Lecture 32: More on harmonic functions

I. Fourier series

This is a nice application of the Theorem on Dirichlet's problem from Lecture 31. Moreover, it gives a better idea of what the harmonic functions of that Theorem look like.

Let $f \in C^0_{\mathbb{R}}([0,2\pi])$, $f(0) = f(2\pi)$. The Theorem just mentioned guarantees the existence of $u \in C^0(\overline{D_1})$ satisfying:

(a) $u|_{D_1}$ is harmonic, hence of the form

$$u(re^{i\theta}) = \mathrm{Re}\left(\sum_{n\geq 0} \alpha_n (re^{i\theta})^n\right)$$

$a_0 = 2\,\mathrm{Re}\,\alpha_0$, $a_{n>0} = \mathrm{Re}\,\alpha_n$, $b_n = -\mathrm{Im}\,\alpha_n$

(*) $$= \frac{a_0}{2} + \sum_{n\geq 1} a_n r^n \cos(n\theta) + \sum_{n\geq 1} b_n r^n \sin(n\theta),$$

where the series are absolutely and uniformly convergent on $\overline{D_r}$ for $r<1$

& (b) $u(e^{i\theta}) = f(\theta)$.

Now using basic trigonometric integrals, (a) gives

$$\frac{1}{\pi}\int_0^{2\pi} \cos(n\theta)\, u(re^{i\theta})\, d\theta = a_n r^n$$
$$\frac{1}{\pi}\int_0^{2\pi} \sin(n\theta)\, u(re^{i\theta})\, d\theta = b_n r^n .$$

Taking $r \to 1^-$ and using (uniform) continuity of u on $\overline{D_1}$ together with (b),

$$\left.\begin{aligned} \frac{1}{\pi}\int_0^{2\pi} \cos(n\theta)\, f(\theta)\, d\theta &= a_n \\ \frac{1}{\pi}\int_0^{2\pi} \sin(n\theta)\, f(\theta)\, d\theta &= b_n . \end{aligned}\right\} (*)$$

(*) may be regarded as the definition for <u>Fourier coefficients of f</u>. Together with (a), this gives a formula for $u|_{D_1}$ as a series. What about f?

This is a nontrivial question, since "the series converges to u on D_1" and "u is continuous on $\overline{D_1}$" do NOT imply that the series converges to u on ∂D_1.

Since f is (uniformly) continuous, $\forall \epsilon > 0\ \exists \delta > 0$ s.t. $|\theta_2 - \theta_1| < \delta \Rightarrow |f(\theta_2) - f(\theta_1)| < \frac{\epsilon}{2}$. This means that for $n > \frac{2\pi}{\delta}$ (i.e. sufficiently large),

$$\left| \frac{1}{\pi} \int_{\frac{2\pi}{n}m}^{\frac{2\pi}{n}(m+1)} \underbrace{f(\theta)}_{\substack{f\left(\frac{2\pi}{n}m\right)+\tilde{f}_m(\theta), \text{ where} \\ |\tilde{f}_m(\theta)| < \frac{\epsilon}{2} \text{ since } \frac{2\pi}{n} < \delta}} \cos(n\theta)\, d\theta \right| = \left| \frac{1}{\pi} \int_{\frac{2\pi}{n}m}^{\frac{2\pi}{n}(m+1)} \tilde{f}_m(\theta) \cos(n\theta)\, d\theta \right| < \frac{1}{\pi} \cdot \frac{2\pi}{n} \cdot \frac{\epsilon}{2} = \frac{\epsilon}{n}$$

$$\Rightarrow |a_n| = \left| \sum_{m=0}^{n-1} \frac{1}{\pi} \int_{\frac{2\pi}{n}m}^{\frac{2\pi}{n}(m+1)} f(\theta) \cos(n\theta)\, d\theta \right| < \epsilon. \qquad (**)$$

