Lecture 33: Extensions and boundary values

I. Schwarz reflection

Let $U \subseteq \mathbb{C}$ be a region and write

$$U^* := \{ z \mid \bar{z} \in U \}.$$

(I don't write $\overline{U}$ because that is already used for closure.)

Lemma: (i) If $f(z) \in Hol(U)$, then $\overline{f(\bar{z})} \in Hol(U^*)$.

(ii) If $u(z) \in \mathcal{H}(U)$, then $u(\bar{z}) \in \mathcal{H}(U^*)$.

Proof (i): There are a few ways to see this: e.g.

- $$\frac{\partial}{\partial \bar{z}} \overline{f(\bar{z})} = \overline{\frac{\partial}{\partial z} f(\bar{z})} = \overline{\underbrace{\cancel{\frac{\partial f}{\partial \bar{z}}}}_{0} \frac{\partial z}{\partial z} + \frac{\partial f}{\partial z} \underbrace{\cancel{\frac{\partial \bar{z}}{\partial z}}}_{0}} = 0$$
- locally $f(z) = \sum \alpha_n z^n$
  $$\Rightarrow f(\bar{z}) = \sum \alpha_n \bar{z}^n \Rightarrow \overline{f(\bar{z})} = \sum \bar{\alpha}_n z^n.$$

(ii): In a nbhd. of any pt. $z_0 \in U$, $u = Re(f_0)$

Then $f_0(z) = u(z) + i v_0(z) \Rightarrow \overline{f_0(\bar{z})} = u(\bar{z}) - i v_0(\bar{z})$ holo.

$$\Rightarrow u(\bar{z}) \text{ harmonic.}$$

$\square$

Now we will assume henceforth that

$$U = U^*,$$

i.e. U is symmetric. Since U is open it must contain a point in $h \cup h^*$, say $z_0 \in h$, hence also $\bar{z}_0 \in h^*$. Since it is connected, it contains a path from z_0 to $\bar{z}_0$ hence (by the intermediate value thm.!) a point of $\mathbb{R}$. Again using openness, U contains an interval $(a,b) \subset \mathbb{R}$. Write

$$U \cap h =: U^+$$
$$U \cap \mathbb{R} =: \delta$$
$$U \cap h^* =: U^- \quad (= (U^+)^*).$$

The rough idea of Schwarz reflection is this:

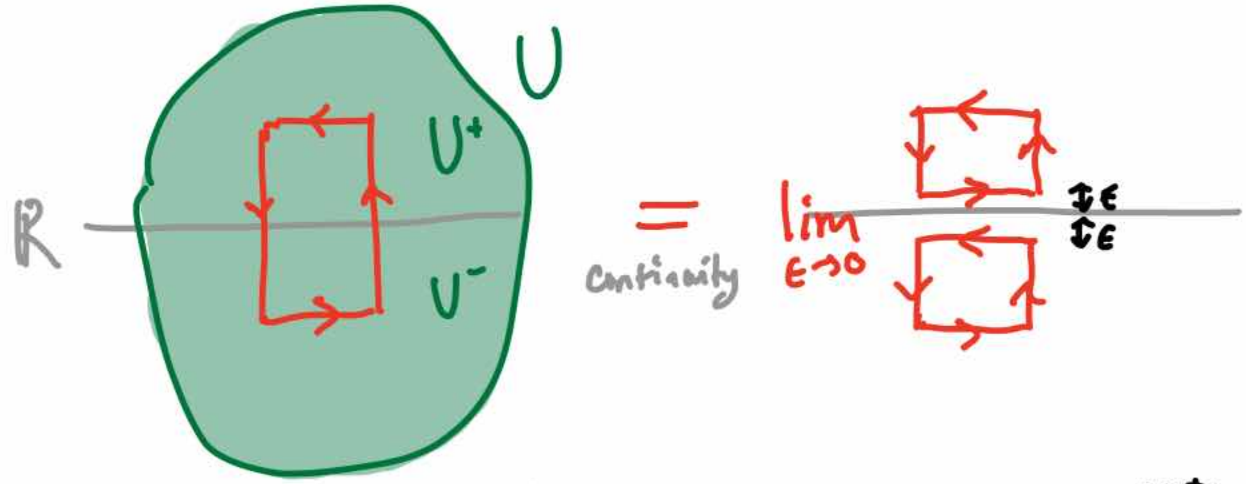
Proposition Let $f \in C^0(U^+ \cup \delta)$ with $f|_{U^+}$ holomorphic, and $f|_\delta$ real. Then $\exists\, g \in \mathrm{Hol}(U)$ s.t. $g|_{U^+ \cup \delta} = f$.

Proof: For $z \in U^-$ set $g(z) := \overline{f(\bar{z})}$,

$z \in U^+ \cup \delta$ set $g(z) := f(z)$.

