Lecture 34 : Functions w/ prescribed principal parts

I. Constructing meromorphic functions in $\mathbb{C}$

We begin by recalling the estimate on the Taylor remainder for $f \in \mathcal{Hol}(U)$ at $a \in U$ (= region). In U,

$$(*) \qquad f(z) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(z-a)^k + \underbrace{f_{n+1}(z)}_{\in \mathcal{Hol}(U)} (z-a)^{n+1}.$$

Now writing $D := \mathbb{D}(a,r) \subset U$, $z \in D \Rightarrow$

$$f_{n+1}(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f_{n+1}(\zeta)\,d\zeta}{\zeta - z}$$

$$(\text{by } (*)) \qquad = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)\,d\zeta}{(\zeta-a)^{n+1}(\zeta-z)} - \frac{1}{2\pi i}\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}\int_{\partial D}\frac{d\zeta}{(\zeta-a)^{n-k+1}(\zeta-z)}.$$

The $k=n$ term at the end is (omitting a constant)

$$\frac{1}{z-a}\int_{\partial D}\left(\frac{1}{\zeta-z} - \frac{1}{\zeta-a}\right)d\zeta \quad \left(=\ 0\right);$$

the other terms are (up to const.) derivatives of this in $\underline{a}$ & therefore zero as well. So

$$|f_{n+1}(z)| \le \frac{M}{r^n(r-|z-a|)}, \quad \text{where } M = \|f\|_{\partial D}$$

and the estimate on the remainder term $f_{n+1}(z)(z-a)^{n+1}$ in $(*)$ is

$$|f_{n+1}(z)||z-a|^{n+1} \le \frac{M|z-a|^{n+1}}{r^n(r-|z-a|)}.$$

In the special case where z is restricted to $D_0 := D(a, r/2) \subset D$, with a slight weakening and assuming $a=0$, this becomes

$$|f_{n+1}(z)||z|^{n+1} \le \frac{M|z|^{n+1}}{\left(\frac{r}{2}\right)^n\left(\frac{r}{2}\right)} = M\left(\frac{2|z|}{r}\right)^{n+1}.$$

We want to build meromorphic functions from principal parts. For $F \in \mathcal{M}er(U)$ with a pole at α, and Laurent expansion about α

$$F(z) = \underbrace{\sum_{-m \le n < 0} A_n (z-\alpha)^n}_{=: P(z)} + \text{holomorphic},$$

P is called the principal part of F at α.

So the idea will be to add a holomorphic function on U to a sum of principal parts at distinct points $\{b_\nu\} \subset U$. When this set is infinite, the convergence question is interesting. It is also clear, by the definition of a meromorphic function (poles must be isolated in U) that the $\{b_\nu\}$ may not have an accumulation point in U;

but they will have at least one on $\partial U \cup \{\infty\}$ if they are infinite in number. In what follows we take $U = \mathbb{C}$.

So let $\{b_\nu\}_{\nu \geq 1} \subset \mathbb{C}$ be a sequence with $|b_\nu| \to \infty$, $\{P_\nu\}$ a sequence of polynomials with zero constant term; we may assume $b_\nu \neq 0$ $(\forall \nu)$ w/o loss of generality. The goal is to sum $\sum_\nu P_\nu\left(\frac{1}{z-b_\nu}\right)$, or (failing that)

$$\sum_\nu \left\{ P_\nu\left(\frac{1}{z-b_\nu}\right) - \text{holo}_\nu \right\}$$

which will at least have the same principal parts.

(centered at 0)

Set $M_\nu := \left\| P_\nu\left(\frac{1}{z-b_\nu}\right) \right\|_{\overline{D_{|b_\nu|/2}}}$, and (for now) let n_ν be an arbitrary increasing sequence of natural numbers. Expanding $P_\nu\left(\frac{1}{z-b_\nu}\right)$ about the origin in $D_{\frac{|b_\nu|}{2}+\epsilon}$, write p_ν for the n_ν^{th} partial sum. The remainder is bounded by (using $r = \frac{|b_\nu|}{2}$)

$$\left| P_\nu\left(\frac{1}{z-b_\nu}\right) - p_\nu(z) \right| \leq M_\nu \left(\frac{4|z|}{|b_\nu|}\right)^{n_\nu+1} \quad \forall z \in D_{\frac{|b_\nu|}{4}}.$$