So $a_n \to 0$ as $n \to \infty$. (Similar for b_n)

But this isn't good enough for convergence, and indeed if $\mathcal{S} \subset [0, 2\pi]$ is <u>any</u> set of measure zero, then $\exists f \in C^0([0, 2\pi])$ whose Fourier series diverge (unboundedly!) on $\mathcal{S}$. A famous theorem of <u>Carleson</u> implies that the Fourier series at least converges pointwise almost everywhere, but still this is a bit shocking

(A) when you first learn Fourier series from physicists who repeat the mantra that $f \in C^k \Rightarrow a_n \sim \frac{1}{n^{k+2}}$

(B) in light of our theorem on Dirichlet for D_1.

The problem is that, while <u>u limits to $f(e^{i\theta})$ at each point $e^{i\theta_0} \in \partial D_1$</u>, this statement amounts to Abel summability of the Fourier series at Θ_0, which is weaker than ordinary summability!

To fix this, suppose now that f is <u>everywhere differentiable, with bounded derivative</u> (weaker than C^1).

Then $\|f'\|_{[0,2\pi]} \leq M$, and so if $\epsilon = \frac{2\pi M}{n}$ then we can take $\delta = \frac{\epsilon}{M} = \frac{2\pi}{n} \underset{(**)}{\Longrightarrow} |n a_n| \leq \frac{2\pi M}{\epsilon} \cdot \epsilon = 2\pi M$ $(\forall n)$.

(& same for b_n).

If f is C^1, then

$\int$ by parts

$$|a_n| = \left|\frac{1}{n}\int_0^{2\pi} f(\theta)\cos(n\theta)\,d\theta\right| = \left|-\frac{1}{\pi n}\int_0^{2\pi} f'(\theta)\sin(n\theta)\,d\theta\right|$$

$[b_n]$ $[\sin]$ $[+]$ $[\cos]$

$$\leq \frac{\epsilon}{n}$$

[same technique (appl. to f') as in the derivation of $(**)$.]

and we conclude that $|n a_n| \to 0$.

$[b_n]$

In the first case ($n a_n$ bounded) we can use Littlewood's theorem, in the second case ($n a_n \to 0$) Tauber's theorem (proved in Lecture 8), to assert that $(\forall \theta_0)$

$$f(\theta_0) = \lim_{r \to 1^-} u(re^{i\theta_0})$$

$$= \lim_{r \to 1^-} \left(\frac{a_0}{2} + \sum_{n \geq 1} \left[a_n \cos(n\theta_0) + b_n \sin(n\theta_0) \right] r^n \right)$$

$$= \frac{a_0}{2} + \sum_{n \geq 1} \left[a_n \cos(n\theta_0) + b_n \sin(n\theta_0) \right]$$

(and that this last expression converges).

Conversely, by Abel's theorem, whenever the last expression converges, the $\lim_{r \to 1^-}$ must equal it; and since the $\lim_{r \to 1^-}$ gives $u(re^{i\theta_0}) = f(\theta_0)$, we have the

$\boxed{\text{Theorem}}$ Let $f \in C^0_{\mathbb{R}}([0, 2\pi])$ and a_n, b_n be its Fourier coefficients. Then:

(i) Whenever $\frac{a_0}{2} + \sum_{n \geq 1} a_n \cos(n\theta) + b_n \sin(n\theta)$ (the Fourier series of f) converges, it converges to $f(\theta)$.

(ii) The Fourier series converges everywhere if f is everywhere differentiable, with bounded derivative.

II. Harmonic differentials

Though we only want to deal with 1-forms on $\mathbb{C}$,
I'll start by putting this in a broader context:

If M is a Riemannian n-manifold, one can take an oriented orthonormal basis $\{\theta_i\}$ for local coframes[†] and define

$$\theta_{i_1} \wedge \cdots \wedge \theta_{i_k} \wedge \underbrace{*(\theta_{i_1} \wedge \cdots \wedge \theta_{i_k})}_{\text{defn.}} := \theta_1 \wedge \cdots \wedge \theta_n .$$

$*$ is called the Hodge star operator.