Since $\overline{f(\bar{z})} = f(z)$ on δ, $g \in C^0(U)$. Clearly $g|_{U^+ \cup U^-}$ is holomorphic (use the lemma). We need to check that g is holo. on δ. So....

apply Morera to g's integrals over rectangles



which vanish due to holomorphicity of g on $U^+ \cup U^-$. $\square$

The problem is that this isn't a very strong result: we don't want to assume that f extends continuously to $[U^+ \cup] \mathcal{S}$, just that $\mathrm{Im}(f)$ extends continuously to $\mathcal{S}$ with the value zero there. To do this we first need to prove a reflection principle for harmonic functions:

Theorem 1 Let $v \in C^0(U^+ \cup \mathcal{S})$, with $v|_{U^+}$ harmonic & $v|_{\mathcal{S}} \equiv 0$. Then v extends to a harmonic function on U s.t. $v(\bar{z}) = -v(z)$.

Proof: Set $V(z) := \begin{cases} v(z), & z \in U^+ \\ 0, & z \in \mathcal{S} \\ -v(\bar{z}), & z \in U^- \end{cases}$;

by hypothesis this belongs to $C^0(U) \cap \mathcal{H}(U^+ \cup U^-)$.

We need to check "harmonicity on $\mathcal{S}$". Let $x_0 \in \mathcal{S}$, $D := D(x_0, \epsilon) \subset U$, and put $(\forall \alpha \in D)$

$$P_V(\alpha) := \frac{1}{2\pi}\int_{\partial D} V(z)\,\mathrm{Re}\left(\frac{z+\alpha}{z-\alpha}\right) d\theta. \qquad (z = \epsilon e^{i\theta})$$

By the solution to Dirichlet in D_1, we know that this extends continuously to V on the boundary) (since V is C^0). Moreover,

$$P_V \in \mathcal{H}(D_1) \implies V - P_V \in \mathcal{H}(D \cap U^+)$$

and

$$V - P_V \Big|_{\partial D \cap U^+} \equiv 0.$$

Also, if $\alpha \in \mathbb{R}$ then

$$\begin{aligned} P_V(\alpha) &= \underbrace{\frac{1}{2\pi}\int_0^{\pi} v(e^{i\theta})\,\mathrm{Re}\left(\frac{e^{i\theta}+\alpha}{e^{i\theta}-\alpha}\right)d\theta}_{v(z)\text{ part}} \\ &\quad \underbrace{- \frac{1}{2\pi}\int_{\pi}^{2\pi} v(e^{-i\theta'})\,\mathrm{Re}\left(\frac{e^{i\theta'}+\alpha}{e^{i\theta'}-\alpha}\right)d\theta'}_{-v(\bar{z})\text{ part}} \\ &= \frac{1}{2\pi}\int_0^{\pi} v(e^{i\theta})\,\mathrm{Re}\left(\frac{e^{i\theta}+\alpha}{e^{i\theta}-\alpha}\right)d\theta \\ &\quad + \frac{1}{2\pi}\int_{\pi}^{0} v(e^{i\theta})\,\underbrace{\mathrm{Re}\left(\frac{e^{-i\theta}+\alpha}{e^{-i\theta}-\alpha}\right)}_{=\mathrm{Re}\left(\frac{e^{i\theta}+\alpha}{e^{i\theta}-\alpha}\right)}d\theta \\ &= 0. \end{aligned}$$

$$\left(\begin{aligned} &\text{2nd term:} \\ &\theta = 2\pi - \theta' \\ &d\theta = -d\theta' \end{aligned}\right)$$

So $P_V\big|_{\mathcal{S}\cap D} \equiv 0$, while meanwhile (by defn.) $V\big|_{\mathcal{S}} \equiv 0$

$$\Rightarrow \quad V - P_V \Big|_{\underbrace{\partial(D \cap U^+)}_{= (\partial D \cap U^+) \cup (\gamma \cap D)}} \equiv 0$$

$\Rightarrow \quad V - P_V$ is harmonic on $D \cap U^+$,

continuous on $\overline{D \cap U^+}$,

and $\equiv 0$ on the boundary

$\Rightarrow \quad V - P_V \equiv 0$ on $D \cap U^+$.

MMP

$\mathcal{I}_{max}$ & $\mathcal{I}_{min}$

Repeating the argument for $D \cap U^-$, we finally have

$$V = P_V \qquad \text{on} \qquad D.$$

Since P_V was harmonic there, so is V. We have established harmonicity in a neighborhood of each point of γ, & so we are done. $\square$

Examples // $v(z) = xy$ and $e^x \sin(y)$ are examples of harmonic functions which are $\equiv 0$ on $\mathbb{R}$. //

$x + iy$

Now we are ready for the holomorphic analogue.