That is, the series we want to sum is dominated by the power series

$$\sum_\nu \frac{M_\nu 4^{n_\nu+1}}{|b_\nu|^{n_\nu+1}} z^{n_\nu} =: \sum_\nu a_{n_\nu} z^{n_\nu}.$$

which converges absolutely and uniformly on $\overline{D_R}$ iff $\limsup_{\nu\to\infty} \sqrt[n_\nu]{|a_{n_\nu}|} < \frac{1}{R}$. In particular, if

$$(**) \qquad \limsup_{\nu\to\infty} \frac{M_\nu^{1/n_\nu}}{|b_\nu|} = 0$$

then taking $N(R)$ s.t. $\nu \geq N(R) \Rightarrow |b_\nu| > 4R$,

$$\sum_{\nu \geq N(R)} \left(P_\nu\left(\frac{1}{z-b_\nu}\right) - p_\nu(z) \right) \text{ converges unif. on } \overline{D_R}.$$

Consequently the tails of the series converge to holomorphic functions on arbitrarily large balls, and so the entire series converges to a meromorphic function on $\mathbb{C}$ with the desired principal parts.

To arrange condition $(**)$, we simply demand that

$$n_\nu > \log M_\nu,$$

since then

$$\log \frac{M_\nu^{1/n_\nu}}{|b_\nu|} = \underbrace{\frac{\log M_\nu}{n_\nu}}_{<1} - \log|b_\nu| \longrightarrow -\infty.$$

(Note that since the condition that $n_{\nu+1} > n_\nu$ is already in place, we are really assuming something more like $n_\nu > \nu + \log M_\nu$.)

II. Runge's theorem

On a disk, an arbitrary holomorphic function is the limit of a sequence of polynomials. This does <u>not</u> work on an arbitrary region — already $\frac{1}{z}$ on D_1^* furnishes a counterexample, since there is no polynomial with $\int_{|z|=\frac{1}{2}} P(z)\,dz = 2\pi i$ (or even nonzero). But if we replace "polynomial" by "rational function", we get a nice result:

Theorem Given $\overset{\text{compact}}{K} \subset \overset{\text{open}}{U} \subset \mathbb{C}$, $\underbrace{E \subseteq \hat{\mathbb{C}} \setminus K}_{\text{subset w/ points in every connected component}}$, and $f \in \mathrm{Hol}(U)$. Then $\forall \epsilon > 0$ there exists a rational function $R(z)$ with polar set $\subset E$ and $|f(z) - R(z)| < \epsilon$ on K.

With this in hand, we'll be able to prove a more general result on "constructing meromorphic functions from principal parts". We begin with the following easy

<u>Lemma 1</u>: Given a path $\gamma : [0,1] \to \mathbb{C} \setminus K$, $f \in C^0(\gamma)$, $F(z) := \int_\gamma \frac{f(w)}{w-z}\,dw$. Then $\forall \epsilon > 0$, $\exists$ rat'l fcn. $R(z)$ with polar set $\subset \gamma$, and $|F(z) - R(z)| < \epsilon$ on K.

Proof: Let $r \in (0, d(K,\gamma))$, $c \geq \max\{\|z\|_K, \|\gamma\|_{[0,1]}, \|f\|_\gamma\}$, $\alpha, \beta \in \gamma$. Then for $z \in K$,

$$\left|\frac{f(\alpha)}{\alpha - z} - \frac{f(\beta)}{\beta - z}\right| \leq \frac{1}{r^2}\left|\underbrace{\beta f(\alpha) - \alpha f(\beta) - z\{f(\alpha) - f(\beta)\}}_{= f(\alpha)(\beta - \alpha) + (\alpha - z)(f(\alpha) - f(\beta))}\right|$$

$$\leq \frac{c}{r^2}|\beta - \alpha| + \frac{2c}{r^2}|f(\alpha) - f(\beta)|.$$

By (uniform) continuity of γ & f, $\exists$ partition $0 = t_0 < t_1 < \cdots < t_n = 1$ s.t. $\left|\frac{f(\gamma(t))}{\gamma(t) - z} - \frac{f(\gamma(t_j))}{\gamma(t_{j-1}) - z}\right| < \frac{\epsilon}{L(\gamma)}$ $\forall t \in [t_{j-1}, t_j]$. Setting