Presently this yields

$$\left.\begin{aligned} dx \wedge * dx &= dx \wedge dy \implies * dx = dy \\ dy \wedge * dy &= dx \wedge dy \implies * dy = -dx \end{aligned}\right\} \text{ note } ** = -1$$

so
$$*(A\,dx + B\,dy) = -B\,dx + A\,dy$$
$$* dz = *(dx + i\,dy) = -i\,dx + dy = -i(dx + i\,dy) = -i\,dz$$
$$* d\bar{z} = *(dx - i\,dy) = i\,dx + dy = i(dx - i\,dy) = i\,d\bar{z} .$$

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† that is, a basis (over C^∞ functions) for differential 1-forms in a neighborhood, which is orthonormal in the metric on the cotangent space at each point.

Hence $i*(A\,dz + B\,d\bar{z}) = A\,dz - B\,d\bar{z}$, and in particular

$$i*A\,dz = A\,dz.$$

Now let ω be a real 1-form:

$$\omega = a\,dx + b\,dy = A\,dz + B\,d\bar{z},$$ a, b real-valued C^∞ functions.

Suppose $d\omega = 0 = d(*\omega)$ ($\overset{\text{def.}}{\Longleftrightarrow}$ ω is a harmonic differential).

Then $\phi := \omega + i*\omega = 2A\,dz$ satisfies

$$d\phi = 0 = -2\frac{\partial A}{\partial \bar{z}}\,dz\wedge d\bar{z}, \quad \text{i.e. } \frac{\partial A}{\partial \bar{z}} = 0 \quad (A \text{ holo.}).$$

Since $\omega = \bar{\omega} \Rightarrow B = \bar{A}$, we have $\omega = \mathrm{Re}(\underbrace{2A\,dz}_{\text{holo. 1-form}})$.

Conversely, taking $\phi = 2A\,dz$ with $\partial A/\partial\bar{z} = 0$, write

$$\phi = \tfrac{1}{2}(\phi + \bar{\phi}) + i\cdot\tfrac{1}{2i}(\phi - \bar{\phi}) =: \omega + i\eta.$$ We have

(real 1-forms)

$$0 = d\phi = d\omega + i\,d\eta \Rightarrow d\omega = 0 = d\eta.$$

But $i*\phi = \phi \Rightarrow \omega + i\eta = i(*\omega + i*\eta) = -*\eta + i*\omega$

$$\Rightarrow \eta = *\omega$$

$$\Rightarrow d(*\omega) = 0 \quad (\Rightarrow \omega \text{ harmonic}).$$

We have proved

$\boxed{\text{Proposition 1}}$ The harmonic differentials ω are precisely the real parts of holo. differentials, with $*\omega$ equal to the imag. part.

Now again let ω be harmonic. Using Prop. 1, $\omega = \mathrm{Re}(f\,dz)$, f holomorphic. But f is <u>locally</u> the derivative of something else analytic, say F. (This is OK "globally" in any simply-connected region in which f is defined & holomorphic.) So (locally) $\omega = \mathrm{Re}(dF) = d(\mathrm{Re}(F))$.

By differentiating the Cauchy-Riemann equations once, $\mathrm{Re}(F)$ is a harmonic function. [$\mathrm{Re}(F)_x = \mathrm{Im}(F)_y \Rightarrow \mathrm{Re}(F)_{xx} = \mathrm{Im}(F)_{xy}$, $\mathrm{Re}(F)_y = -\mathrm{Im}(F)_x \Rightarrow \mathrm{Re}(F)_{yy} = -\mathrm{Im}(F)_{xy}$; now subtract.]

