular for f $\iff$
$\exists\, D(P,r) =: D$ and $\tilde{f} \in \mathcal{H}ol(D)$ s.t. $f|_{D \cap U} = \tilde{f}|_{D \cap U}$.

Corollary 2 For any region $U \subsetneq \mathbb{C}$, $\exists f \in \mathcal{H}ol(U)$ s.t. no $P \in \partial U$ is regular for f.

Proof (idea): Construct a sequence $\{a_n\}$ which has every point of ∂U as a limit point, using smaller and smaller grids (cf. [Greene–Krantz, pp. 268–70]). $\square$



Proof of Theorem 2: Suppose first that

- $\{z \mid |z| > R\} \subset U$
- $|a_n| \leq R \; (\forall n)$.

Let $\{z_n\}$ consist of the a_n's w/mults. $\{m_n\}$, & $\{w_n\} \subset \mathbb{C} \setminus U$ satisfy $|w_n - z_n| = d(z_n, \mathbb{C}\setminus U) \left(\underset{n\to\infty}{\longrightarrow} 0\right)$.

The functions $E_n\left(\frac{z_n - w_n}{z - w_n}\right)$ have (simple) zeroes at $z = z_n$ (where the argument $= 1$) & nowhere else.

Let $K \subset U$ be compact, and set $\delta_0 := d(\mathbb{C}\setminus U, K) \,(>0)$.

We have $\left\| \frac{z_n - w_n}{z - w_n} \right\|_K \leq \frac{|z_n - w_n|}{\delta_0}$, so that for any $\delta \in (0,1)$

$\exists N$ s.t. $n \geq N \Rightarrow \left\| \frac{z_n - w_n}{z - w_n} \right\| < \delta \xrightarrow[\text{Lemma 4}]{} \left\| E_n\left(\frac{z_n - w_n}{z - w_n}\right) - 1 \right\| \leq \delta^{n+1}$.

Therefore $\sum \left\{ E_n\left(\frac{z_n - w_n}{z - w_n}\right) - 1 \right\}$ is absolutely & uniformly convergent on K

$\underset{\text{Prop. 1}}{\Longrightarrow} f(z) := \prod_{n \geq 1} E_n\left(\frac{z_n - w_n}{z - w_n}\right) \in \mathcal{H}ol(U)$, with the right zeroes.

Now taking $|z|$ sufficiently large that (for any given $\epsilon \in (0, \frac{1}{2})$) $\left|\frac{z_n - w_n}{z - w_n}\right| < \epsilon$ $(\forall n)$, we have (Lemma 4) $\left|E_n\left(\frac{z_n - w_n}{z - w_n}\right) - 1\right| < \epsilon^{n+1}$ $(\forall n)$. But then

$$\left|\sum_{n \geq 1} \log E_n(\cdots)\right| \leq \sum_{n \geq 1} \left|\log E_n(\cdots)\right|$$

is dominated by (a const. multiple of)

$$\sum_{n \geq 1} \left|E\left(\frac{z_n - w_n}{z - w_n}\right) - 1\right| \leq \sum \epsilon^{n+1} < 2\epsilon$$

and so $\longrightarrow 0$ as $|z| \to \infty$. Hence, $f(z) \to 1$ as $|z| \to \infty$.

Finally, let $U \subsetneq \mathbb{C}$ be arbitrary and $\{a_n\} \subset U$ be w/o limit point in U, etc.; and consider $\overline{D} = \overline{D}(a, r) \subset U$ a ball not meeting $\{a_n\}$. Applying the FLT $T(z) := \frac{1}{z - a}$ recovers the above situation, and the fact that $\lim_{z \to \infty} f(z) = 1$ in the above means that the pullback via T (call this g) will have $\lim_{z \to a} g(z) = 1$ hence a removable singularity at a. $\square$