$$R(z) := \sum_{j=1}^{n} \frac{f(\gamma(t_{j-1}))(\gamma(t_j) - \gamma(t_{j-1}))}{\gamma(t_{j-1}) - z},$$ we have

constant w.r.t. integral

$$|F(z) - R(z)| = \left|\sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \left\{\frac{f(\gamma(t))}{\gamma(t) - z} - \frac{f(\gamma(t_{j-1}))}{\gamma(t_{j-1}) - z}\right\} \gamma'(t)\, dt\right|$$

$$< \cancel{\frac{\epsilon}{L(\gamma)} \cdot L(\gamma)} = \epsilon. \qquad \square$$

Define a distance function ρ on $C^0(K)$ by

$$\rho(f,g) = \|f - g\|_K.$$

Since the uniform limit of a sequence of functions on K is C^0, this makes $C^0(K)$ into a complete metric space. Define

$$B(E) := \overline{\{\text{rational functions with poles} \subset E\}}$$

where the bar denotes closure in $C^0(K)$ w.r.t. ρ: i.e. uniform limits of the restrictions of such rational functions to K. $B(E)$ is obviously closed under $\times, +$, scalar mult.

Lemma 2: $\frac{1}{z-\alpha} \in B(E)$ for every $\alpha \notin K$.

Proof: Assume first that $\infty \notin E$; set $V := \{a \mid \frac{1}{z-a} \in B(E)\}$.

Clearly $E \subset V \subset \mathbb{C}\setminus K$.

Claim: If $a \in V$ & $|b-a| < d(a,K)$, then $b \in V$ ($\Rightarrow V$ open).

[Pf: $B(E) \ni \frac{1}{z-a} \underset{\substack{\text{closed} \\ \text{(under } +, \times, \cdot)}}{\Longrightarrow} B(E) \ni \frac{1}{z-a} \cdot \sum_{n=0}^{N} \left(\frac{b-a}{z-a}\right)^n$ which conv. uniformly on K

$$\Rightarrow B(E) \ni \frac{1}{z-a} \cdot \sum_{n=0}^{\infty} \left(\frac{b-a}{z-a}\right) = \frac{1}{z-b}.\ ]$$

Given $b \in \partial V$, let $V \ni a_n \to b$. Since V is open, $b \notin V$

$\underset{\text{Claim}}{\Longrightarrow} |b-a_n| \geq d(a_n,K) \underset{n\to\infty}{\Longrightarrow} 0 = d(b,K) \underset{K \text{ compact}}{\Longrightarrow} b \in K \Rightarrow \partial V \subset K$.

Let H be any component of $\mathbb{C}\setminus K$. By assumption $H \cap E \neq \emptyset$, so $H \cap V \neq \emptyset$. But $\partial V \subset K \Rightarrow H \cap \partial V = \emptyset$. If H has points both not in V and in V, path connectedness shows $H \cap \partial V \neq \emptyset$, a contradiction; so in fact $H \subset V$. Since H was arbitrary, $\mathbb{C}\setminus K \subset V$ and so $V = \mathbb{C}\setminus K$, and we're done in this case.

Next suppose $\infty \in E$, and set $E_0 := E\setminus\{\infty\} \cup \{a_0\}$ for $a_0 \in (\mathbb{C}\setminus K)^{\text{ub}}$ with $|a_0| > 2\|z\|_K$. (This is in $\mathbb{C}$ and still meets each component of $\hat{\mathbb{C}}\setminus K$.) By the above, $B(E_0) \ni \frac{1}{z-\alpha}$ $\forall \alpha \in \mathbb{C}\setminus K$. Moreover, $\left\|\frac{z_0}{a_0}\right\|_K \leq \frac{1}{2} \Longrightarrow$ $\frac{1}{z-a_0} = -\frac{1}{a_0}\sum_{n=0}^{\infty}\left(\frac{z}{a_0}\right)^n$ is a uniform limit on K of rational functions w/ pole at ∞, hence $\in B(E)$. Thus $B(E) \supset B(E_0)$ and we're done. $\square$

Proof of Runge's Thm.: We want to show that

(#) If f is analytic in a nbhd. (say, U) of K, then $f|_K \in B(E)$.