Theorem 2 Let $F \in \mathcal{H}ol(U^+)$ satisfy

$$\lim_{U^+ \ni z \to x_0} \operatorname{Im}(F(z)) = 0 \quad (\forall\, x_0 \in \mathcal{S}).$$

Then $\phi(x) := \lim_{U^+ \ni z \to x} \operatorname{Re}(F(z))$ exists $\forall\, x \in \mathcal{S}$, and

$$G(z) := \begin{cases} F(z), & z \in U^+ \\ \phi(z), & z \in \mathcal{S} \\ \overline{F(\bar{z})}, & z \in U^- \end{cases}$$

extends F to a holomorphic function on U.

Proof: Let $x_0 \in \mathcal{S}$, $D := D(x_0, \epsilon) \subset U$.

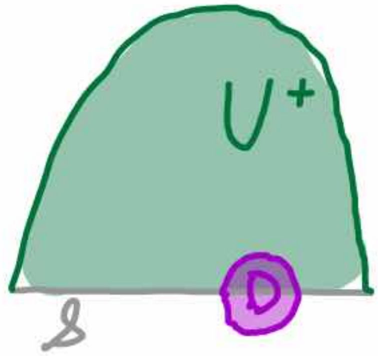
Set $v := \operatorname{Im}(F) \in \mathcal{H}(D \cap U^+)$.

Clearly v satisfies the hypotheses of Theorem 1, so extends to $v \in \mathcal{H}(D)$ satisfying $v(\bar{z}) = -v(z)$. We choose a harmonic conjugate $-u := \int * dv$ and compute

$$\operatorname{Im}(F - \underbrace{(u+iv)}_{\text{holo.}}) = \operatorname{Im}(F) - v = 0 \quad \text{on } D \cap U^+$$

$\Rightarrow F - (u+iv)$ constant there. Changing u by a constant, we get that $u+iv\big|_{D \cap U^+} = F$. So

$\phi(x)$ exists $(= u|_{D\cap \ell})$, and
on



$G_0(z) := u(z) + iv(z) \in \mathcal{H}ol(D)$

extends F to D. By the Lemma,

also $\overline{G_0(\bar z)} \in \mathcal{H}ol(D)$. Since $v \equiv 0$ on $D \cap \ell$,

G_0 is real there $\Longrightarrow \left(G_0(z) - \overline{G_0(\bar z)}\right)\Big|_{D\cap\ell} \equiv 0$

$$\Longrightarrow G_0(z) \equiv \overline{G_0(\bar z)} \text{ on } D.$$

We conclude that $G_0 = G|_D$, so that G (see stmt. of theorem) is holomorphic on D, which is all that was in question. $\square$

$\boxed{\text{Corollary}}$ Suppose $f \in C^0(\overline{h})$ with $f|_h$ holomorphic and $f|_{(a,b)} \equiv 0$. Then $f \equiv 0$ on h.

Proof: Let $x_0 \in (a,b)$ & $(x_0-\epsilon, x_0+\epsilon) \subset (a,b)$; restrict f to $D(x_0,\epsilon) \cap \overline{h}$. By Thm. 2, we get an extension g to $D(x_0,\epsilon)$ which has $\underbrace{g|_{(x_0-\epsilon,x_0+\epsilon)}}_{\text{has accumulation point}} \equiv 0 \Longrightarrow g \equiv 0$

$$\Longrightarrow f|_{D\cap h} \equiv 0$$

$$\Longrightarrow f \equiv 0. \quad \square$$

II. Some remarks on the Dirichlet problem

We begin with an application of the result on harmonic functions on a punctured disk from the end of Lecture 32.

Proposition If $u \in \mathcal{H}(D_\rho^*)$ is bounded, then u extends to a harmonic function on D_ρ.

Proof: We had that for $0 < r < \rho$

$$\int_{C_r} u \, d\theta = \beta + \underbrace{\left(\int_{C_r} {}^* du \right)}_{\text{constant in } r} \log r .$$

Since u is bounded, the LHS is bounded as $r \to 0^+$, and so $\int_{C_r} {}^* du$ must $= 0$. Hence u has a well-defined harmonic conjugate $\int {}^* du$ on D_ρ^* and is thus the real part of a global holomorphic f (on D_ρ^*).

Expanding f in Laurent series

$$f(z) = \sum_{n \in \mathbb{Z}} \underset{\substack{\shortparallel \\ a_n + i b_n}}{\alpha_n} z^n \implies u(\underset{\substack{\shortparallel \\ re^{i\theta}}}{z}) = \sum_{n \in \mathbb{Z}} (a_n \cos n\theta - b_n \sin n\theta) r^n$$

makes it clear that f can't have a pole (we'd have $u(re^{i\theta}) = \frac{1}{r^m} \left(\sum_{l \geq 0} P_l(\theta) r^l \right) \Big)$; while if f had an essential singularity then it comes close to arbitrary complex values in any D_ϵ^*, hence u comes arbitrarily close to arbitrary (larger & larger) <u>real</u> values. So f must have $\alpha_n = 0 \ \forall n < 0$, and the result follows. $\square$

<u>Example</u> // ("Solve the Dirichlet problem for $U = D_1^*$, with boundary value $f \in C^0_{\mathbb{R}}(\partial U)$ given w/ $f(0) = 0$.")