Conversely, suppose that u is harmonic and $\omega := du$ (locally). Using $\Delta = *d*d$ [HW], $d(*\omega) = d*du = 0$; and

(insert $-**$)

$d\omega = \underbrace{dd}_{0}\omega = 0$. So ω is harmonic.

Proposition 2 <u>The harmonic differentials are (locally) precisely the differentials of harmonic functions.</u>

or in simply connected regions

If u is a harmonic function in a region then we may define the harmonic conjugate of u as the integral of $*du$. [Easy exercise: this agrees with the previous defn.]

This both explains why harmonic conjugates aren't well-def'd. in general in multiply connected regions, and illustrates

Why the differential-form point of view is better!

If $u_1, u_2 \in \mathcal{H}(U)$ (U any region in $\mathbb{C}$) then on any simply-connected subregion U_0, u_j has a conjugate function v_j, with $u_j + iv_j \in \mathcal{Hol}(U_0)$.

$$\Rightarrow \Omega^1(U_0) \ni \Omega := (u_1+iv_1)\,d(u_2+iv_2)$$
$$= i\{v_1 du_2 + u_1 dv_2\} + \mathrm{Re}(\Omega)$$

$$\Rightarrow 0 = \int_{\partial K} v_1 du_2 + u_1 dv_2 \quad \forall K \subset U_0 \text{ 2-chain}$$

$$\Rightarrow 0 = \int_{\partial K} u_1 dv_2 - u_2 dv_1 = \underbrace{\int_{\partial K} u_1 * du_2 - u_2 * du_1}_{\text{nothing special to } U_0 \text{ here}}$$

subtract $u_2 dv_1 + v_1 du_2 = d(u_2 v_1)$ (exact 1-form)

$$\Rightarrow 0 = \int_\gamma u_1 * du_2 - u_2 * du_1 \quad \forall \gamma \subset U. \quad (\#)$$
$$\gamma \underset{\text{hom}}{\equiv} 0$$

<u>An application</u>

[HW]: $\begin{cases} * dr = r\, d\theta \\ * d\theta = -\frac{1}{r} dr \end{cases}$

If $u \in \mathcal{H}(D_R^*)$, then taking $U = D_R^*$, $\gamma = C_{r_2} - C_{r_1}$ $(0 < r_1, r_2 < R)$

we want to apply (#) to

$$u_1 = u \longrightarrow * du_1 = *\left(\frac{\partial u}{\partial r} dr + \frac{\partial u}{\partial \theta} d\theta\right) = r \frac{\partial u}{\partial r} d\theta - \frac{1}{r}\frac{\partial u}{\partial \theta} dr$$

$$u_2 = \log(r) \longrightarrow * du_2 = \frac{1}{r}\, r\, d\theta = d\theta$$
$(\in \mathcal{H}(D_R^*))$ and $u_3 = 1 \longrightarrow * du_3 = 0$.

$$\Longrightarrow \quad 0 = \int_\gamma u_1 * du_2 - u_2 * du_1$$

(#)

$$= \int_\gamma u \, d\theta - \int_\gamma \log(r) \underbrace{* du}_{\text{just } r \frac{\partial u}{\partial r} d\theta \text{, but the specific form is irrelevant}}$$

and

$$0 = \int_\gamma u_1 * du_3 - u_3 * du_1$$

$$= - \int_\gamma * du$$

$$\Longrightarrow \begin{cases} \int_{C_r} u \, d\theta - \log(r) \int_{C_r} * du & (=: \beta) \\ \int_{C_r} * du & (=: \alpha) \end{cases}$$

are both constant in $r \in (0, R)$.

$$\Longrightarrow \boxed{\int_{C_r} u \, d\theta = \beta + \alpha \log(r) \quad \left(\forall \, r \in (0, R) \right)}$$

which generalizes the MVT.†

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† recall MVT says that if $u \in \mathcal{H}(\underline{D_R})$, ← not just D_R^*

then $\alpha = 0$ and $\beta = u(0)$.