Let γ be a closed path in $U \setminus K$ with winding number 1 about every point of K. (That this is feasible follows from the construction of a homology basis for the connected open set $U \setminus K$ earlier in this course.) Then $f|_K = \int_\gamma \frac{f(w)}{w-z}\,dw$, and Lemma 1 $\Rightarrow \exists$ of R with poles in ($\gamma \subset$) $\mathbb{C}\setminus K$ s.t. $\|f-R\|_K < \epsilon$. But the fact that $B(E)$ is an algebra and $\frac{1}{z-\alpha} \in B(E)$ ($\forall \alpha \in \mathbb{C}\setminus K$) by lemma 2 $\Rightarrow R \in B(E)$. By taking ϵ smaller & smaller we get a sequence $R_n \longrightarrow f$ unif. on K, so that $f \in B(E)$ too. $\square$

III. Mittag-Leffler's theorem

We are now ready for the generalization of §I.

Theorem 2 Let $G \subset \mathbb{C}$ be an open set, $\{a_k\} \subset G$ a sequence of distinct points with no point of accumulation in G, and $S_k(z) = \sum_{j=1}^{m_k} \frac{A_{jk}}{(z-a_k)^j}$ the sequence of desired principal parts. Then $\exists\, f \in Mer(G)$ with polar set $\{a_k\}_{k=1}^\infty$ and principal parts S_k (at a_k).

Proof: By taking $K_n := \{z \mid d(z, \mathbb{C}\setminus G) \geq \frac{1}{n}\} \cap \overline{D}_n$,

we get a sequence of compact subsets $K_n \subset G$ with

$G = \bigcup_{n=1}^{\infty} K_n$, $K_n \subset \overset{\circ}{K}_{n+1}$ ← (interior), and

(†) each component of $\hat{\mathbb{C}}\setminus K_n$ containing a component of $\hat{\mathbb{C}}\setminus G$.

In each K_n there are finitely many $\{a_k\}$, and we define

$$I_1 := \{k \in \mathbb{N} \mid a_k \in K_1\}, \quad I_n := \{k \in \mathbb{N} \mid a_k \in K_n \setminus K_{n-1}\},$$

$$f_n(z) := \sum_{k \in I_n} S_k(z) \quad (n \geq 1).$$

Clearly f_n is rational & holomorphic in a neighborhood of K_{n-1}. Because of (†), Runge's Thm. $\Rightarrow \exists$ rational fcn. $R_n(z)$ with poles in $\hat{\mathbb{C}}\setminus G$, s.t. $\|f_n - R_n\|_{K_{n-1}} < (\frac{1}{2})^n$. Hence

$$f_1(z) + \sum_{k \geq 2} \left(f_k(z) - R_k(z)\right)$$

has tails converging uniformly on each K_n (by Weierstraß M-test w/ $M_k = (\frac{1}{2})^k$), and therefore on every compact subset of G; it thus defines a meromorphic function with the desired principal parts. $\square$

IV. Examples

These results are nice, but the main applications are on $\mathbb{C}$ and don't necessarily use them (just the method of subtracting off a polynomial from each principal part).

1 $\quad f(z) = \dfrac{\pi^2}{\sin^2 \pi z} \overset{\text{at } 0}{=} \dfrac{\pi^2}{\left(\pi z - \frac{(\pi z)^3}{3!} + \dots\right)^2} = \dfrac{1}{z^2} + \dfrac{\alpha}{z} + \text{holo.}$

But f even $\Rightarrow$ $\alpha = 0$ $\Rightarrow$ $PP_0(f) = \dfrac{1}{z^2} \xRightarrow{\text{periodicity}}$ $PP_n(f) = \dfrac{1}{(z-n)^2}$. Moreover, $\sum\limits_{n\in\mathbb{Z}} \dfrac{1}{(z-n)^2}$ is convergent by comparison to $\sum \frac{1}{n^2}$ (essentially), and so

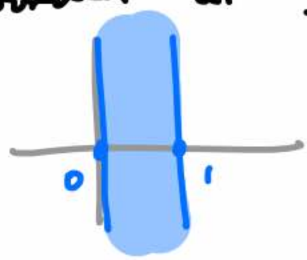
$$g(z) := f(z) - \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2} \in \mathrm{Hol}(\mathbb{C}) \quad !$$