[Note that $\partial U = \{0\} \amalg \partial D_1$.]

Suppose u is a solution: $u \in C^0(\overline{D}_1)$ with $u(0) = 0$, $u|_{\partial D_1}$ agreeing with f, $u|_{D_1}$ harmonic. Then in particular u is bounded in D_1^* $\underset{\text{Prop.}}{\Longrightarrow}$ u extends to a harmonic function on D_1 (which, by continuity, must satisfy $u(0) = 0$). Thus by the MVT, $0 = u(0) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\, d\theta$, so the problem is only soluble if the $\int$ of f around the boundary is zero (and clearly this condition is sufficient). //

Assuming the Riemann mapping theorem, we should always be able to solve Dirichlet on a simply connected region $U \subsetneq \mathbb{C}$ <u>provided $\overline{U}$ is compact</u>. The latter condition is unpleasant, and so to get around it we need to first extend Dirichlet on D_1 to the setting where f is piecewise continuous and bounded on ∂D_1. There is in fact nothing in the proof on p. 9 of lecture 31 (of Dirichlet) that used continuity on $\underline{\gamma_1}$ (just on γ_2), so we have:

$\boxed{\text{Theorem}}$ Let f be a real, bounded, piecewise continuous function on ∂D_1, and define $u(z)$ by the Poisson formula. Then $u \in C^0(\overline{D}_1 \setminus \{\text{pts of discontinuity of } f\})$, and $u|_{D_1} \in \mathcal{H}(D_1)$.

Remark: You need a version of the footnote Lemma of Lect. 31 for piecewise continuous functions, which is just obtained by working on the intervals where f is continuous separately and then summing the results. //

Assuming RMT plus a result on extension of the mapping to the boundary, we get the

$\boxed{\text{Corollary}}$ Let $U \subsetneq \mathbb{C}$ be simply connected, & $f : \partial U \to \mathbb{R}$ bdd. / pcwise continuous. Then there exists $u \in \mathcal{H}(U)$ extending continuously to the boundary with value f at all points of continuity of f.

Proof: RMT $\Rightarrow \exists$ of $F : D_1 \xrightarrow{\cong} U$ (analytic isomorphism)

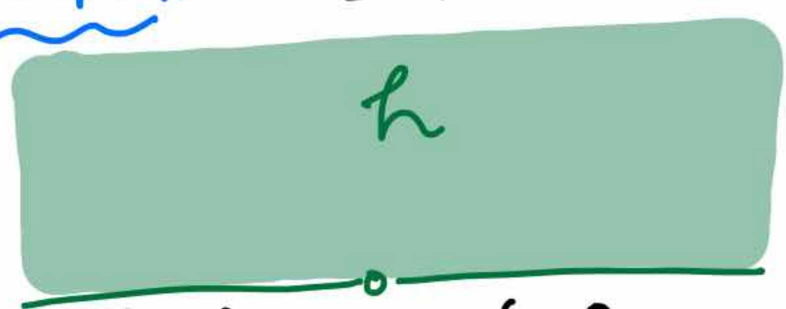
extends to continuous $\Big\}$

$\tilde{F} : \bar{D}_1 \xrightarrow{\cong} \bar{U}$ (homeomorphism)

Composing $\partial \bar{D} \xrightarrow[\tilde{F}]{} \partial \bar{U} \xrightarrow[f]{} \mathbb{R}$ and applying Poisson formula gives $u : D_1 \to \mathbb{R}$ harmonic, which (precomposing w/ F^{-1}) yields the sol'n. on U. $\square$

The main reason for bringing this up before RMT is so we can try to solve Dirichlet in some easy cases:

Example // Does there exist a $h \xrightarrow[\text{function}]{\text{harmonic}} \mathbb{R}$?



$f = 20$ $\qquad$ $f = 0$ $\qquad$ (with these bdry. values)

Don't even bother with the disk:

the point is that $\log(z) = \log|z| + i\arg(z)$, hence $\log|z|, \arg(z)$ are harmonic. And $\arg(z)$ is just θ, which is 0 where $f = 0$ and π where $f = 20$.

So $\frac{20}{\pi} \arg(z) =: u(z)$ does it.

In fact, this example on the upper half-plane is useful for solving problems elsewhere by explicitly constructing conformal maps to h [HW].