Now g is periodic with period 1, and since

$$|\sin \pi z|^2 = \frac{e^{i\pi z} - e^{-i\pi z}}{2i} \, \frac{e^{-i\pi \bar z} - e^{i\pi \bar z}}{-2i} = \frac{e^{-2\pi y} + 2 + e^{2\pi y} - e^{2i\pi x} - 2 - e^{-2i\pi x}}{4}$$

$$= \cosh^2 \pi y - \cos^2 \pi x \longrightarrow \infty \quad \text{as } |y| \to \infty,$$

g is bounded on strips $\xRightarrow{\text{periodicity}}$ g bounded on $\mathbb{C}$



$\xRightarrow{\text{Liouville}}$ g constant.

But since the limit as $|y| \to \infty$ of both functions $f(z)$ & $\sum \frac{1}{(z-n)^2}$ is zero, g must then <u>be</u> zero.

Therefore

$$\boxed{\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}}$$

② $f(z) = \pi \cot(\pi z)$. Consider

$$\underset{n\in\mathbb{Z}}{{\sum}'^{\,\leftarrow (n\neq 0)}} \Big(\underbrace{\frac{1}{z-n}}_{PP} + \underbrace{\frac{1}{n}}_{\text{(polynomial correction)}} \Big) = \sum_{n\in\mathbb{Z}}{}' \frac{z}{n(z-n)} ,$$

which is unif. conv. on compact sets hence may be differentiated termwise:

$\frac{d}{dz}$

$$-\sum_{n\in\mathbb{Z}}{}' \frac{1}{(z-n)^2} = \frac{-\pi^2}{\sin^2 \pi z} + \frac{1}{z^2} .$$

So $$\frac{d}{dz}\big(\underbrace{\pi \cot \pi z}_{*}\big) = -\pi^2 \csc^2 \pi z = \frac{-\pi^2}{\sin^2 \pi z}$$

$$= \frac{d}{dz}\Big\{ \frac{1}{z} + \underbrace{\sum_{n\in\mathbb{Z}}{}' \Big(\frac{1}{z-n} + \frac{1}{n}\Big)}_{= \sum_{n\geq 1} \frac{1}{z-n} + \frac{1}{z+n}} \Big\}$$

$$= \frac{d}{dz}\Big\{ \underbrace{\frac{1}{z} + \sum_{n\geq 1} 2z/(z^2-n^2)}_{*} \Big\}$$

where both starred functions are odd, hence have 0 const. term and

$$\boxed{\pi \cot(\pi z) \underset{\text{on } \mathbb{C}}{=} \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2}} \underset{\text{on } D_1}{=} \frac{1}{z} + \sum_{m\geq 1} a_{2m-1} z^{2m-1}$$

$$\Rightarrow a_{2m-1} = \frac{1}{2\pi i} \oint_{|z|=\frac{1}{2}} \frac{\pi \cot(\pi z)}{z^{2m}}\, dz = \sum_{n=1}^{\infty} \frac{1}{2\pi i} \oint_{|z|=\frac{1}{2}} \frac{2z^{1-2m}}{z^2-n^2}\, dz$$

$$= \sum_{n=1}^{\infty} \mathrm{Res}_0 \left(\frac{2/(z^2-n^2)}{z^{2m-1}} \right) = \sum_{n=1} -\frac{2}{n^2} \cdot \frac{1}{n^{2(m-1)}}$$

$$2/(z^2-n^2) = \frac{-2/n^2}{1-z^2/n^2} = -\frac{2}{n^2} \sum_{k\geq 0} \frac{z^{2k}}{n^{2k}}$$

(Res picks out the term $k = m-1$)

$$= -2 \sum_{n=1}^{\infty} \frac{1}{n^{2m}} = -2\zeta(2m) .$$

But we computed before that $a_{2m-1} = \dfrac{\pi^{2m} B_{2m} (-4)^m}{(2m)!}$.

So $$\boxed{\zeta(2m) = \frac{2^{2m-1}|B_{2m}|}{(2m)!}\pi^{2m}},$$ a rational multiple of π^{2m}.